SPECTRA OF ALGEBRAS OF ENTIRE FUNCTIONS ON BANACH SPACES

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(Communicated by N. Tomczak-Jaegermann)

ABSTRACT. We obtain an explicit description of the spectrum (set of closed maximal ideals) of $H_b(X)$, algebra of analytic functions on a Banach space X which are bounded on bounded subsets. We show that the spectrum of $H_b(X)$ admits a natural linear structure. Some applications to the algebra of uniformly continuous and bounded analytic functions on the unit ball $B \subset X$ are indicated.

Let A be a complex commutative topological algebra. Let us denote by $M(A)$ the spectrum (set of closed maximal ideals = set of continuous characters = set of continuous complex-valued homomorphisms) of A . Recall that A is *semisimple* if the complex homomorphisms from $M(A)$ separate points of A . It is well known that every semisimple commutative Fréchet algebra A is isomorphic to some subalgebra of continuous functions on $M(A)$ endowed with a natural topology. More exactly, for every $a \in A$ there exists a function $\hat{a} : M(A) \rightarrow \mathbb{C}$ defined by $\hat{a}(\phi) := \phi(a)$. The weakest topology on $M(A)$ such that all functions $\hat{a}$, $a \in A$, are continuous is called the *Gelfand topology*. The Gelfand topology coincides with the weak-star topology of the strong dual space A' , restricted to $M(A)$. If A is a Banach algebra, $M(A)$ is a weak-star compact subset of the unit ball of A' .

If A is a uniform algebra of continuous functions on a metric space G , then for any $x \in G$ the *point evaluation functional* $\delta(x) : f \mapsto f(x)$ belongs to $M(A)$.

The purpose of this paper is to describe the spectrum of the Fréchet algebra $H_b(X)$ of entire analytic functions of bounded type on a Banach space X and to study some related questions of infinite-dimensional holomorphy.

The problem of description of the spectrum of $H_b(X)$ was first studied by Aron, Cole and Gamelin [3, 4]. Using the Aron-Berner extension operation [2, 10], they showed, in particular, that X'' belongs to the spectrum of $H_b(X)$. In [5] it is proved that this inclusion is proper if there exists a polynomial on X which is not weakly continuous on bounded sets. This approach was generalized for algebra-valued analytic functions by García et al. in [18]. Some analytic structure on the set of maximal ideals was considered in [5] (for generalization for algebra-valued functions see [17]). In [22] Mujica investigated ideals of analytic functions of bounded type on

Received by the editors July 8, 2004 and, in revised form, January 28, 2005, February 1, 2005, and March 15, 2005.

2000 *Mathematics Subject Classification*. Primary 46J15, 46J20; Secondary 46E15, 46E25, 46G20, 46G25.

Key words and phrases. Infinite-dimensional holomorphy, spectra of algebras.

This research was supported in part by NSERC research grant OGP 005616.

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Tsirelson's space T and showed that each character on $H_b(T)$ is a point evaluation functional. Homomorphisms of H_b were studied by Carando, García and Maestre in [9]. In [1] Alencar et al. considered maximal ideals of algebras of symmetric analytic functions on ℓ_p .

In this paper we show that every element of the spectrum of $H_b(X)$ can be represented by a sequence of functionals $(u_k)_{k=1}^\infty$ such that each u_k belongs to a Banach space E_k , where $E_1 = X''$ and E_n coincides with a special subspace of linear functionals on n -homogeneous polynomials. It is also shown that the spectrum of $H_b(X)$ contains the linear space of all finite sequences $(u_1, \dots, u_m, 0, 0, \dots)$. Finally, some related examples are considered.

For background on analytic functions on infinite-dimensional spaces, we refer the reader to [13] or to [21]. For details on the Aron-Berner extension we refer to [8].

