INTERPOLATION THEOREM ON LORENTZ SPACES OVER WEIGHTED MEASURE SPACES

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ABSTRACT. In 1997 Ferreyra proved that it is impossible to extend the Stein-Weiss theorem in the context of Lorentz spaces. In this paper we obtain an interpolation theorem on Lorentz spaces over weighted measure spaces.

1. INTRODUCTION

The aim of the paper is to prove an interpolation theorem on Lorentz spaces over weighted measure spaces.

In 1966, Calderón [2] and Hunt [4] proved an interpolation theorem on Lorentz spaces $L_{p,q}$. On the other hand, in 1958, Stein and Weiss [7] proved an interpolation theorem on L_p spaces which allows one to change measures simultaneously with changing exponents. Thus it is natural to seek an interpolation theorem on $L_{p,q}$ spaces which allows one to change measures simultaneously with changing exponents. However, in 1997, Ferreyra [3] gave a negative result for such an interpolation.

In [5], the following result was obtained; notation is given in section 2.

Let $i = 0, 1$. Suppose that $0 < p_0 < p_1 \leq \infty$, $0 < r_i \leq \infty$ with $r_0 \neq r_1$, $0 < q_i \leq \infty$, $0 < s_i \leq \infty$ and $\mathbb{D} = (-\alpha, -\alpha) \in \mathbb{R}^2$ with $\alpha > 1$. Put $1/p = (1 - \theta)/p_0 + \theta/p_1$, $1/r = (1 - \theta)/r_0 + \theta/r_1$ for $0 < \theta < 1$. Let v_i, w_i be nonnegative measurable functions. If T is a quasi-linear operator bounded from $L_{p_i, q_i; v_i}$ to $L_{r_i, s_i; w_i}$, then there exists a constant C such that $\|Tf\|_{r, s; \mathbb{D}; w} \leq C\|f\|_{p, q; v}$ for every linear combination f of two characteristic functions with disjoint supports, where $q \leq s$ and $v^{1/p} = v_0^{(1-\theta)/p_0} v_1^{\theta/p_1}$, $w^{1/r} = w_0^{(1-\theta)/r_0} w_1^{\theta/r_1}$.

In this paper, we prove an interpolation theorem of the above type. Under the hypothesis that T is nonnegative, and bounded from $L_{p_i, q_i; v}$ to $L_{r_i, s_i; w_i}$, we obtain the inequality $\|Tf\|_{r, \infty; w} \leq C\|f\|_{p, 1; v}$. Note that the domain space is a Lorentz space $L_{p, 1; v}$ with fixed weight v . This conclusion is weak in the sense that the target $L_{r, \infty}$ is large and the domain $L_{p, 1}$ is small in the framework of $L_{p, q}$ classes.

The plan of the paper is as follows. In preliminary section 2 we give definitions and lemmas. In section 3 we state our main theorem and give its proof.

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2. PRELIMINARIES

Let $(M, \mathfrak{M}, \mu)$ be a resonant measure space and let f be a complex-valued or real-valued measurable function defined on M .

Definition 2.1. Let w be a nonnegative measurable function on M . Then the distribution function of f with respect to w is defined as follows:

$$\lambda_{f,w}(y) = w(\{x \in M : |f(x)| > y\}), \quad y > 0,$$

where $w(A) = \int_A w d\mu$ for every $A \in \mathfrak{M}$. Then the nonincreasing rearrangement of f with respect to w is defined as follows:

$$f_w^*(t) = \inf\{y : \lambda_{f,w}(y) \leq t\}, \quad t > 0.$$

The average function of f with respect to w is defined as follows:

$$f_w^{**}(t) = \frac{1}{t} \int_0^t f_w^*(y) dy, \quad t > 0.$$

Definition 2.2. Let w be a nonnegative measurable function on M . Then the weighted Lorentz space $L_{p,q;w}$ is defined as the set of all measurable functions f on M such that $\|f\|_{p,q;w} < \infty$, where

$$\|f\|_{p,q;w} = \begin{cases} \left(\frac{q}{p} \int_0^\infty [t^{1/p} f_w^*(t)]^q \frac{dt}{t} \right)^{1/q}, & 1 \leq p < \infty, 0 < q < \infty, \\ \sup_{t>0} t^{1/p} f_w^*(t), & 1 \leq p \leq \infty, q = \infty. \end{cases}$$

