FAILURE OF KRULL-SCHMIDT FOR INVERTIBLE LATTICES OVER A DISCRETE VALUATION RING

ESTHER BENEISH

(Communicated by Martin Lorenz)

ABSTRACT. Let p be a prime greater than 3, and let N be the semi-direct product of a group H of order p by a cyclic C group of order $p - 1$, which acts faithfully on H . Let R be the localization of $\mathbb{Z}$ at p . We show that the Krull-Schmidt Theorem fails for the category of invertible RN -lattices.

1. INTRODUCTION

Let G be a finite group and let RG be the group ring of G with coefficients in a Dedekind domain R . An RG -lattice M is defined to be a finitely generated R -torsion-free RG -module. M is said to be a permutation lattice if it is R -free and has an R -basis permuted by G . M is said to be an invertible or a permutation projective lattice, if it is a direct summand of a permutation lattice. This note was motivated by a question of A. Merkurjev about the existence of a category of invertible lattices over a discrete valuation ring, for which the Krull-Schmidt Theorem failed. The question arose in the study of the problem of the uniqueness of a direct sum decomposition of the motive of a projective homogeneous variety into indecomposable objects in the category of Chow motives. This category contains a subcategory equivalent to the category of invertible lattices for a certain finite group. Failure of Krull-Schmidt for this subcategory implies failure of uniqueness of direct sum decompositions for the motives; see [4].

Let p be a prime greater than 3, and let N be the semi-direct product of a group H of order p by a cyclic group C of order $p - 1$, where C acts faithfully on H by conjugation. Let R denote the localization of the ring of integers, $\mathbb{Z}$, at the prime p . We show that the Krull-Schmidt Theorem fails for the category of invertible RN -lattices.

2. INVERTIBLE LATTICES

Let $N = H \rtimes C$ be as defined in the introduction. The ZN -lattice ZN/H is isomorphic to ZC , and ZH has the structure of a ZN -lattice via the isomorphism

$$ZN/C \cong ZH.$$

Received by the editors September 8, 2004 and, in revised form, October 24, 2004 and February 5, 2005.

2000 *Mathematics Subject Classification*. Primary 20C10, 20C11.

Key words and phrases. Permutation lattices, invertible lattices, Krull-Schmidt.

This research was partially supported by NSF grant #DMS-0244766.

Let I_H be the augmentation ideal of ZH . We have the following ZN -exact sequences:

$$0 \rightarrow I_H \rightarrow ZH \rightarrow Z \rightarrow 0.$$

Tensoring by I_H over Z and setting $V = I_H \otimes I_H$ we obtain

$$(2.1) \quad 0 \rightarrow V \rightarrow ZH \otimes I_H \rightarrow I_H \rightarrow 0.$$

One checks directly that $\text{Res}_C^N I_H \cong ZC$, and so the following isomorphisms are given by Frobenius reciprocity:

$$ZH \otimes I_H \cong ZN \otimes_{ZC} \text{Res}_C^N I_H \cong ZN \otimes_{ZC} ZC \cong ZN.$$

Therefore, (2.1) becomes

$$(2.2) \quad 0 \rightarrow V \rightarrow ZN \rightarrow I_H \rightarrow 0.$$

For any ZN -lattice M , we let $M^* = \text{Hom}(M, Z)$. Let q be a prime different from p , and let C_q be a q -Sylow subgroup of N . Without loss of generality, we may assume that C_q is contained in C . Thus $H^1(C_q, I_H) \cong H^1(C_q, ZC) = 0$. We also have $H^1(H, I_H) \cong Z/pZ$, and hence $H^1(H, I_H^*) \cong Z/pZ$. Let α generate $H^1(H, I_H^*)$. By [6, Proposition 12.5] there exists a ZN -lattice W and an exact sequence

$$0 \rightarrow I_H^* \rightarrow W^* \rightarrow ZN/H \rightarrow 0$$

such that the image of α in $H^1(H, W^*)$ is 0, and hence W^* is H^1 -trivial since $H^1(N, I_H^*)$ injects into $H^1(H, I_H^*)$. Since N is meta-cyclic this implies that W^* is invertible by [3, Proposition 2, section 1]. Dualizing the above sequence we obtain

$$(2.3) \quad 0 \rightarrow ZN/H \rightarrow W \rightarrow I_H \rightarrow 0$$

with W invertible.