For a given complex Banach space X , $\mathcal{P}(^n X)$ (resp. $\mathcal{P}(^{\leq n} X)$) denotes the Banach space of all continuous n -homogeneous complex-valued polynomials on X (resp. the Banach space of all continuous n -degree complex-valued polynomials on X). $\mathcal{P}_f(^n X)$ denotes the subspace of n -homogeneous polynomials of *finite type*, that is the subspace generated by all polynomials of the form $P(x) = (\gamma(x))^n$ with $\gamma \in X'$ and $\mathcal{P}_c(^n X)$ is the closure of $\mathcal{P}_f(^n X)$ with the topology of uniform convergence on bounded subsets of X . It is well known [6] that if X' has the approximation property, then $\mathcal{P}_c(^n X)$ coincides with $\mathcal{P}_{wu}(^n X)$, the space of n -homogeneous polynomials which are weakly uniformly continuous on bounded subsets of X .

Recall that for every polynomial $P \in \mathcal{P}(^n X)$ there exists a (necessarily unique) symmetric n -linear form A_P , associated with P such that $A_P(x, \dots, x) = P(x)$. We will write $A_P(x_1^{k_1}, \dots, x_n^{k_n})$ instead of $A_P(\underbrace{x_1, \dots, x_1}_{k_1}, \dots, \underbrace{x_n, \dots, x_n}_{k_n})$. We will use

the fact that $\mathcal{P}(^n X)$ is isomorphic to the dual space of the symmetric projective n -fold tensor product $\bigotimes_{s, \pi}^n X$ of X .

Let us denote by $A_n(X)$ the closure of the algebra, generated by polynomials from $\mathcal{P}(^{\leq n} X)$ with respect to the uniform topology on bounded subsets. It is clear that $A_1(X) \cap \mathcal{P}(^n X) = \mathcal{P}_c(^n X)$ and $A_n(X)$ is a Fréchet algebra of entire analytic functions on X for every n . The closure of the algebra of all polynomials $\mathcal{P}(X)$ with respect to the uniform topology on bounded subsets is denoted by $H_b(X)$ and is called the *algebra of entire functions of bounded type* on X . It is well known that $H_b(X)$ consists of all entire functions that are bounded on bounded subsets. The closure of the algebra of all polynomials with respect to the uniform topology on the unit ball B , $H_{uc}^\infty(B)$, is the algebra of all analytic functions on B which are uniformly continuous and bounded. We will use the short notation M_b and M_{uc} for the spectra $M(H_b(X))$ and $M(H_{uc}^\infty(B))$ respectively.

According to [3], every continuous functional $\phi \in H_b(X)'$ can be represented by $\phi = \sum_{k=0}^\infty \phi_k$, where $\phi_k = \pi_k(\phi)$ is the restriction of ϕ to $\mathcal{P}(^k X)$. The infimum of all $r > 0$, $R(\phi)$ such that ϕ is continuous with respect to the norm of uniform convergence on the ball rB is called the *radius function* of ϕ . It is known [3] that

$$R(\phi) = \limsup_{n \rightarrow \infty} \|\phi_n\|^{1/n}.$$

For every polynomial $P \in \mathcal{P}(^{mk} X)$ we denote by $P_{(m)}(u)$ the polynomial from $\mathcal{P}(^k \bigotimes_{s, \pi}^m X)$ such that $P_{(m)}(x^{\otimes m}) = P(x)$, where $x^{\otimes m} = \underbrace{x \otimes \dots \otimes x}_{m \text{ times}}$.

Lemma 1. *Let $\phi \in H_b(X)'$ such that $\phi(P) = 0$ for every $P \in \mathcal{P}({}^m X) \cap A_{m-1}(X)$, where m is a fixed positive integer and $\phi_m \neq 0$. Then there is $\psi \in M_b$ such that $\psi_k = 0$ for $k < m$ and $\psi_m = \phi_m$. The radius function $R(\psi) = \|\phi_m\|^{1/m}$.*

Proof. Since $\phi_m \neq 0$, there is an element $w \in (\bigotimes_{s,\pi}^m X)''$, $w \neq 0$ such that for any m -homogeneous polynomial P , $\phi(P) = \phi_m(P) = \tilde{P}_{(m)}(w)$, where $\tilde{P}_{(m)}$ is the Aron-Berner extension of the linear functional $P_{(m)}$ from $\bigotimes_{s,\pi}^m X$ to $(\bigotimes_{s,\pi}^m X)''$ and $\|w\| = \|\phi_m\|$. For an arbitrary n -homogeneous polynomial Q we set

$$(1) \quad \psi(Q) = \begin{cases} \tilde{Q}_{(m)}(w) & \text{if } n = mk \text{ for some } k \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\tilde{Q}_{(m)}$ is the Aron-Berner extension of the k -homogeneous polynomial $Q_{(m)}$ from $\bigotimes_{s,\pi}^m X$ to $(\bigotimes_{s,\pi}^m X)''$.