In general, however, $\|\cdot\|_{p,q;w}$ is not a norm since the Minkowski inequality may fail. But by replacing f_w^* with f_w^{**} in the above definition of $\|f\|_{p,q;w}$, we obtain a norm $\|\cdot\|_{(p,q);w}$ for all $q \geq 1$:

$$\|f\|_{(p,q);w} = \begin{cases} \left(\frac{q}{p} \int_0^\infty [t^{1/p} f_w^{**}(t)]^q \frac{dt}{t} \right)^{1/q}, & 1 < p < \infty, 0 < q < \infty, \\ \sup_{t>0} t^{1/p} f_w^{**}(t), & 1 < p \leq \infty, q = \infty. \end{cases}$$

We state the following from Bennett and Sharpley [1].

Lemma 2.3. If $1 < p \leq \infty$ and $1 \leq q \leq \infty$, then

$$\|f\|_{p,q;w} \leq \|f\|_{(p,q);w} \leq \frac{p}{p-1} \|f\|_{p,q;w}.$$

Proof. The first inequality is an immediate consequence of Definition 2.2 and the fact that $f_w^* \leq f_w^{**}$. The second follows from the Hardy inequality. $\square$

As mentioned in the introduction, we include the definition of Lorentz-Zygmund spaces with norm $\|\cdot\|_{p,q;\mathbb{D};w}$. We note that the case where $\mathbb{D} = (0, 0)$ corresponds to the weighted Lorentz spaces given in Definition 2.2.

Definition 2.4. Let w be a nonnegative measurable function on M . Then the weighted Lorentz-Zygmund space $L_{p,q;w}(\log L)_{\mathbb{D}}$, $\mathbb{D} = (\alpha, \beta) \in \mathbb{R}^2$, is defined as the set of all measurable functions f on M such that $\|f\|_{p,q;\mathbb{D};w} < \infty$, where

$$\|f\|_{p,q;\mathbb{D};w} = \begin{cases} \left(\frac{q}{p} \int_0^\infty [t^{1/p} f_w^*(t)]^q (\log t)^{\mathbb{D}} \frac{dt}{t} \right)^{1/q}, & 0 < p, q < \infty, \\ \sup_{t>0} t^{1/p} f_w^*(t) (\log t)^{\mathbb{D}}, & 0 < p \leq \infty, q = \infty. \end{cases}$$

Here,

$$(\log t)^{\mathbb{D}} = \begin{cases} (1 - \log t)^{\alpha}, & 0 < t < 1, \\ (1 + \log t)^{\beta}, & 1 \leq t < \infty. \end{cases}$$

Let $(N, \mathfrak{N}, \nu)$ be another resonant measure space. Let T be an operator from a class of measurable functions on M to that on N .

Definition 2.5. An operator T is called quasi-linear if there is a constant $K \geq 1$ such that

$$|T(f+g)| \leq K(|Tf| + |Tg|), \quad |T(\lambda f)| = |\lambda| \cdot |Tf|$$

hold ν -a.e. on N for all f and g in the domain of T and for all scalars λ . If we can take $K = 1$, T is called sublinear. T is called nonnegative when $Tf \geq 0$, ν -a.e. for all f in the domain of T .

We recall the following result from Bennett and Sharpley [1, Lemma 5.1, p. 231]. This is necessary for the limiting argument given in the proof of Theorem 3.1 in the next section 3.

Lemma 2.6. *Let X and Y be rearrangement-invariant Banach function spaces over resonant measure spaces (M, μ) and (N, ν) , respectively. Let T be a nonnegative sublinear operator defined on a dense linear subspace D of X and taking values in Y . If*

$$(2.1) \quad \|Tf\|_Y \leq C\|f\|_X$$

for all $f \in D$, then T has a unique extension to a nonnegative sublinear operator from X to Y , for which (2.1) holds for all $f \in X$.

Finally, we prove the following.

Lemma 2.7. *There are constants C_1 and C_2 such that*

$$C_1 \sum_{k=-\infty}^{\infty} 2^{k/p} f_w^*(2^k) \leq \|f\|_{p,1;w} \leq C_2 \sum_{k=-\infty}^{\infty} 2^{k/p} f_w^*(2^k).$$

Proof. Let $t \in [2^k, 2^{k+1}]$. Since f_w^* is nonincreasing, $f_w^*(2^{k+1}) \leq f_w^*(t) \leq f_w^*(2^k)$.

