THE NUMERICAL INDEX OF THE L_p SPACE

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(Communicated by N. Tomczak-Jaegermann)

ABSTRACT. We give a partial answer to the problem of computing the numerical index of $L_p[0, 1]$ for $1 < p < \infty$.

1. INTRODUCTION

Given a Banach space X over $\mathbb{R}$ or $\mathbb{C}$, we write B_X for the closed unit ball and S_X for the unit sphere of X . The dual space is denoted by X^* , and $B(X)$ is the Banach algebra of all bounded linear operators on X . The *numerical range* of an operator $T \in B(X)$ is the subset $V(T)$ of the scalar field defined by

$$V(T) = \{x^*(Tx) : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\}.$$

The *numerical radius* is then given by

$$v(T) = \sup\{|\lambda| : \lambda \in V(T)\}.$$

Clearly, v is a semi-norm on $B(X)$, and $v(T) \leq \|T\|$ for every $T \in B(X)$. It was shown by Glickfeld [8] (and essentially by Bohnenblust and Karlin [3]) that if X is a complex space, then $e^{-1}\|T\| \leq v(T)$ for every $T \in B(X)$ where $e = \exp 1$, so that for complex spaces v is always a norm and is equivalent to the operator norm $\|\cdot\|$. Thus it is natural to consider the so-called *numerical index* of the Banach space X , namely the constant $n(X)$ defined by

$$n(X) = \inf\{v(T) : T \in S_{B(X)}\}.$$

Obviously, $n(X)$ is the greatest constant $k \geq 0$ such that $k\|T\| \leq v(T)$ for every $T \in B(X)$. Note that for any complex Banach space X , $e^{-1} \leq n(X) \leq 1$.

The concept of the numerical index was first suggested by G. Lumer [10] in a lecture at the North British Functional Analysis Seminar in 1968. At that time, it was known that if X is a complex Hilbert space (with $\dim X > 1$), then $n(X) = \frac{1}{2}$, and if it is real, then $n(X) = 0$ so that for real spaces, $0 \leq n(X) \leq 1$.

Later, Duncan, McGregor, Pryce and White [5] determined the range of values of the numerical index. More precisely they proved that

$$\begin{aligned} \{n(X) : X \text{ real Banach space}\} &= [0, 1], \\ \{n(X) : X \text{ complex Banach space}\} &= [e^{-1}, 1]. \end{aligned}$$

Received by the editors October 14, 2004 and, in revised form, February 10, 2005.
 2000 *Mathematics Subject Classification*. Primary 46B20, 47A12.
Key words and phrases. Numerical index.

Recently, Finet, Martín and Payá [7] also studied the values of the numerical index from the isomorphic point of view, and they proved that

$$\mathcal{N}(c_0) = \mathcal{N}(l_1) = \mathcal{N}(l_\infty) = \begin{cases} [0, 1] & \text{in the real case,} \\ [e^{-1}, 1] & \text{in the complex case,} \end{cases}$$

where $\mathcal{N}(X) := \{n(X, p) : p \in \mathcal{E}(X)\}$, $\mathcal{E}(X)$ denotes the set of all equivalent norms on the Banach space X and $n(X, p)$ is the numerical index of X equipped with the norm p .

The authors in [5] proved that the extreme case $n(X) = 1$ occurs for a large class of interesting spaces including all real or complex L-spaces and M-spaces. Lately, López, Martín and Payá [11] studied some real Banach spaces with numerical index 1. In fact, they proved that an infinite-dimensional real Banach space with numerical index 1 and satisfying the Radon-Nikodym property contains l_1 . This result is a partial answer to the conjecture [12]: Every infinite-dimensional Banach space with numerical index 1 contains either l_1 or c_0 . Very recently, Martín and Payá [13] studied the numerical index of vector-valued function spaces, and they proved that if K is a compact Hausdorff space and μ is a positive measure, then the Banach spaces $C(K, X)$ and $L_1(\mu, X)$ have the same numerical index as the Banach space X . For general information and background on numerical ranges, we refer to the books by Bonsall and Duncan [1], [2]. Further developments in the Hilbert space may be found in [9].