Let (u_α) be a net from $\bigotimes_{s,\pi}^m X$ that converges to w in the weak-star topology of $(\bigotimes_{s,\pi}^m X)''$, where α belongs to an index set $\mathfrak{A}$. We can assume that u_α has a representation $u_\alpha = \sum_{j=1}^\infty x_{j,\alpha}^{\otimes m}$ for some $x_{j,\alpha} \in X$. Let us show that $\psi(PQ) = \psi(P)\psi(Q)$ for any homogeneous polynomials P and Q . Let us suppose first that $\deg(PQ) = mr + l$ for some integers $r \geq 0$ and $m > l > 0$. Then P or Q has degree equal to $mk + s$, $k \geq 0$, $m > s > 0$. Thus, by the definition, $\psi(PQ) = 0$ and $\psi(P)\psi(Q) = 0$. Suppose that $\deg(PQ) = mr$ for some integer $r \geq 0$. If $\deg P = mk$ and $\deg Q = mn$ for $k, n \geq 0$, then $\deg(PQ) = m(k+n)$ and $\psi(PQ) = (\tilde{PQ})_{(m)}(w) = \tilde{P}_{(m)}(w)\tilde{Q}_{(m)}(w) = \psi(P)\psi(Q)$.

Now let $\deg P = mk + l$ and $\deg Q = mn + r$, $l, r > 0$, $l + r = m$. Write $\nu = 1/(\deg P + \deg Q)! = 1/(m(k+n+1))!$. Let A_{PQ} denote the symmetric multilinear map, associated with PQ . Then

$$\begin{aligned} & A_{PQ}(x_1, \dots, x_{m(k+n+1)}) \\ &= \nu \sum_{\sigma \in S_{m(k+n+1)}} A_P(x_{\sigma(1)}, \dots, x_{\sigma(mk+l)}) A_Q(x_{\sigma(mk+l+1)}, \dots, x_{\sigma(m(k+n+1))}), \end{aligned}$$

where $S_{m(k+n+1)}$ is the group of permutations on $\{1, \dots, m(k+n+1)\}$. Thus for $\alpha_1, \dots, \alpha_{k+n+1} \in \mathfrak{A}$ we have

$$\begin{aligned} \psi(PQ) &= (\tilde{PQ})_{(m)}(w) = \lim_{\alpha_1, \dots, \alpha_{k+n+1}} \tilde{A}_{PQ(m)}(u_{\alpha_1}, \dots, u_{\alpha_{k+n+1}}) \\ &= \lim_{\alpha_1, \dots, \alpha_{k+n+1}} \tilde{A}_{PQ(m)} \left(\sum_{j=1}^\infty x_{j,\alpha_1}^{\otimes m}, \dots, \sum_{j=1}^\infty x_{j,\alpha_{k+n+1}}^{\otimes m} \right) \\ &= \nu \sum_{\sigma \in S_{m(k+n+1)}} \lim_{\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(k+n+1)}} \sum_{j_1, \dots, j_{k+n+1}=1}^\infty A_P(x_{j_{\sigma(1)}, \alpha_{\sigma(1)}}^m, \dots, x_{j_{\sigma(k)}, \alpha_{\sigma(k)}}^m, \\ &\quad x_{j_{\sigma(k+1)}, \alpha_{\sigma(k+1)}}^l) A_Q(x_{j_{\sigma(k+1)}, \alpha_{\sigma(k+1)}}^r, x_{j_{\sigma(k+2)}, \alpha_{\sigma(k+2)}}^m, \dots, x_{j_{\sigma(k+n+1)}, \alpha_{\sigma(k+n+1)}}^m). \end{aligned}$$

