PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 134, Number 7, Pages 1937–1946
S 0002-9939(05)08275-4
Article electronically published on December 16, 2005

ON THE CONVERGENCE OF MAXIMAL MONOTONE OPERATORS

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(Communicated by Jonathan M. Borwein)

Dedicated to F.E. Browder for the impact of his work on nonlinear analysis

Abstract. We study the convergence of maximal monotone operators with the help of representations by convex functions. In particular, we prove the convergence of a sequence of sums of maximal monotone operators under a general qualification condition of the Attouch–Brezis type.

1. Introduction

Maximal monotone operators represent one of the cornerstones of modern nonlinear analysis. They have been used to model several nonlinear phenomena, and their properties make them valuable in the study of evolution equations and for surjectivity results. For these reasons they have been the subject of several monographs ([2], [4], [5], [23]), and they appear in many books.

One of the incentives for obtaining representations of maximal monotone operators by convex functions is the hope of finding new results about such operators by using tools from convex analysis. Other motivations stem from the analogies between many results concerning the class of maximal monotone operators with corresponding results about closed proper convex functions. Up to now, the representations introduced in [6], [7], [8], [13], [14], [20] have enabled one to devise simple proofs of known results, but they have not been used to establish new results. In the present note we give a general convergence result for sums of maximal monotone operators. It relies on a series of papers by the authors and their collaborators ([13], [14], [15], [16], [17], [18], [19], [21], [22]). It encompasses previous results by Attouch–Moudafi–Riahi [1], Pennanen–Revalski–Théra [11], Pennanen–Rockafellar–Théra [12] and settles a conjecture which remained open for some time.

Received by the editors January 28, 2005 and, in revised form, February 1, 2005.
2000 *Mathematics Subject Classification.* Primary 47H05; Secondary 26B25.
Key words and phrases. Bounded convergence, convergence, convex function, maximal monotone operator, monotone operator, qualification condition.
The second author was partially supported by Grant CEEX-05-D11-36.

2. A new representation

Let us first recall some basic facts about bounded convergence (also called bounded-Hausdorff convergence, or Attouch–Wets convergence or epi-distance convergence); we refer to [15], [16] for more leisurely recent expositions and bibliographical references. It is the conjunction of two notions requiring persistence and stability in rather demanding (but realistic, compared to crude Hausdorff convergence) ways. These notions are usually defined with the help of truncated Hausdorff excesses and distances; we adopt here an equivalent formulation. Given subsets A, A_n ($n \in \mathbb{N}$) of the n.v.s. X, we write $A \subset b\text{–}\liminf_n A_n$ if for any bounded sequence (x_n) of A we have $(d(x_n, A_n)) \to 0$. Similarly, we write $b\text{–}\limsup A_n \subset A$ if for any bounded sequence (x_n) of X such that $x_n \in A_n$ for each $n \in \mathbb{N}$, we have $(d(x_n, A)) \to 0$. We say that the sequence (A_n) *boundedly converges* to A, and we write $(A_n) \xrightarrow{b} A$ if $A \subset b\text{–}\liminf_n A_n$ and $b\text{–}\limsup A_n \subset A$. Recall also that for $f, f_n : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, the sequence (f_n) *boundedly converges* to f, and we write $(f_n) \xrightarrow{b} f$ if $(\operatorname{epi} f_n) \xrightarrow{b} \operatorname{epi} f$, where $\operatorname{epi} f := \{(x, s) \in X \times \mathbb{R} \mid f(x) \le s\}$ is the *epigraph* of f. We write $f \ge b\text{–}\limsup f_n$ if $\operatorname{epi} f \subset b\text{–}\liminf \operatorname{epi} f_n$ and $f \le b\text{–}\liminf f_n$ if $\operatorname{epi} f \supset b\text{–}\limsup f_n$. As usual, the *domain* of f is $\operatorname{dom} f := \{x \in X \mid f(x) < \infty\}$, and f is said to be *proper* if its domain is nonempty and if f does not take the value $-\infty$. The *conjugate* f^* of f is defined by $f^*(x^*) := \sup\{\langle x, x^*\rangle - f(x) \mid x \in X\}$, and the *sublevel* (resp. *strict sublevel*) set at height $r \in \mathbb{R}$ is $[f \le r] := \{x \in X \mid f(x) \le r\}$ (resp. $[f < r] := \{x \in X \mid f(x) < r\}$).

In the sequel X (and any other space) is a reflexive Banach space and $\Gamma(X)$ denotes the class of proper lower semicontinuous (lsc for short) convex functions defined on X. If A is a subset of X, the *indicator function* ι_A of A is the function whose value is 0 on A and $+\infty$ on $X \setminus A$.