The computation of $n(L_p)$ for $1 < p < \infty$, $p \neq 2$ is much more complicated, in fact it is an open problem since 1968.

In this paper we give a partial answer to this problem (Theorem 2.1). Actually, we prove that for $1 \leq p < \infty$, the numerical index of the Banach space $L_p([0, 1], \mu)$, where μ is the Lebesgue measure on the unit interval, is equal to the numerical index of the l_p space. It is also known that the numerical index of the Banach space l_p is the limit of the sequence of numerical index of finite-dimensional subspaces l_p^m , $m = 1, 2, \dots$, [6]. The computation of the numerical index of the l_p^m -space then gives a complete answer to the problem of the numerical index of the L_p -space.

2. MAIN RESULTS

Theorem 2.1. *For $1 < p < \infty$, the numerical index of the Banach space $L_p[0, 1]$ is equal to the numerical index of l_p , i.e.,*

$$n(L_p[0, 1]) = n(l_p).$$

Before, we prove this theorem, we need to introduce some notation.

Let μ be the Lebesgue measure on $\Omega = [0, 1]$. For each integer $n \geq 1$ we denote by π_{2^n} the partition of Ω into 2^n dyadic intervals: $[0, \frac{1}{2^n}]$, $[\frac{1}{2^n}, \frac{2}{2^n}]$, $\dots$, $[\frac{2^n-1}{2^n}, 1]$. By V_n we denote the subspace of $L_p(\Omega, \mu)$ defined by

$$V_n = \left\{ \sum_{k=1}^{2^n} a_k 1_{[\frac{k-1}{2^n}, \frac{k}{2^n}]} : a_k \in \mathbb{K} \right\},$$

where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. P_n denotes the projection of $L_p(\Omega, \mu)$ onto V_n defined by

$$P_n(f) = \sum_{k=1}^{2^n} \left[\frac{1}{2^{-n}} \int_{[\frac{k-1}{2^n}, \frac{k}{2^n}]} f(t) d\mu(t) \right] 1_{[\frac{k-1}{2^n}, \frac{k}{2^n}]},$$

and V denotes the union of all subspaces V_n of $L_p(\Omega, \mu)$. We recall that V is a dense subspace of $L_p(\Omega, \mu)$ ([4], p. 140), that is, for each $f \in L_p(\Omega, \mu)$ the sequence $(P_n f)_n$ converges to f in $L_p(\Omega, \mu)$.

Lemma 2.2. *For each integer $n \geq 1$ and $T \in B(V_n)$ there exists $\tilde{T} \in B(V_{n+1})$ satisfying the following conditions:*

- (i) $\tilde{T}|_{V_n} = T$,
- (ii) $\|\tilde{T}\| = \|T\|$,
- (iii) $v(\tilde{T}) = v(T)$.

Proof. Let $E_k^{2^n}$ denote the interval $[\frac{k-1}{2^n}, \frac{k}{2^n}[$ for $k = 1, 2, \dots, 2^n$.

STEP 1. *We shall first prove that the conditions in Lemma 2.2 are satisfied for $n = 1$.*

Let $T \in B(V_1)$ be represented by its matrix $\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$ on the unit normal basis $\left\{ \frac{1_{E_1^{2^1}}}{\mu(E_1^{2^1})^{\frac{1}{p}}}, \frac{1_{E_2^{2^1}}}{\mu(E_2^{2^1})^{\frac{1}{p}}} \right\}$ of V_1 . Since $\mu(E_1^{2^1})^{\frac{1}{p}} = \mu(E_2^{2^1})^{\frac{1}{p}} = 2^{-\frac{1}{p}}$, the same matrix represents T on the basis $\{1_{E_1^{2^1}}, 1_{E_2^{2^1}}\}$. Consider the operator $\tilde{T} \in B(V_2)$ represented by its matrix

$$\begin{bmatrix} t_{11} & 0 & t_{12} & 0 \\ 0 & t_{11} & 0 & t_{12} \\ t_{21} & 0 & t_{22} & 0 \\ 0 & t_{21} & 0 & t_{22} \end{bmatrix}$$

on the normal basis $\left\{ \frac{1_{E_k^{2^2}}}{\mu(E_k^{2^2})^{\frac{1}{p}}} \right\}_{k=1, \dots, 4}$ of V_2 . Also, since $\mu(E_k^{2^2}) = 2^{-\frac{2}{p}}$, $k = 1, \dots, 4$, the same matrix represents $\tilde{T}$ on the basis $\{1_{E_k^{2^2}}\}_{k=1, \dots, 4}$.