Fix some $\sigma \in S_{m(k+n+1)}$ and fix all $x_{j_{\sigma(i)}, \alpha_{\sigma(i)}}$, for $i \leq k$ and for $i > k+1$. Then

$$\begin{aligned} & \sum_{j_1, \dots, j_{k+n+1}=1}^\infty \lim_{\alpha_{\sigma(k+1)}} A_P(x_{j_{\sigma(1)}, \alpha_{\sigma(1)}}^m, \dots, x_{j_{\sigma(k)}, \alpha_{\sigma(k)}}^m, x_{j_{\sigma(k+1)}, \alpha_{\sigma(k+1)}}^l) \\ & \times A_Q(x_{j_{\sigma(k+1)}, \alpha_{\sigma(k+1)}}^r, x_{j_{\sigma(k+2)}, \alpha_{\sigma(k+2)}}^m, \dots, x_{j_{\sigma(k+n+1)}, \alpha_{\sigma(k+n+1)}}^m) = 0 \end{aligned}$$

because for fixed $x_{k_{\sigma(i)}, \alpha_{\sigma(i)}}$, $i \leq k$,

$$P_{\sigma}(y) := \sum_{j_1, \dots, j_k, j_{k+2}, \dots, j_{k+n+1}=1}^{\infty} A_P(x_{j_{\sigma(1)}, \alpha_{\sigma(1)}}^m, \dots, x_{j_{\sigma(k)}, \alpha_{\sigma(k)}}^m, y^l)$$

is an l -homogeneous polynomial and for fixed $x_{k_{\sigma(i)}, \alpha_{\sigma(i)}}$, $i > k+1$,

$$Q_{\sigma}(y) := \sum_{j_1, \dots, j_k, j_{k+2}, \dots, j_{k+n+1}=1}^{\infty} A_Q(y^r, x_{j_{\sigma(k+2)}, \alpha_{\sigma(k+2)}}^m, \dots, x_{j_{\sigma(k+n+1)}, \alpha_{\sigma(k+n+1)}}^m)$$

is an r -homogeneous polynomial. Thus $P_{\sigma}Q_{\sigma} \in A_{m-1}(X)$. Hence

$$\lim_{\alpha} (P_{\sigma}Q_{\sigma})_{(m)}(u_{\alpha}) = \psi(P_{\sigma}Q_{\sigma}) = 0$$

for every fixed σ . Thus $\psi(PQ) = 0$. On the other hand, $\psi(P)\psi(Q) = 0$ by the definition of ψ . So $\psi(PQ) = \psi(P)\psi(Q)$.

Thus we have defined the multiplicative function ψ on homogeneous polynomials. We can extend it by linearity and distributivity to a linear multiplicative functional on the algebra of all continuous polynomials $\mathcal{P}(X)$. If ψ_n is the restriction of ψ to $\mathcal{P}^n(X)$, then $\|\psi_n\| = \|w\|^{n/m}$ if n/m is a positive integer and $\|\psi_n\| = 0$ otherwise. Hence $\psi = \sum_{n=0}^{\infty} \psi_n$ is a continuous linear multiplicative functional on $H_b(X)$ by [3, 2.4. Theorem] and the radius function of ψ can be computed by

$$R(\psi) = \limsup_{n \rightarrow \infty} \|\psi_n\|^{1/n} = \limsup_{n \rightarrow \infty} \|w\|^{n/mn} = \|w\|^{1/m} = \|\phi_m\|^{1/m}$$

as required. $\square$

For each fixed $x \in X$, the translation operator T_x is defined on $H_b(X)$ by

$$(T_x f)(y) = f(y+x), \quad f \in H_b(X).$$

It is not complicated to check that $T_x f \in H_b(X)$ and for fixed $\phi \in H_b(X)'$ the function $x \mapsto \phi(T_x f)$, $x \in X$, belongs to $H_b(X)$ (see [3]).