A multifunction $M : X \rightrightarrows X^*$ is said to be *monotone* if for any (w, w^*), (x, x^*) in its graph $\operatorname{gph} M$, one has $\langle w - x, w^* - x^*\rangle \ge 0$. It is *maximal monotone*, and we write $M \in \mathfrak{M}(X)$ if there is no monotone operator whose graph strictly contains $\operatorname{gph} M$. Given a monotone operator $M : X \rightrightarrows X^*$, following [8], [13] and [14], one can associate to it the two convex functions f_M and p_M on $X \times X^*$ given by

$$f_M := (c_M^*)^\intercal, \quad p_M := c_M^{**}, \quad \text{where } c_M := c + \iota_M,$$

with M identified with $\operatorname{gph} M$ and $c := \langle \cdot, \cdot \rangle$, the coupling function on $X \times X^*$. It is straightforward to pass from the convergence of a sequence (M_n) of $\mathfrak{M}(X)$ to the convergence of the associated functions (c_{M_n}). However, to get the convergence of the sequence (f_{M_n}) of the Fitzpatrick representatives, one would need a continuity property of the Legendre–Fenchel transform applied to nonconvex functions. To the best of our knowledge, only the following result provides such a property.

Lemma 2.1 ([17, Cor. 19])**.** *Let (f_n) be a family of proper functions from X to $\mathbb{R} \cup \{\infty\}$ which is equi-hypercoercive in the sense that $\lim_{\|x\| \to \infty} f_n(x)/\|x\| = \infty$ uniformly for $n \in \mathbb{N}$. Suppose $(f_n) \xrightarrow{b} f$, where f is bounded below on bounded subsets. Then $(f_n^*) \xrightarrow{b} f^*$. Moreover, f^* is bounded on bounded subsets and $(f_n^*) \to f^*$ uniformly on bounded sets.*

The stringent equi-hypercoercivity assumption is not satisfied by the sequence (c_{M_n}). Thus, we introduce another representative function. For a monotone operator M it is given by

$$q_M := \overline{\mathrm{conv}}(c_M + \tfrac{1}{2}\|\cdot\|^2) := (c_M + \tfrac{1}{2}\|\cdot\|^2)^{**},$$

where $\|(x,x^*)\| := (\|x\|^2 + \|x^*\|^2)^{1/2}$. In order to relate it to previously defined representative functions, we will make use of the function $\delta : X \times X^* \to \mathbb{R}$ given by

$$\delta(x,x^*) := \langle x, x^*\rangle + \tfrac{1}{2}\|(x,x^*)\|^2 = \langle x, x^*\rangle + \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|x^*\|^2 .$$

In [20] and [21] good use is made of this function in view of the fact that $\delta(x,x^*) \geq 0$ with equality if, and only if, $x^* \in -J(x)$, where J is the usual duality mapping given by

$$J(x) := \{x^* \in X^* : \|x^*\| = \|x\|\,,\ \langle x, x^*\rangle = \|x\|^2\}.$$

With the new representation, δ will play a role similar to the one played by the coupling function c with respect to the previous representations.

Lemma 2.2. *Let $M \in \mathfrak{M}(X)$ and $q_M := \overline{\mathrm{conv}}(c_M + \frac{1}{2}\|\cdot\|^2)$. Then*

$$\iota_M + \delta = c_M + \tfrac{1}{2}\|\cdot\|^2 \geq q_M \geq p_M + \tfrac{1}{2}\|\cdot\|^2 \geq f_M + \tfrac{1}{2}\|\cdot\|^2 \geq \delta. \tag{2.1}$$

Moreover $f_M = (q_M - \frac{1}{2}\|\cdot\|^2)^*$ and $p_M = (q_M - \frac{1}{2}\|\cdot\|^2)^{**}$. Furthermore,

$$M = \{(x,x^*) \mid q_M(x,x^*) = \delta(x,x^*)\}.$$

Proof. The first inequality in (2.1) is obvious. The second one is due to the fact that $c_M + \frac{1}{2}\|\cdot\|^2 \geq p_M + \frac{1}{2}\|\cdot\|^2$ and $p_M + \frac{1}{2}\|\cdot\|^2 \in \Gamma(X \times X^*)$. The other two inequalities follow from the relations $p_M \geq f_M \geq c$ ([14, Thm. 5]).

From the first inequality we deduce that $(q_M - \frac{1}{2}\|\cdot\|^2)^* \geq c_M^* = f_M^{\intercal}$, while from the second one we deduce that $(q_M - \frac{1}{2}\|\cdot\|^2)^* \leq p_M^* = f_M^{\intercal}$. Equality ensues. Taking conjugates, we get the second equality $p_M = (q_M - \frac{1}{2}\|\cdot\|^2)^{**}$.