For (i), we have

$$\begin{aligned} \tilde{T}(1_{E_1^{2^1}}) &= \tilde{T}(1_{E_1^{2^2}}) + \tilde{T}(1_{E_2^{2^2}}) \\ &= t_{11}1_{E_1^{2^2}} + t_{21}1_{E_3^{2^2}} + t_{11}1_{E_2^{2^2}} + t_{21}1_{E_4^{2^2}} = T(1_{E_1^{2^1}}). \end{aligned}$$

Also, $\tilde{T}(1_{E_2^{2^1}}) = T(1_{E_2^{2^1}})$. This means that $\tilde{T}|_{V_1} = T$.

For (ii), let $x = \sum_{k=1}^4 x_k \frac{1_{E_k^{2^2}}}{\mu(E_k^{2^2})^{\frac{1}{p}}} \in V_2$. Clearly, $\|x\|^p = \sum_{k=1}^4 |x_k|^p$, and we have

$$\tilde{T}(x) = \begin{bmatrix} t_{11} & 0 & t_{12} & 0 \\ 0 & t_{11} & 0 & t_{12} \\ t_{21} & 0 & t_{22} & 0 \\ 0 & t_{21} & 0 & t_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} t_{11}x_1 + t_{12}x_3 \\ t_{11}x_2 + t_{12}x_4 \\ t_{21}x_1 + t_{22}x_3 \\ t_{21}x_2 + t_{22}x_4 \end{bmatrix}.$$

From this we obtain

$$\begin{aligned} \|\tilde{T}(x)\|^p &= |t_{11}x_1 + t_{12}x_3|^p + |t_{11}x_2 + t_{12}x_4|^p + |t_{21}x_1 + t_{22}x_3|^p + |t_{21}x_2 + t_{22}x_4|^p \\ &= \left\| \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} \right\|^p + \left\| \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} \right\|^p \\ &\leq \|T\|^p (|x_1|^p + |x_2|^p + |x_3|^p + |x_4|^p). \end{aligned}$$

This means that $\|\tilde{T}\| \leq \|T\|$ and since $\tilde{T}|_{V_1} = T$ we have $\|T\| \leq \|\tilde{T}\|$. Therefore $\|\tilde{T}\| = \|T\|$.

For (iii), we have

$$v(\tilde{T}) = \sup\{|x_x^*(\tilde{T}x)| : x \in S_{V_2}\} = |x_x^*(\tilde{T}x)|$$

for some

$$x = \sum_{k=1}^4 x_k \frac{1_{E_k^{2^2}}}{\mu(E_k^{2^2})^{\frac{1}{p}}}, \quad \|x\|^p = \sum_{k=1}^4 |x_k|^p = 1, \quad x_x^* = \sum_{k=1}^4 \varepsilon_k |x_k|^{p-1} \frac{1_{E_k^{2^2}}}{\mu(E_k^{2^2})^{\frac{1}{q}}}$$

where ε_k is a scalar number such that $\varepsilon_k x_k = |x_k|$. From the previous expression of $\tilde{T}(x)$ we have

$$\begin{aligned} x_x^*(\tilde{T}x) &= \left(\varepsilon_1 |x_1|^{p-1}, \dots, \varepsilon_4 |x_4|^{p-1} \right) \begin{bmatrix} t_{11}x_1 + t_{12}x_3 \\ t_{11}x_2 + t_{12}x_4 \\ t_{21}x_1 + t_{22}x_3 \\ t_{21}x_2 + t_{22}x_4 \end{bmatrix} \\ &= \left(\varepsilon_1 |x_1|^{p-1}, \varepsilon_3 |x_3|^{p-1} \right) \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} \\ &\quad + \left(\varepsilon_2 |x_2|^{p-1}, \varepsilon_4 |x_4|^{p-1} \right) \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}. \end{aligned}$$