For fixed $\phi, \theta \in H_b(X)'$ the convolution product $\phi * \theta$ in $H_b(X)$ is defined by

$$(\phi * \theta)(f) = \phi(\theta(T_x f)), \quad f \in H_b(X).$$

Let $\phi, \theta \in M_b$. According to [3, 4.7. Corollary], there exist nets $(x_{\alpha}), (y_{\beta}) \subset X$ such that

$$(2) \quad \phi(P) = \lim_{\alpha} P(x_{\alpha}), \quad \theta(P) = \lim_{\beta} P(y_{\beta})$$

for every polynomial P . We will write the condition (2) by $x_{\alpha} \xrightarrow{P} \phi$ and $y_{\beta} \xrightarrow{P} \theta$. Thus for every polynomial P we have: $(\phi * \theta)(P) = \lim_{\beta} \lim_{\alpha} P(x_{\alpha} + y_{\beta})$. Note that M_b is a semigroup with respect to the convolution product and $\phi * \theta \neq \theta * \phi$ in general (see [5, Remark 3.5]). We denote $\phi_1 * \dots * \phi_n$ briefly by $\bigstar_{k=1}^n \phi_k$.

Let I_k be the minimal closed ideal in $H_b(X)$, generated by all m -homogeneous polynomials, $0 < m \leq k$. Evidently, I_k is a proper ideal (contains no unit) so it is contained in a closed maximal ideal (see [21, p. 228]). Let

$$\Phi_k := \{\phi \in M_b : \ker \phi \supset I_k\}.$$

We set $\Phi_0 := M_b$. The functional $\delta(0)$, that is, point evaluation at zero, belongs to Φ_k for every $k > 0$.

Corollary 2. *If $A_m(X) \neq A_{m-1}(X)$ for some $m > 1$, then there exists $\psi \in \Phi_{m-1}$ such that $\psi \notin \Phi_m$.*

Proof. Let $P \in \mathcal{P}^m(X)$ and $P \notin A_{m-1}(X)$. Since $A_{m-1}(X)$ is a closed subspace of $H_b(X)$, by the Hahn-Banach Theorem there exists a linear functional $\phi \in H_b(X)'$ such that $\phi(Q) = 0$ for every $Q \in A_{m-1}(X)$ and $\phi(P) \neq 0$. So $\phi_k \equiv 0$ for $k < m$ and $\phi_m(P) \neq 0$. By Lemma 1 there exists $\psi \in M_b$ such that $\psi_k = \phi_k$ for $k = 1, \dots, m$. Thus $\psi \in \Phi_{m-1}$, but $\psi \notin \Phi_m$. $\square$

Note that $A_1(c_0) = A_n(c_0)$ for every n , but $A_k(\ell_p) = A_m(\ell_p)$ for $k \neq m$ if and only if $k < p$ and $m < p$. Moreover, if X admits a polynomial which is not weakly sequentially continuous, then the chain of algebras $\{A_k(X)\}$ does not stabilize and if X contains ℓ_1 , then $A_k(X) \neq A_m(X)$ for $k \neq m$ [19, 12].

Lemma 3. *If $\phi, \psi \in M_b$ and $\psi \in \Phi_{k-1}$, then $\phi * \psi(P) = \phi(P) + \psi(P)$ for every $P \in \mathcal{P}^k(X)$.*

Proof. Let (x_α) and (y_β) be nets in X such that $x_\alpha \xrightarrow{\mathcal{P}} \phi$ and $y_\beta \xrightarrow{\mathcal{P}} \psi$. For any fixed y_β and $0 < n < k$, $A_P(x^{k-n}, y_\beta^n)$ is a $(k-n)$ -homogeneous polynomial. Thus

$$\phi(A_P(x^{k-n}, y_\beta^n)) = \lim_{\alpha} A_P(x_\alpha^{k-n}, y_\beta^n) = 0.$$

Therefore,

$$\begin{aligned} \phi * \psi(P) &= \lim_{\beta, \alpha} P(x_\alpha + y_\beta) \\ &= \sum_{n+m=k} \lim_{\beta, \alpha} A_P(x_\alpha^n, y_\beta^m) = \sum_{n+m=k} \lim_{\beta} \left(\lim_{\alpha} A_P(x_\alpha^n, y_\beta^m) \right) \\ &= \lim_{\beta} \left(\lim_{\alpha} A_P(x_\alpha, \dots, x_\alpha) + A_P(y_\beta, \dots, y_\beta) \right) = \phi(P) + \psi(P). \end{aligned}$$

$\square$

Lemma 4. *If $P \in \mathcal{P}^k(X)$, $\phi_j \in \Phi_{j-1}$, then for every $m > k$, $\bigstar_{j=1}^m \phi_j(P) = \bigstar_{j=1}^k \phi_j(P)$.*