When (x,x^*) is such that $q_M(x,x^*) = \delta(x,x^*)$, one has $q_M(x,x^*) - \frac{1}{2}\|(x,x^*)\|^2 = \delta(x,x^*) - \frac{1}{2}\|(x,x^*)\|^2 = \langle x, x^*\rangle$, hence $p_M(x,x^*) = \langle x, x^*\rangle$ and so $(x,x^*) \in M$ ([14, Thm. 5]). Conversely, when $(x,x^*) \in M$, one has $(\iota_M + \delta)(x,x^*) = \delta(x,x^*)$ and the inequalities in (2.1) are equalities. □

Lemma 2.3. *Let $M \in \mathfrak{M}(X)$ and $f \in \Gamma(X \times X^*)$ be such that $f_M \leq f \leq p_M$. Then for every $(y,y^*) \in X \times X^*$ there exists $(z,z^*) \in M$ such that $z^* - y^* \in J(y-z)$. Moreover, any $(z,z^*) \in M$ satisfying this relation satisfies the estimate $\|(z,z^*) - (y,y^*)\| \leq (\sqrt{2}+1)\cdot d((y,y^*),M)$. Furthermore one has*

$$\|y-z\|^2 = \|y^* - z^*\|^2 \leq f(y,y^*) - \langle y, y^*\rangle .$$

Proof. Let $(y,y^*) \in X \times X^*$. The first assertion is given by [20, Thms. 10.3, 10.6] which asserts that $\operatorname{gph} M + \operatorname{gph}(-J) = X \times X^*$, so that there exists $(z,z^*) \in M$ such that $(y-z, z^*-y^*) \in \operatorname{gph} J$. Let $t := \|y-z\| = \|y^*-z^*\|$; by definition of J one has $\langle y-z, y^*-z^*\rangle = -t^2$. Let us pick some $(u,u^*) \in M$ and use the monotonicity of M to write

$$\begin{aligned}
0 &\leq \langle u-z, u^*-z^*\rangle = \langle u-y+y-z, u^*-y^*+y^*-z^*\rangle \\
&= \langle u-y, u^*-y^*\rangle + \langle y-z, u^*-y^*\rangle + \langle u-y, y^*-z^*\rangle + \langle y-z, y^*-z^*\rangle \\
&\leq \|u-y\|\cdot\|u^*-y^*\| + \|y-z\|\cdot\|u^*-y^*\| + \|u-y\|\cdot\|y^*-z^*\| - t^2.
\end{aligned}$$

Thus, setting $m := \left(\|u-y\|^2 + \|u^*-y^*\|^2\right)^{1/2} = \|(y,y^*)-(u,u^*)\|$, it follows that

$$t^2 \le t(\|u^*-y^*\| + \|u-y\|) + \|u-y\| \cdot \|u^*-y^*\| \le \sqrt{2}mt + \tfrac{1}{2}m^2,$$

and so

$$t \le \tfrac{1}{2}\big(\sqrt{2}m + 2m\big) = m\big(1 + \tfrac{1}{2}\sqrt{2}\big).$$

Hence $\|(z,z^*)-(y,y^*)\| = t\sqrt{2} \le (\sqrt{2}+1)m$. Taking the infimum over $(u,u^*) \in M$, we obtain the announced estimate.

We also note that the choice of (z,z^*) and the definition of J yield

$$\begin{aligned}\tfrac{1}{2}\|y-z\|^2 + \tfrac{1}{2}\|y^*-z^*\|^2 &= -\langle y-z, y^*-z^*\rangle \\ &= \langle y,z^*\rangle + \langle z,y^*\rangle - c_M(z,z^*) - \langle y,y^*\rangle \\ &\le f_M(y,y^*) - \langle y,y^*\rangle \le f(y,y^*) - \langle y,y^*\rangle .\end{aligned}$$

The proof is complete. □

3. Convergence results

Let us first study the passage from convergence of representative functions to convergence of the associated operators. We write $(M_n) \xrightarrow{b} M$ instead of $(\operatorname{gph} M_n) \xrightarrow{b} \operatorname{gph} M$.

Proposition 3.1. *Let $M, M_n \in \mathfrak{M}(X)$ for $n \in \mathbb{N}$. Consider $f, f_n \in \Gamma(X \times X^*)$ with $f_M \le f \le p_M$ and $f_{M_n} \le f_n \le p_{M_n}$ for every $n \in \mathbb{N}$.*

(a) *If $f \ge b\text{–}\limsup f_n$, then $M \subset b\text{–}\liminf M_n$.*
(b) *If $f \le b\text{–}\liminf f_n$, then $b\text{–}\limsup M_n \subset M$.*
(c) *If $(f_n) \xrightarrow{b} f$, then $(M_n) \xrightarrow{b} M$.*