Then

$$\begin{aligned} |x_x^*(\tilde{T}x)| &\leq v(T) \left\| (\varepsilon_1 |x_1|^{p-1}, \varepsilon_3 |x_3|^{p-1}) \right\|_q \left\| (x_1, x_3) \right\|_p \\ &\quad + v(T) \left\| (\varepsilon_2 |x_2|^{p-1}, \varepsilon_4 |x_4|^{p-1}) \right\|_q \left\| (x_2, x_4) \right\|_p \leq v(T). \end{aligned}$$

Since $\tilde{T}|_{V_1} = T$, we also have $v(T) \leq v(\tilde{T})$ and consequently $v(\tilde{T}) = v(T)$.

STEP 2. For every integer $n \geq 1$ and $T \in B(V_n)$, there exists $\tilde{T} \in B(V_{n+1})$ satisfying conditions (i), (ii) and (iii).

Indeed, let $n \geq 1$ and let $T \in B(V_n)$ be represented by its matrix

$$\begin{bmatrix} t_{11} & \cdots & t_{12^n} \\ \vdots & & \vdots \\ t_{2^n 1} & \cdots & t_{2^n 2^n} \end{bmatrix}$$

on the normal basis $\left\{ \frac{1_{E_k^{2^n}}}{\mu(E_k^{2^n})^{\frac{1}{p}}} \right\}_{k=1, \dots, 2^n}$ of V_n . As in Step 1, we check that the operator $\tilde{T} \in B(V_{n+1})$ defined by its matrix

$$\begin{bmatrix} \begin{pmatrix} t_{11} & 0 \\ 0 & t_{11} \end{pmatrix} & \cdots & \begin{pmatrix} t_{12^n} & 0 \\ 0 & t_{12^n} \end{pmatrix} \\ \vdots & & \vdots \\ \begin{pmatrix} t_{2^n 1} & 0 \\ 0 & t_{2^n 1} \end{pmatrix} & \cdots & \begin{pmatrix} t_{2^n 2^n} & 0 \\ 0 & t_{2^n 2^n} \end{pmatrix} \end{bmatrix}$$

as in the normal basis $\left\{ \frac{1_{E_k^{2^{n+1}}}}{\mu(E_k^{2^{n+1}})^{\frac{1}{p}}} \right\}_k$, as in $\left\{ 1_{E_k^{2^{n+1}}} \right\}_k$ of V_{n+1} , satisfies the three conditions (i), (ii) and (iii). $\square$

Proof of Theorem 2.1. We shall prove that the numerical index of $L_p[0, 1]$ is less than the numerical index of l_p .

Let $n \geq 1$ and let $U := T_n \in B(V_n)$. Following the previous lemma we can find a sequence of operators $\{T_m\}_{m \geq n}$, $T_m : V_m \rightarrow V_m$, satisfying the following conditions:

- (i) $T_{m+1}|_{V_m} = T_m$,
- (ii) $\|T_{m+1}\| = \|T_m\|$,
- (iii) $v(T_{m+1}) = v(T_m)$.

Let $x \in L_p[0, 1]$ and let $x_m \in V_m$ for $m \geq n$ such that x_m converges to x in $L_p[0, 1]$. The sequence $\{T_m(x_m)\}_{m \geq n}$ converges in $L_p[0, 1]$. Indeed, for $m, m' \geq n$ with $m' \geq m$, we have

$$T_{m'}(x_{m'}) - T_m(x_m) = T_{m'}(x_{m'} - x_m) + (T_{m'} - T_m)(x_m).$$

Since $T_{m'}|_{V_m} = T_m$ and $\|T_{m'}\| = \|T_n\|$, we obtain

$$\|T_{m'}(x_{m'}) - T_m(x_m)\| = \|T_{m'}(x_{m'} - x_m)\| \leq \|T_n\| \|x_{m'} - x_m\|.$$

Let $T(x) = \lim_m T_m(x_m)$ (note that $T(x)$ is independent of the choice of $\{x_m\}_m$). Clearly, T is a bounded linear operator on $L_p[0, 1]$. Moreover,

- (i) $T|_{V_n} = T_n$,
- (ii) $\|T\| = \|T_n\|$,
- (iii) $v(T) = v(T_n)$.