Proof. Since $\phi_j \in \Phi_{j-1}$, $\phi_j(P) = 0$ for every $j > k$. $\square$

Given a sequence $(\phi_n)_{n=1}^\infty \subset M_b$, $\phi_n \in \Phi_{n-1}$, the infinite convolution $\bigstar_{n=1}^\infty \phi_n$ denotes a linear multiplicative functional on the algebra of all polynomials $\mathcal{P}(X)$ such that $\bigstar_{n=1}^\infty \phi_n(P) = \bigstar_{n=1}^k \phi_n(P)$ if $P \in \mathcal{P}^k(X)$ for an arbitrary k . This multiplicative functional uniquely determines a functional in M_b (which we denote by the same symbol $\bigstar_{n=1}^\infty \phi_n$) if it is continuous.

The point evaluation operator δ maps X into M_b by $x \mapsto \delta(x)$, $\delta(x)(f) = f(x)$. The operator $\tilde{\delta}$ is the extension of δ onto X'' , i.e. $\tilde{\delta}(x'')(f) = \tilde{f}(x'')$ for every $x'' \in X''$.

Theorem 5. *There exists a sequence of dual Banach spaces $(E_n)_{n=1}^\infty$ and a sequence of maps $\delta^{(n)} : E_n \rightarrow M_b$ such that $E_1 = X''$, $E_n = \mathcal{P}^n(X)' \cap I_{n-1}^\perp$, $\delta^{(1)} = \tilde{\delta}$ and such that an arbitrary complex homomorphism $\phi \in M_b$ has a representation*

$$(3) \quad \phi = \bigstar_{n=1}^\infty \delta^{(n)}(u_n)$$

for some $u_n \in E_n$, $n = 1, 2, \dots$

Proof. Put $E_1 = X''$. Then $\delta^{(1)}(x'') = \tilde{\delta}(x'') \in M_b$ for every $x'' \in X''$. Suppose that spaces E_k and maps $\delta^{(k)}$ are constructed for $k < n$. Denote by E_n the set $\{\pi_n(\phi) : \phi \in \Phi_{n-1}\}$, where $\pi_n(\phi) = \phi_n$ is the restriction of ϕ onto the subspace $\mathcal{P}({}^nX)$. In other words, E_n consists of linear continuous functionals on $\mathcal{P}({}^nX)$ that vanish on all polynomials in $\mathcal{P}({}^nX) \cap A_{n-1}$. If $A_n = A_{n-1}$, then $E_n \equiv 0$. Otherwise, by Corollary 2, there are nonzero points in E_n .

By Lemma 3, for $P \in \mathcal{P}({}^nX)$ and $\phi, \psi \in \Phi_{n-1} \subset M_b$, $\pi_n(\phi * \psi)(P) = \phi * \psi(P) = \phi(P) + \psi(P) = \pi_n\phi(P) + \pi_n\psi(P)$. Hence $\pi_n(\phi * \psi) = \pi_n(\phi) + \pi_n(\psi)$. For an arbitrary complex number a , $a\phi \in H_b(X)'$ and $\pi_k(a\phi) = a\pi_k(\phi)$. So $a\phi$ vanishes on all homogeneous polynomials of degree less than n . By Lemma 1 there exists $\psi \in M_b$ such that $\psi_k = a\phi_k$ for $1 \leq k \leq n$. Thus $\psi \in \Phi_{n-1}$ and $a\phi_n = \psi_n \in E_n$. Hence E_n is a linear space and polynomials from $\mathcal{P}({}^nX)$ are acting on E_n as linear functionals. Put $W_n = \mathcal{P}({}^nX)/(I_{n-1} \cap \mathcal{P}({}^nX))$. Then W_n is a Banach space of linear functionals on E_n and the functionals from W_n separate points of E_n . Let us define a norm on E_n , $\|\cdot\|_n$ as the supremum of values of a vector from E_n on the unit ball of W_n . Therefore $W'_n = (\mathcal{P}({}^nX)/(I_{n-1} \cap \mathcal{P}({}^nX)))' = \mathcal{P}({}^nX)' \cap I_{n-1}^\perp \supset E_n$. On the other hand, if $u \in \mathcal{P}({}^nX)' \cap I_{n-1}^\perp$, then by Lemma 1, $u = \pi_n(\phi)$ for some $\phi \in M_b$ and so $u \in E_n$. Thus $E_n = W'_n$.