Proof. (a) Let $((x_n,x_n^*)) \subset M$ be bounded. Then $w_n := (x_n, x_n^*, \langle x_n, x_n^*\rangle) \in \operatorname{epi} f$ and (w_n) is also bounded. From our hypothesis, there exists a sequence (w_n') such that $w_n' := (y_n, y_n^*, t_n) \in \operatorname{epi} f_n$ for every n and $\|w_n - w_n'\| \to 0$. By Lemma 2.3 applied to M_n and (y_n, y_n^*), there exists $(z_n, z_n^*) \in M_n$ such that

$$\tfrac{1}{2}\|y_n - z_n\|^2 + \tfrac{1}{2}\|y_n^* - z_n^*\|^2 \le f_n(y_n,y_n^*) - \langle y_n, y_n^*\rangle \le t_n - \langle y_n, y_n^*\rangle =: \varepsilon_n.$$

Because $\|x_n - y_n\| \to 0$, $\|x_n^* - y_n^*\| \to 0$, $|t_n - \langle x_n, x_n^*\rangle| \to 0$ and the sequence $((x_n,x_n^*))$ is bounded, we obtain easily that $\varepsilon_n \to 0$. Hence $\|y_n - z_n\| \to 0$, $\|y_n^* - z_n^*\| \to 0$, whence $d\left((x_n,x_n^*), M_n\right) \to 0$.

(b) is obtained similarly; (c) is an immediate consequence of (a) and (b). □

The preceding implications can be changed into equivalences if one substitutes the new representatives q_{M_n} for the original representatives f_n.

Proposition 3.2. *Let $M_n \in \mathfrak{M}(X)$ for $n \ge 0$. Then the following assertions hold:*

(a) *if $(d\left((0,0), M_n\right))$ is bounded, then $M_0 \supset b\text{–}\limsup M_n \Leftrightarrow q_{M_0} \le b\text{–}\liminf q_{M_n}$;*
(b) *$M_0 \subset b\text{–}\liminf M_n \Leftrightarrow q_{M_0} \ge b\text{–}\limsup q_{M_n}$;*
(c) *$(M_n) \xrightarrow{b} M_0 \Leftrightarrow (q_{M_n}) \xrightarrow{b} q_{M_0}$.*

Proof. During the proof we denote by f_n the function $\iota_{M_n} + \delta$ for $n \ge 0$. Of course, we have that $q_{M_n} = f_n^{**}$.

(a) Assume that $(d((0,0),M_n))_{n\geq 0}$ is bounded; this means that there exists a bounded sequence $((x_n,x_n^*))$ with $(x_n,x_n^*)\in M_n$ for every $n\geq 0$. Taking an arbitrary $(x,x^*)\in M_n$, and using the monotonicity of M_n, we have that

$$\begin{aligned}\delta(x,x^*) &= \tfrac{1}{2}\|(x,x^*)\|^2 + \langle x-x_n, x^*-x_n^*\rangle + \langle x_n,x^*\rangle + \langle x,x_n^*\rangle - \langle x_n,x_n^*\rangle\\ &\geq \tfrac{1}{2}\|(x,x^*)\|^2 - \|x_n\|\,\|x^*\| - \|x\|\,\|x_n^*\| - \|x_n\|\,\|x_n^*\|\\ &\geq \tfrac{1}{2}\|(x,x^*)\|^2 - \alpha\|(x,x^*)\| - \beta =: \zeta(x,x^*)\end{aligned}$$

for some $\alpha,\beta\geq 0$ (independent of $n\geq 0$). Hence

$$f_n := \iota_{M_n} + \delta \geq \zeta \quad \forall n\geq 0. \tag{3.1}$$

Assume that $M_0 \supset b\text{–}\limsup M_n$. Then $M_0\times\mathbb{R}_+ \supset b\text{–}\limsup(M_n\times\mathbb{R}_+)$, and so $\iota_{M_0}\leq b\text{–}\liminf \iota_{M_n}$. Because δ is Lipschitz on bounded sets, using [3, Thm. 7.1.5] (more precisely with a similar proof) we have that $f_0\leq b\text{–}\liminf f_n$. From (3.1) we have that the family of functions $\{f_n \mid n\geq 0\}$ is equi-hypercoercive and equi-bounded from below. Moreover, $(x_n,x_n^*,\delta(x_n,x_n^*))\in \operatorname{epi} f_n$ for every n, and so $(d(0,\operatorname{epi} f_n))_{n\geq 0}$ is bounded. Using [17, Lemma 17, Prop. 18] we obtain that $f_0^*\geq b\text{–}\limsup f_n^*$. Now using [17, Thm. 14(a)] we obtain that $f_0^{**}\leq b\text{–}\liminf f_n^{**}$, that is $q_{M_0}\leq b\text{–}\liminf q_{M_n}$.