The first conditions (i) and (ii) are clear. For (iii), let $\varepsilon > 0$. There exists $x \in L_p[0, 1]$ such that

$$v(T) - \varepsilon \leq |x_x^*(Tx)|.$$

Here $x_x^* = \varepsilon_x |x|^{p-1}$, where ε_x is the sign function of x , that is, $\varepsilon_x(t)x(t) = |x(t)|$ for all $t \in [0, 1]$. But $T(x) = \lim_m T_m(x_m)$ for x_m converges to x in $L_p[0, 1]$ and $x_m \in S_{V_m}$ for all m . Since the norm of $L_p[0, 1]$ is Fréchet differentiable [4], the sequence $x_{x_m}^*$ converges to x_x^* in $L_q[0, 1]$. Consequently

$$v(T) - \varepsilon \leq |x_x^*(Tx)| = \lim_m |x_{x_m}^*(T_m x_m)| \leq \lim_m v(T_m) = v(T_n),$$

which means that $v(T) \leq v(T_n)$. Since $T|_{V_n} = T_n$, we have also $v(T_n) \leq v(T)$ and therefore $v(T) = v(T_n)$. It follows from this that

$$\{v(U) : U \in S_{V_n}\} \subset \{v(T) : T \in S_{L_p[0, 1]}\},$$

and then

$$n(L_p[0, 1]) \leq n(V_n).$$

Since V_n is isometric to $l_p^{2^n}$, we obtain

$$n(L_p[0, 1]) \leq n(l_p^{2^n}).$$

According to [6, Theorem 2.5] we get

$$n(L_p[0, 1]) \leq \lim_n n(l_p^{2^n}) = n(l_p).$$

Also, it was proved [6, Theorem 3.1] that

$$n(L_p(\mu)) \geq n(l_p)$$

for any Banach space $L_p(\mu)$, $1 \leq p < \infty$. This completes the proof of Theorem 2.1. $\square$

D. Li and C. Finet asked (in a communication) if the numerical index of the real Banach space l_p is equal to 0 for some $p \neq 2$.

In the following we prove that the numerical index of all Banach spaces $l_p^m, p \neq 2, m = 1, 2, \dots$, cannot be equal to 0.

Theorem 2.3. *For every real number $p \neq 2$ and $m = 2, 3, \dots$, the numerical index of the Banach space l_p^m is positive.*

Proof. It is enough to give the proof for the case $1 < p < 2$, since $n(X) = n(X^*)$ when X is a reflexive space. Fix a real number $p, 1 < p < 2$, and let m be an integer with $m \geq 2$. Consider

$$y = \frac{e_i + \varepsilon t e_j}{(1 + t^p)^{\frac{1}{p}}} \text{ and } y^* = \frac{e_i^* + \varepsilon t^{p-1} e_j^*}{(1 + t^p)^{\frac{p-1}{p}}}, \quad 0 \leq t \leq 1, \quad \varepsilon \in \{-1, 1\}.$$

Here $\{e_i\}$ is the canonical basis of l_p^m . Clearly, $\|y\|_{l_p^m} = \|y^*\|_{l_q^m} = y^*(y) = 1$. For $T = (t_{lk})_{lk} \in B(l_p^m)$ and by a simple calculus we have

$$y^*(Ty) = \frac{1}{1 + t^p} (t_{ii} + \varepsilon t t_{ij} + \varepsilon t^{p-1} t_{ji} + t^p t_{jj}).$$

From this we obtain

$$|t_{ii} + \varepsilon t t_{ij} + \varepsilon t^{p-1} t_{ji} + t^p t_{jj}| \leq (1 + t^p) v(T).$$

Then

$$\max_{\varepsilon \in \{-1, 1\}} |t_{ii} + \varepsilon t t_{ij} + \varepsilon t^{p-1} t_{ji} + t^p t_{jj}| = |t_{ii} + t^p t_{jj}| + |t t_{ij} + t^{p-1} t_{ji}| \leq (1 + t^p) v(T).$$