On the one hand, we obtain

$$\begin{aligned} \|f\|_{p,1;w} &= \frac{1}{p} \int_0^{\infty} t^{1/p} f_w^*(t) \frac{dt}{t} \\ &= \frac{1}{p} \sum_{k=-\infty}^{\infty} \int_{2^k}^{2^{k+1}} t^{1/p} f_w^*(t) \frac{dt}{t} \\ &\leq \frac{1}{p} \sum_{k=-\infty}^{\infty} \int_{2^k}^{2^{k+1}} t^{1/p} f_w^*(2^k) \frac{dt}{t} \\ &= \sum_{k=-\infty}^{\infty} f_w^*(2^k) (2^{(k+1)/p} - 2^{k/p}) \\ &= (2^{1/p} - 1) \sum_{k=-\infty}^{\infty} 2^{k/p} f_w^*(2^k). \end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}\|f\|_{p,1;w} &\geq \frac{1}{p} \sum_{k=-\infty}^{\infty} \int_{2^k}^{2^{k+1}} t^{1/p} f_w^*(2^{k+1}) \frac{dt}{t} \\ &= \sum_{k=-\infty}^{\infty} f_w^*(2^{k+1}) (2^{(k+1)/p} - 2^{k/p}) \\ &= (1 - 2^{-1/p}) \sum_{k=-\infty}^{\infty} 2^{(k+1)/p} f_w^*(2^{k+1}).\end{aligned}$$

□

3. MAIN THEOREM

We can now prove the following.

Theorem 3.1. *Let $i = 0, 1$. Suppose that $1 \leq p_0 < p_1 \leq \infty$, $1 \leq r_i \leq \infty$ with $r_0 \neq r_1$, $0 < q_i, s_i \leq \infty$. Put $1/p = (1 - \theta)/p_0 + \theta/p_1$, $1/r = (1 - \theta)/r_0 + \theta/r_1$ for $0 < \theta < 1$. Let v, w_i be nonnegative measurable functions. If T is a nonnegative sublinear operator from $L_{p_i, q_i; v}$ to $L_{r_i, s_i; w_i}$, then there exists a constant C such that*

$$\|Tf\|_{r, \infty; w} \leq C \|f\|_{p, 1; v},$$

where $w^{1/r} = w_0^{(1-\theta)/r_0} w_1^{\theta/r_1}$.

Remark 3.2. We do not know if this theorem is true in the case where T is not assumed to be nonnegative.

Proof of Theorem 3.1. In our proof, the letter C will denote a constant not necessarily the same at each occurrence. Let f be a measurable function on M . We consider pairwise disjoint measurable subsets of M ,

$$E_n = \{x \in M : f_v^*(2^{n+1}) < |f(x)| \leq f_v^*(2^n)\}, \quad n = 0, \pm 1, \pm 2, \dots$$

We may assume that $|f(x)| < \infty$, μ -a.e. $x \in M$. If $|f(x)| \equiv \infty$ on some set of positive measure, then we have $f_v^*(t) \equiv \infty$ near $t = 0$, which implies that $\|f\|_{p, 1; v} = \infty$. Moreover, we may also assume that $f_v^*(t)$ tends to 0 as $t \rightarrow \infty$. If $f_v^*(t)$ does not tend to 0, then $\|f\|_{p, 1; v} = \infty$.

Let f_n be defined by

$$f_n(x) = \begin{cases} f(x), & x \in E_n, \\ 0, & \text{otherwise.} \end{cases}$$

Then we consider the following three cases according to the value of $|f(x)|$. When $|f(x)| = 0$, x does not belong to any E_n , so $f_n(x) = 0$ for every n . Next, when $0 < |f(x)| < \infty$, x belongs to some E_n because we have assumed that $f_v^*(t)$ tends to 0 as $t \rightarrow \infty$. In this case, $f_n(x) = f(x)$ and $f_m(x) = 0$ for $m \neq n$. Finally, when $|f(x)| = \infty$, x does not belong to any E_n because we have assumed that $|f(x)| < \infty$, μ -a.e. $x \in M$. In this final case, $f_n(x) = 0$ for every n . Therefore, we may describe the function f as the following sum of the functions f_n (see [6]):

$$(3.1) \quad f(x) = \sum_{n=-\infty}^{\infty} f_n(x), \quad \mu\text{-a.e. } x \in M.$$

We remark that if $E_n = \emptyset$, then $f_n(x) \equiv 0$ on M . Another important remark is that we have $v(E_n) \leq 2 \cdot 2^n$ because of the construction of the sets E_n , which is used in the estimate (3.6) below. Because of Lemma 2.6, we may assume that all but finitely many f_n 's in (3.1) are identically zero. The set of such functions f is a dense linear subspace of $L_{p,1;v}$, $1 < p < \infty$, and we can apply a limiting argument.