Conversely, assume that $q_{M_0}\leq b\text{–}\liminf q_{M_n}$, and take a bounded sequence $((x_n,x_n^*))_{n\geq 1}$ with $(x_n,x_n^*)\in M_n$ for every $n\geq 1$. Setting $v_n := (x_n,x_n^*,\delta(x_n,x_n^*))\in \operatorname{epi} f_n\subset \operatorname{epi} q_{M_n}$, $(v_n)_{n\geq 1}$ is also bounded. Therefore, there exists a sequence $(w_n)\subset \operatorname{epi} q_{M_0}$ such that $\|v_n-w_n\|\to 0$. Let $w_n := (y_n,y_n^*,t_n)$; by Lemma 2.2, $\delta(y_n,y_n^*)\leq q_{M_n}(y_n,y_n^*)\leq t_n$. By Lemma 2.3, for every n, there exists $(z_n,z_n^*)\in M_n$ such that

$$\begin{aligned}\tfrac{1}{2}\|y_n-z_n\|^2 + \tfrac{1}{2}\|y_n^*-z_n^*\|^2 &\leq p_{M_n}(y_n,y_n^*) - \langle y_n,y_n^*\rangle\\ &\leq q_{M_n}(y_n,y_n^*) - \delta(y_n,y_n^*) \leq t_n - \delta(y_n,y_n^*)\\ &\leq |t_n-\delta(x_n,x_n^*)| + |\delta(x_n,x_n^*)-\delta(y_n,y_n^*)|\,.\end{aligned}$$

Because $((x_n,x_n^*))$ is bounded and $\|x_n-y_n\|\to 0$, $\|x_n^*-y_n^*\|\to 0$, it follows that $|\delta(x_n,x_n^*)-\delta(y_n,y_n^*)|\to 0$, and so $\|y_n-z_n\|\to 0$, $\|y_n^*-z_n^*\|\to 0$. We get that $d((x_n,x_n^*),M_0)\to 0$. Hence $M_0\supset b\text{–}\limsup M_n$.

(b) Observe that $(d((0,0),M_n))_{n\geq 0}$ is bounded when $M_0\subset b\text{–}\liminf M_n$. The rest of the proof is similar to that of (i) (noting that from (3.1) we obtain that $f_n^*\leq \zeta^*$ for every $n\geq 0$, and so the condition of [17, Thm. 14(b)] is satisfied).

(c) is an immediate consequence of (a) and (b). □

4. Partial operators and their convergence

Given another Banach space Y, let $F: X\times Y \rightrightarrows X^*\times Y^*$ be monotone. In this section we consider the operator $G: X\rightrightarrows X^*$ defined by

$$\operatorname{gph} G := \{(x,x^*)\in X\times X^* \mid \exists y^*\in Y^* \ :\ (x,0,x^*,y^*)\in \operatorname{gph} F\}. \tag{4.1}$$

It follows easily that G is monotone (see [19, Lemma 3.3]), but not necessarily maximal monotone. We intend to give conditions ensuring that G is maximal monotone when F is maximal monotone and to study convergence questions related to this construction. We first provide a preliminary result of independent interest.

Theorem 4.1. *Let $F : X \times Y \rightrightarrows X^* \times Y^*$ be a monotone multifunction. Assume that $0 \in \operatorname{int}(\operatorname{conv}(\operatorname{Pr}_Y(\operatorname{dom} F)))$. Then*

$$\forall p > 0,\ \exists q > 0,\ \forall (x, 0, x^*, y^*) \in \operatorname{gph} F\ :\ \|x\| \le p,\ \|x^*\| \le p \Rightarrow \|y^*\| \le q.$$

Proof. Let f_F be the Fitzpatrick function associated to F. Because F is monotone, we have that $f_F \le c_F$ ([8], [14, Prop. 4]), and so $\operatorname{gph} F = \operatorname{dom} c_F \subset \operatorname{dom} f_F$. It follows that $\operatorname{Pr}_Y(\operatorname{dom} F) = \operatorname{Pr}_Y(\operatorname{gph} F) \subset \operatorname{Pr}_Y(\operatorname{dom} f_F)$, and so, $\operatorname{conv}(\operatorname{Pr}_Y(\operatorname{dom} F)) \subset \operatorname{Pr}_Y(\operatorname{dom} f_F)$. Using the Robinson–Ursescu Theorem, we find some $r, \rho, m > 0$ such that, denoting by U_Z the closed unit ball of a normed vector space Z, we have

$$\forall y \in rU_Y,\ \exists (u, u^*, v^*) \in \rho U_{X \times X^* \times Y^*}\ :\ f_F(u, y, u^*, v^*) \le m.$$