Lemma 2.1. *There is an isomorphism of ZN -lattices*

$$V \oplus W \cong ZN \oplus ZN/H.$$

Proof. We form the following pullback diagram with sequences (2.2) and (2.3):

$$\begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \uparrow & & \uparrow & \\ 0 & \rightarrow & V & \rightarrow & ZN & \rightarrow & I_H \rightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \rightarrow & V & \rightarrow & M & \rightarrow & W \rightarrow 0 \\ & & & & \uparrow & & \uparrow \\ & & & & ZN/H & \rightarrow & ZN/H \\ & & & & \uparrow & & \uparrow \\ & & & & 0 & & 0 \end{array}$$

We have $(I_H)^K = 0$ for all subgroups K of N so that V is H^1 -trivial. Since N is meta-cyclic, this implies that V is invertible by [3, Proposition 2, section 1] By [3, Lemma 9, section 1], the middle horizontal and vertical sequences split, which proves the lemma. $\square$

Notation 2.2. For a ZN -lattice M , let M_p denote its localization at the prime p , and let $\hat{M}$ denote its p -adic completion.

Remark 2.3. We have

$$\hat{Z}N/H \cong \hat{Z}[x]/(x^{p-1} - 1) \cong \bigoplus_{k=1}^{p-1} \hat{Z}[x]/(x - \theta^k) \cong \bigoplus_{k=1}^{p-1} Z_k$$

where $Z_k \cong \hat{Z}[x]/(x - \theta^k)$ and θ is a primitive $(p-1)^{st}$ root of 1 in $\hat{Z}$. So Z_k is a $\hat{Z}N$ -module of $\hat{Z}$ -rank 1 with trivial H -action and such that if c generates C , then $c.1 = \theta^k$. Therefore,

$$\hat{Z}N \cong \hat{Z}H \otimes \hat{Z}C \cong \bigoplus_{k=1}^{p-1} \hat{Z}H \otimes Z_k.$$

Note that for each k , $\hat{Z}H \otimes Z_k$ is indecomposable since $\text{Res}_H^N \hat{Z}H \otimes Z_k \cong \hat{Z}H$, and $\hat{Z}H$ is an indecomposable $\hat{Z}H$ -module by [2, Corollary 19.24].

Theorem 2.4. *Let R denote the localization of Z at the prime p . Then the Krull-Schmidt Theorem fails for invertible RN -lattices.*

Proof. By [1, Theorem 2.3] we have

$$\hat{V} \cong \left(\bigoplus_{k=2}^{p-1} \hat{Z}H \otimes Z_k \right) \oplus Z_1$$

Therefore by Lemma 2.1 and Remark 2.3

$$\hat{W} \cong \left(\bigoplus_{k=2}^{p-1} Z_k \right) \oplus \hat{Z}H \otimes Z_1.$$

Let Q be the field of rational numbers. For each k dividing $p-1$, let ω_k be a primitive k^{th} root of 1 over Q . Then

$$QN/H \cong \bigoplus_{k|p-1} Q(\omega_k)$$

and the $Q(\omega_k)$ are the irreducible components of QN/H . Now ZN/H is isomorphic to ZC as a ZN -module, and since RC is a maximal R -order in QC we have

$$RN/H \cong \bigoplus_{k|p-1} R[\omega_k]$$

by [2, Proposition 31.2]. Therefore,

$$RN \cong \bigoplus_{k|p-1} RH \otimes R[\omega_k].$$

Now for each k we have

$$\hat{Z}[\omega_k] \cong \hat{Z}[x]/\phi_k(x)$$

where $\phi_k(x)$ is the k^{th} cyclotomic polynomial. As above we let $\theta \in Q_p$ be a primitive $(p-1)^{st}$ root of 1 over Q , where Q_p is the completion of Q at the prime p . Set $\omega_k = \theta^{(p-1)/k}$ and let $J_k = \{i \in \mathbb{Z} : 1 \leq i < k, (i, k) = 1\}$. Then $\phi_k(x) = \prod_{j \in J_k} (x - \omega_k^j)$. Therefore,