By using the sublinearity of T (Definition 2.5), the Minkowski inequality (Definition 2.2) and Lemma 2.3, we have

$$(3.2) \quad \|Tf\|_{(r,\infty);w} \leq \sum_{n=-\infty}^{\infty} \|Tf_n\|_{(r,\infty);w} \leq \frac{r}{(r-1)} \sum_{n=-\infty}^{\infty} \|Tf_n\|_{r,\infty;w}.$$

(We note again that the summations in the inequalities in (3.2) have only finitely many nonzero terms.)

We now estimate the quantity $\|Tf_n\|_{r,\infty;w}$. Because T is assumed to be a sublinear operator from $L_{p_i,q_i;v}$ to $L_{r_i,s_i;w_i}$,

$$t^{1/r_i} (Tf_n)^*_{w_i}(t) \leq C \|f_n\|_{p_i,q_i;v}, \quad i = 0, 1.$$

By the change of variables $t = \lambda_{Tf_n,w_i}(s)$, we have

$$(3.3) \quad s(\lambda_{Tf_n,w_i}(s))^{1/r_i} \leq C \|f_n\|_{p_i,q_i;v}, \quad i = 0, 1.$$

We use the Hölder inequality and the relation that $1/r = (1-\theta)/r_0 + \theta/r_1$, $w^{1/r} = w_0^{(1-\theta)/r_0} w_1^{\theta/r_1}$ to obtain

$$\begin{aligned} & s(\lambda_{Tf_n,w}(s))^{1/r} \\ &= s \left(\int_{\{x; |Tf_n(x)| > s\}} w(x) d\mu(x) \right)^{1/r} \\ (3.4) \quad & \leq s \left(\int_{\{x; |Tf_n(x)| > s\}} w_0(x) d\mu(x) \right)^{(1-\theta)/r_0} \left(\int_{\{x; |Tf_n(x)| > s\}} w_1(x) d\mu(x) \right)^{\theta/r_1} \\ &= \left[s(\lambda_{Tf_n,w_0}(s))^{1/r_0} \right]^{1-\theta} \left[s(\lambda_{Tf_n,w_1}(s))^{1/r_1} \right]^{\theta}. \end{aligned}$$

From (3.3) and (3.4),

$$s(\lambda_{Tf_n,w}(s))^{1/r} \leq C \|f_n\|_{p_0,q_0;v}^{1-\theta} \|f_n\|_{p_1,q_1;v}^{\theta}.$$

Thus we obtain

$$(3.5) \quad \|Tf_n\|_{r,\infty;w} \leq C \|f_n\|_{p_0,q_0;v}^{1-\theta} \|f_n\|_{p_1,q_1;v}^{\theta}.$$

As for the quantity $\|f_n\|_{p_0,q_0;v}$, we have

$$\begin{aligned} \|f_n\|_{p_0,q_0;v} &= \left(\frac{q_0}{p_0} \int_0^\infty [t^{1/p_0} (f_n)^*_v(t)]^{q_0} \frac{dt}{t} \right)^{1/q_0} \\ &\leq \left(\frac{q_0}{p_0} \int_0^{v(E_n)} [t^{1/p_0} f_v^*(2^n)]^{q_0} \frac{dt}{t} \right)^{1/q_0} \\ &= f_v^*(2^n) (v(E_n))^{1/p_0}. \end{aligned}$$

Similarly as for the quantity $\|f_n\|_{p_1,q_1;v}$,

$$\|f_n\|_{p_1,q_1;v} \leq f_v^*(2^n) (v(E_n))^{1/p_1}.$$

By using the relation that $1/p = (1 - \theta)/p_0 + \theta/p_1$, we obtain from (3.5),

$$(3.6) \quad \|Tf_n\|_{r,\infty;w} \leq Cf_v^*(2^n)(v(E_n))^{1/p} \leq Cf_v^*(2^n) \cdot (2 \cdot 2^n)^{1/p}.$$

Consequently, from (3.2), (3.6) and Lemma 2.7 we obtain

$$\|Tf\|_{r,\infty;w} \leq C \sum_{n=-\infty}^{\infty} f_v^*(2^n) \cdot 2^{n/p} \leq C\|f\|_{p,1;v}.$$

□

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