Let $p > 0$ and $(x, 0, x^*, y^*) \in \operatorname{gph} F$ with $\|x\| \le p, \|x^*\| \le p$. Fix $y \in rU_Y$ and take $(u, u^*, v^*) \in \rho U_{X \times X^* \times Y^*}$ such that $f_F(u, y, u^*, v^*) \le m$. Then

$$\begin{aligned} m &\ge \langle (x, 0), (u^*, v^*) \rangle + \langle (u, y), (x^*, y^*) \rangle - \langle (x, 0), (x^*, y^*) \rangle \\ &= \langle x, u^* \rangle + \langle u, x^* \rangle + \langle y, y^* \rangle - \langle x, x^* \rangle \\ &\ge \langle y, y^* \rangle - p \cdot \rho - \rho \cdot p - p \cdot p, \end{aligned}$$

and so $\langle y, y^* \rangle \le m + p(p + 2\rho)$. It follows that $\|y^*\| \le q := r^{-1}[m + p(p + 2\rho)]$. The conclusion follows. □

Note that the preceding result is valid for X, Y arbitrary Banach spaces (or even for barreled n.v.s.) because f is convex and lsc w.r.t. the strong topologies on X, Y, X^*, Y^*.

As a corollary we derive the well-known result on the local boundedness of monotone operators on the interior of their domains; just take $X := \{0\}$ in the previous result.

The proof above shows that given some $f \in \Gamma(X \times Y)$ such that $f_F \le f$ one has the implication (4.2) $\Rightarrow$ (4.3) where

$$rU_Y \subset \operatorname{Pr}_Y([f \le m] \cap \rho(U_X \times Y \times U_{X^*} \times U_{Y^*})) \tag{4.2}$$

and, setting $q(p) := r^{-1}[m + p(p + 2\rho)]$ for $p \in \mathbb{P} := (0, +\infty)$,

$$(x, 0, x^*, y^*) \in \operatorname{gph} F\ :\ \|x\| \le p,\ \|x^*\| \le p \Rightarrow \|y^*\| \le q(p). \tag{4.3}$$

The above implication yields the following result.

Corollary 4.2. *Let $F : X \times Y \rightrightarrows X^* \times Y^*$ be a maximal monotone multifunction. Assume that $r, \rho, m > 0$ are such that*

$$rU_Y \subset \operatorname{Pr}_Y([q_F \le m] \cap \rho(U_X \times Y \times U_{X^*} \times U_{Y^*})). \tag{4.4}$$

Then (4.3) *holds and G given by* (4.1) *is maximal monotone.*

Proof. By Lemma 2.2 we have that $f_F \le p_F \le q_F$, and so $[q_F \le m] \subset [p_F \le m] \subset [f_F \le m]$. Hence (4.4) $\Rightarrow$ (4.2), with $f = f_F$ or $f = p_F$ and so (4.3) holds. The maximality of G follows from [19, Prop. 3.4]. □

Theorem 4.3. *Let $F, F_n : X \times Y \rightrightarrows X^* \times Y^*$ be maximal monotone multifunctions and let $G, G_n : X \rightrightarrows X^*$ be the monotone multifunctions associated with them as defined in* (4.1). *Assume that $Y = \mathbb{R}_+(\operatorname{Pr}_Y(\operatorname{dom} F))$ and $(F_n) \xrightarrow{b} F$. Then G and G_n for large n are maximal monotone and $(G_n) \xrightarrow{b} G$.*

Proof. From Proposition 3.2 we have that $(q_{F_n}) \xrightarrow{b} q_F$. From (2.1) we obtain that $\operatorname{gph} F \subset \operatorname{dom} q_F \subset \operatorname{dom} f_F$, and so

$$\operatorname{Pr}_Y(\operatorname{dom} F) = \operatorname{Pr}_Y(\operatorname{gph} F) \subset \operatorname{Pr}_Y(\operatorname{dom} q_F) \subset \operatorname{Pr}_Y(\operatorname{dom} f_F). \tag{4.5}$$

It follows that $Y = \mathbb{R}_+ (\operatorname{Pr}_Y(\operatorname{dom} F)) = \mathbb{R}_+ (\operatorname{Pr}_Y(\operatorname{dom} q_F))$. Taking into account [18, Rem. 3.1(b)] and [18, Prop. 3.4(c)], we have that there exist some $r, \rho, m > 0$ such that (4.4) holds. Applying [18, Lemma 3.5] with $Y_0' = \{0\}$, we obtain that there exist $n_0 \geq 1$ and $r', \rho', m' > 0$ such that

$$r' U_Y \subset \operatorname{Pr}_Y ([q_{F_n} \leq m'] \cap \rho' (U_X \times Y \times U_{X^*} \times U_{Y^*})) \quad \forall n \geq n_0.$$

Without loss of generality, we may assume (and we do) that $r = r'$, $m = m'$, $\rho = \rho'$, that is (4.4) holds with F replaced by F_n, for every $n \in N_0 := \{n \in \mathbb{N} \mid n \geq n_0\}$. By [19, Prop. 12] or Corollary 4.2 we obtain that G and G_n for $n \geq n_0$ are maximal monotone.