$$\hat{Z}[\omega_k] = \bigoplus_{j \in J_k} Z_j.$$

Consequently,

$$\hat{V} \cong \left(\bigoplus_{k|p-1, k \neq p-1} \hat{Z}H \otimes \hat{Z}[\omega_k] \right) \oplus \left(\bigoplus_{k \in J_{p-1}, k \neq 1} \hat{Z}H \otimes Z_k \right) \oplus Z_1$$

and

$$\hat{W} \cong \left(\bigoplus_{k|p-1, k \neq p-1} \hat{Z}[\omega_k] \right) \oplus \left(\bigoplus_{k \in J_{p-1}, k \neq 1} Z_k \right) \oplus \hat{Z}H \otimes Z_1.$$

To simplify notation set

$$M = \bigoplus_{k|p-1, k \neq p-1} RH \otimes R[\omega_k] \quad \text{and} \quad M' = \bigoplus_{k|p-1, k \neq p-1} R[\omega_k].$$

Since $\hat{Z}$ is a faithfully flat R -module by [5, Corollary 2.2] we have $\hat{V}/\hat{M} \cong V_p/\hat{M}$. Similarly, $\hat{W}/\hat{M}' \cong W_p/\hat{M}'$. Therefore, the RN -lattices $S = V_p/M$ and $S' = W_p/M'$ have the property that

$$\hat{S} = \left(\bigoplus_{(k \in J_{p-1}, k \neq 1)} Z_k \otimes \hat{Z}H \right) \oplus Z_1$$

and

$$\hat{S}' = \left(\bigoplus_{k \in J_{p-1}, k \neq 1} Z_k \right) \oplus \hat{Z}H \otimes Z_1.$$

We have $\hat{V} = \hat{M} \oplus \hat{S}$ and $\hat{W} = \hat{M}' \oplus \hat{S}'$, which implies that $V_p = M \oplus S$ and $W_p = M' \oplus S'$, [2, Proposition 30.17]. Since $V_p \oplus W_p = RN/H \oplus RN$ by Lemma 2.1, S and S' are invertible RN -lattices. Since

$$\hat{Z}[w_{p-1}] = \bigoplus_{k \in J_{p-1}} Z_k,$$

we get

$$\hat{S} \oplus \hat{S}' \cong \hat{Z}H \otimes \hat{Z}[w_{p-1}] \oplus \hat{Z}[w_{p-1}],$$

and so by [2, Proposition 30.17],

$$S \oplus S' \cong RH \otimes R[w_{p-1}] \oplus R[w_{p-1}].$$

Now $R[w_{p-1}]$ is indecomposable since $Q(\omega_{p-1})$ is irreducible, but it is not a direct summand of either S or S' , for if it were, then $\hat{Z}[w_{p-1}]$ would be a direct summand of $\hat{S}$ or $\hat{S}'$ which is a contradiction.

Note that the condition, that the prime p be greater than 3, is necessary for if $p = 3$, then $S = Z[\omega_2]$ and if $p = 2$, then $S = R = Z[\omega_1]$. $\square$

ACKNOWLEDGMENT

The author would like to thank the referee for a very thorough study of the manuscript.

REFERENCES

- [1] E. Beneish, Monomial actions of the symmetric group, J. of Alg. 265 (2003) 405-419. MR1987009 (2004b:20008)
- [2] C. Curtis and I. Reiner, Methods of Representation Theory, Vol. 1, Springer-Verlag, 1982. MR0632548 (82i:20001)
- [3] J.-L. Colliot-Thelene and J.-P. Sansuc, La R-equivalence sur les tores, Ann. Sci. Ecole Normale Sup. (4) 10 (1977), 175-230. MR0450280 (56:8576)
- [4] V. Chernousov and A. Merkurjev, Motivic decomposition of projective homogeneous varieties and the Krull-Schmidt Theorem, preprint.
- [5] I. Reiner, Maximal Orders, Academic Press, 1975.
- [6] D. Saltman, Lectures on Division Algebras, CMBS, Regional Conference series in Mathematics, number 94, 1998. MR1692654 (2000f:16023)

DEPARTMENT OF MATHEMATICS, CENTRAL MICHIGAN UNIVERSITY, MOUNT PLEASANT, MICHIGAN 48859

E-mail address: `benei1e@cmich.edu`