Since

$$|t_{ii} + t^p t_{jj}| \geq |t_{ii}| - t^p |t_{jj}| \quad \text{and} \quad |t t_{ij} + t^{p-1} t_{ji}| \geq t^{p-1} |t_{ji}| - t |t_{ij}|,$$

we then have

$$|t_{ii}| - t^p |t_{jj}| + t^{p-1} |t_{ji}| - t |t_{ij}| \leq (1 + t^p) v(T).$$

Similarly

$$|t_{jj}| - t^p |t_{ii}| + t^{p-1} |t_{ij}| - t |t_{ji}| \leq (1 + t^p) v(T).$$

From the last two inequalities we obtain

$$(1 - t^p)(|t_{ii}| + |t_{jj}|) + (t^{p-1} - t)(|t_{ij}| + |t_{ji}|) \leq 2(1 + t^p) v(T).$$

Since $(1 - t^p) \geq t^p - t$, we have

$$(t^{p-1} - t)(|t_{ii}| + |t_{ij}| + |t_{ji}| + |t_{jj}|) \leq 2(1 + t^p) v(T).$$

Hence

$$(*) \quad \frac{M_p}{2} (|t_{ii}| + |t_{ij}| + |t_{ji}| + |t_{jj}|) \leq v(T),$$

where $M_p = \sup_{t \in [0, 1]} \frac{t^{p-1} - t}{1 + t^p}$. Thus if $v(T) = 0$ for $T \in B(l_p^m)$, we have necessarily $\|T\| = 0$ and consequently $n(l_p^m) = v(T) > 0$ for some $T \in S_{l_p^m}$. $\square$

Remark. Following (*) in the previous proof, v is a norm on $B(l_p)$ for $p \neq 2$. It is still unknown if it is an equivalent norm to the operator norm.

REFERENCES

- [1] F. F. Bonsall and J. Duncan, *Numerical ranges of Operators on Normed Spaces and of elements of Normed Algebras*, London Math. Soc. Lecture Note Ser. 2, (Cambridge Univ. Press, 1971). MR0288583 (44:5779)
- [2] F. F. Bonsall and J. Duncan, *Numerical ranges II*, London Math. Soc. Lecture Note Ser. 10 (Cambridge Univ. Press, 1971). MR0442682 (56:1063)
- [3] H. F. Bohnenblust and S. Karlin, *Geometrical properties of the unit sphere in Banach algebra*, Ann. of Math. 62 (1955), 217-229. MR0071733 (17:177a)
- [4] B. Beauzamy, *Introduction to Banach Spaces and their Geometry*, Second revised edition, North-Holland-Amsterdam, New York, Oxford (1985). MR0889253 (88f:46021)
- [5] J. Duncan, C. M. McGregor, J. D. Pryce and A. J. White, *The numerical index of a normed space*, J. London Math. Soc. (2) 2 (1970), 481-488. MR0264371 (41:8967)
- [6] E. Ed-dari, *On the numerical index of Banach spaces*, to appear.
- [7] C. Finet, M. Martín and R. Payá, *Numerical index and renorming*, Proc. Amer. Math. Soc. 131 (2003), no. 3, 871-877. MR1937425 (2003h:46021)
- [8] B. W. Glickfeld, *On an inequality of Banach algebra geometry and semi-inner product space theory*, Illinois. J. Math. 14 (1970), 76-81. MR0253024 (40:6239)
- [9] K. E. Gustafsan and D. K. M. Rao, *Numerical Range. The Field of Values of Linear Operators and Matrices*, Springer, New York, 1997. MR1417493 (98b:47008)
- [10] G. Lumer, *Semi-inner-product spaces*, Trans. Amer. Math. Soc. 100 (1961), 29-43. MR0133024 (24:A2860)
- [11] G. López, M. Martín and R. Payá, *Real Banach spaces with numerical index 1*, Bull. London Math. Soc. 31 (1999) 207-212. MR1664125 (99k:46024)
- [12] M. Martín, *A survey on the numerical index of Banach space*, Extracta Math 15 (2000), 265-276. MR1823892 (2002b:46027)
- [13] M. Martín and R. Payá, *Numerical index of vector-valued function spaces*, Studia Mathematica 142 (3) (2000) 269-280. MR1792610 (2001i:46017)

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