Let $((x_n, x_n^*))$ be a bounded sequence in the graph of G. By construction and the preceding corollary, there exists a bounded sequence (y_n^*) such that $(x_n, 0, x_n^*, y_n^*) \in F$ for each n. Then there exists a sequence $((u_n, v_n, u_n^*, v_n^*))$ with (u_n, v_n, u_n^*, v_n^*) in the graph of F_n for every n such that

$$\|(u_n, v_n, u_n^*, v_n^*) - (x_n, 0, x_n^*, y_n^*)\| \to 0.$$

Applying Lemma 2.3 to G_n ($n \in N_0$) and (u_n, u_n^*), we find some $(w_n, w_n^*) \in G_n$ such that

$$r_n := \|w_n - u_n\|^2 = \|w_n^* - u_n^*\|^2 = -\langle u_n - w_n, u_n^* - w_n^* \rangle. \tag{4.6}$$

The construction of G_n yields some $z_n^* \in Y^*$ such that $(w_n, 0, w_n^*, z_n^*) \in F_n$. Since F_n is monotone, we have

$$\langle (u_n, v_n) - (w_n, 0), (u_n^*, v_n^*) - (w_n^*, z_n^*) \rangle \geq 0.$$

Thus

$$r_n = -\langle u_n - w_n, u_n^* - w_n^* \rangle \leq \langle v_n, v_n^* - z_n^* \rangle \leq \|v_n\| \cdot \|v_n^* - z_n^*\|. \tag{4.7}$$

Set $p_n := \max\{\|w_n\|, \|w_n^*\|\}$ and let $\beta > 0$ be such that $\|u_n\|, \|u_n^*\|, \|v_n^*\| \leq \beta$ for every n. By Corollary 4.2 we have that $\|z_n^*\| \leq r^{-1}[m + p_n(p_n + 2\rho)]$ for $n \in N_0$. Assume that (p_n) is not bounded. Then $(p_n)_{n \in P} \to \infty$ for an infinite subset P of N_0; we may assume that $p_n \geq \beta$ for $n \in P$. From (4.6) and (4.7) we get

$$(p_n - \beta)^2 \leq \|v_n\| \cdot \left(r^{-1}[m + p_n(p_n + 2\rho)] + \beta\right) \quad \forall n \in P.$$

Dividing both sides of this inequality by p_n^2 and taking the limit for $n \to \infty$, we get the contradiction $1 \leq 0$ because $(v_n) \to 0$. Hence (p_n) is bounded, and so (z_n^*) is bounded, too. From (4.7) we obtain that $(r_n) \to 0$. Since $((u_n - x_n, u_n^* - x_n^*)) \to (0, 0)$, we get $(d((x_n, x_n^*), G_n)) \to 0$.

Now let $((x_n, x_n^*))$ be a bounded sequence such that $(x_n, x_n^*) \in G_n$ for every n. By the construction of G_n, there exists a sequence (y_n^*) such that $(x_n, 0, x_n^*, y_n^*) \in F_n$ for each n. By Corollary 4.2 applied for F replaced by F_n with $n \geq n_0$ (relation (4.4) is satisfied by F_n for $n \in N_0$), we obtain that (y_n^*) is bounded. Then there exists a sequence $((u_n, v_n, u_n^*, v_n^*)) \subset \operatorname{gph} F$ such that

$$\varepsilon_n := \|(u_n, v_n, u_n^*, v_n^*) - (x_n, 0, x_n^*, y_n^*)\| \to 0.$$

Applying Lemma 2.3 to (u_n, u_n^*), we find some $(w_n, w_n^*) \in G$ such that

$$r_n := \|w_n - u_n\|^2 = \|w_n^* - u_n^*\|^2 = -\langle u_n - w_n, u_n^* - w_n^* \rangle.$$

The construction of G yields some $z_n^* \in Y^*$ such that $(w_n, 0, w_n^*, z_n^*) \in F$. Since F is monotone, we have

$$\langle (u_n, v_n) - (w_n, 0), (u_n^*, v_n^*) - (w_n^*, z_n^*) \rangle \geq 0,$$

and so

$$r_n = -\langle u_n - w_n, u_n^* - w_n^* \rangle \leq \langle v_n, v_n^* - z_n^* \rangle \leq \|v_n\| \cdot \|v_n^* - z_n^*\|.$$

Proceeding as above, since $(v_n) \to 0$ we obtain that $(r_n) \to 0$. Since $((u_n - x_n, u_n^* - x_n^*)) \to (0,0)$, we get $(d((x_n, x_n^*), G)) \to 0$. Therefore $(G_n) \xrightarrow{b} G$. □

5. Applications to the construction of new operators

As in [19], having a multifunction $F : X \times Y \rightrightarrows X^* \times Y^*$ and $A \in L(X,Y)$, that is, a continuous linear operator A from X into Y, we consider the multifunction $F_A : X \times Y \rightrightarrows X^* \times Y^*$ whose graph is

$$\operatorname{gph} F_A := \{(x, y, x^*, y^*) \mid (x^* - A^* y^*, y^*) \in F(x, Ax + y)\}.$$

Thus $F_A := S^* \circ F \circ S^{-1}$, where $S : X \times Y \to X \times Y$ is the linear isomorphism given by $S(x,y) = (x, y - Ax)$. This observation easily implies that F_A is monotone when F is so. Observe that $\operatorname{gph} F_A = T(\operatorname{gph} F)$, where $T := S^{-1} \times S^* : X \times Y \times X^* \times Y^* \to X \times Y \times X^* \times Y^*$ is given by $T(x, y, x^*, y^*) := (x, y - Ax, x^* + A^* y^*, y^*)$. The operator T is an isomorphism of normed vector spaces. Now considering a sequence of multifunctions $F_n : X \times Y \rightrightarrows X^* \times Y$ and a sequence (A_n) of $L(X,Y)$, we easily obtain

$$\big[(F_n) \xrightarrow{b} F,\ \|A_n - A\| \to 0\big] \Rightarrow \big((F_n)_{A_n}\big) \xrightarrow{b} F_A.$$

The next result is then an immediate consequence of the preceding theorem, where

$$\operatorname{gph} H := \{(x, x^*) \in X \times X^* \mid \exists y^* \in Y^* : (x^* - A^* y^*, y^*) \in F(x, Ax)\},$$

and H_n is similarly defined. We use the fact that H is obtained from F_A as G is obtained from F. We also use the property that when T, T_n are continuous linear operators with $(\|T_n - T\|) \to 0$ and $(F_n) \xrightarrow{b} F$, then $\big(T_n(\operatorname{gph} F_n)\big) \xrightarrow{b} T(\operatorname{gph} F)$.

Corollary 5.1. *Let $F, F_n \in \mathfrak{M}(X \times Y)$ be such that $(F_n) \xrightarrow{b} F$ and let $A, A_n \in L(X,Y)$ be such that $\|A_n - A\| \to 0$. Assume that $Y = \mathbb{R}_+\{Ax - y \mid (x,y) \in \operatorname{dom} F\}$. Then H and H_n for large n are maximal monotone and $(H_n) \xrightarrow{b} H$.*

A particular case is when $M, M_n \in \mathfrak{M}(X)$, $N, N_n \in \mathfrak{M}(Y)$ and $F := M \times N$, that is, $F(x,y) := M(x) \times N(y)$ and $F_n := M_n \times N_n$.

Corollary 5.2. *Let $M, M_n \in \mathfrak{M}(X)$, $N, N_n \in \mathfrak{M}(Y)$ be such that $(M_n) \xrightarrow{b} M$, $(N_n) \xrightarrow{b} N$ and let $A, A_n \in L(X,Y)$ be such that $\|A_n - A\| \to 0$. Assume that $Y = \mathbb{R}_+(A(\operatorname{dom} M) - \operatorname{dom} N)$. Then $M + A^*NA$ and $M_n + A_n^* N_n A_n$ for large n are maximal monotone and $(M_n + A_n^* N_n A_n) \xrightarrow{b} M + A^*NA$.*

The special cases when $M = M_n = 0$, and when $X = Y$ and $A = A_n = I_X$, the identify mapping on X, are important.

Corollary 5.3. *Let $N, N_n \in \mathfrak{M}(Y)$, be such that $(N_n) \xrightarrow{b} N$, and let $A, A_n \in L(X,Y)$ be such that $\|A_n - A\| \to 0$. Assume that $Y = \mathbb{R}_+(A(X) - \operatorname{dom} N)$. Then A^*NA and $A_n^* N_n A_n$ for large n are maximal monotone and $(A_n^* N_n A_n) \xrightarrow{b} A^*NA$.*

Corollary 5.4. *Let $M, M_n, N, N_n \in \mathfrak{M}(X)$ be such that $(M_n) \xrightarrow{b} M$ and $(N_n) \xrightarrow{b} N$. Assume that $Y = \mathbb{R}_+(\operatorname{dom} M - \operatorname{dom} N)$. Then $M + N$ and $M_n + N_n$ for large n are maximal monotone and $(M_n + N_n) \xrightarrow{b} M + N$.*

Note that the result in the preceding corollary has been proved by Attouch–Moudafi–Riahi [1] for X a Hilbert space under the stronger condition $\operatorname{dom} M \cap \operatorname{int}(\operatorname{dom} N) \neq \emptyset$ and a certain condition (Q). Pennanen–Revalski–Théra [11] showed that the condition (Q) is implied by the condition $\operatorname{dom} M \cap \operatorname{int}(\operatorname{dom} N) \neq \emptyset$. Note that when X is finite dimensional, Corollary 5.4 covers the recent result by Pennanen–Rockafellar–Théra [12].

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