For given $w \in E_n$ let us define $\delta^{(n)}(w)(Q) = \psi(Q)$ on homogeneous polynomials Q by formula (1) and extend it to the unique complex homomorphism on $H_b(X)$ as in Lemma 1. So $\delta^{(n)}$ maps E_n into M_b . For any $\phi \in M_b$ put $u_1 := \phi_1 \in X'' = E_1$, $u_2 := \phi_2 - \pi_2(\delta^{(1)}(u_1))$. It is clear that $u_2 \in E_2$. Suppose that we have defined $u_k \in E_k$, $k < n$. Set

$$(4) \quad u_n := \phi_n - \pi_n \left(\bigstar_{k=1}^{n-1} \delta^{(k)}(u_k) \right).$$

Let us show that $u_n \in E_n$. It is enough to check that for every $P \in \mathcal{P}({}^nX)$ such that $P = P_k P_m$, $\deg P_k = k \neq 0$, $\deg P_m = n \neq 0$ implies $u_n(P) = 0$. Note that for all n -homogeneous polynomials P_n ,

$$\phi_n - \pi_n \left(\bigstar_{k=1}^{n-1} \delta^{(k)}(u_k) \right) (P_n) = \phi_n - \bigstar_{k=1}^{n-1} \delta^{(k)}(u_k)(P_n).$$

From the multiplicativity of ϕ and Lemma 4 it follows that

$$\begin{aligned} u_n(P) &= \phi_n(P_k P_m) - \bigstar_{j=1}^{n-1} \delta^{(j)}(u_j)(P_k P_m) = \phi_k(P_k) \phi_m(P_m) \\ &\quad - \left(\bigstar_{j=1}^{n-1} \delta^{(j)}(u_j)(P_k) \right) \left(\bigstar_{j=1}^{n-1} \delta^{(j)}(u_j)(P_m) \right) \\ &= \left(u_k(P_k) + \bigstar_{j=1}^{k-1} \delta^{(j)}(u_j)(P_k) \right) \left(u_m(P_m) + \bigstar_{j=1}^{m-1} \delta^{(j)}(u_j)(P_m) \right) \\ &\quad - \left(\bigstar_{j=1}^k \delta^{(j)}(u_j)(P_k) \right) \left(\bigstar_{j=1}^m \delta^{(j)}(u_j)(P_m) \right) = 0. \end{aligned}$$

The last equality holds because by the induction assumption, $u_k \in E_k$, $u_m \in E_m$ and hence, by Lemma 3,

$$(5) \quad u_k(P_k) + \bigstar_{j=1}^{k-1} \delta^{(j)}(u_j)(P_k) = \bigstar_{j=1}^k \delta^{(j)}(u_j)(P_k)$$

and

$$u_m(P_m) + \bigstar_{j=1}^{m-1} \delta^{(j)}(u_j)(P_m) = \bigstar_{j=1}^m \delta^{(j)}(u_j)(P_m).$$

Let us consider the functional $\bigstar_{j=1}^{\infty} \delta^{(j)}(u_j)$. Since $u_k \in E_k$, by Lemma 3,

$$\bigstar_{j=1}^{\infty} \delta^{(j)}(u_j)(f) = f(0) + \sum_{n=1}^{\infty} \bigstar_{j=1}^n \delta^{(j)}(u_j)(f_n),$$

where $f = \sum f_n$ is the Taylor series expansion of f . Hence $\bigstar_{j=1}^{\infty} \delta^{(j)}(u_j)$ is well defined on $\mathcal{P}(X)$. On the other hand, applying (4) and (5) we obtain

$$\begin{aligned} \left(\phi - \bigstar_{j=1}^{\infty} \delta^{(j)}(u_j) \right) (P_n) &= \phi_n(P_n) - \bigstar_{j=1}^n \delta^{(j)}(u_j)(P_n) \\ &= u_n(P) + \bigstar_{j=1}^{n-1} \delta^{(j)}(u_j)(P_n) - \bigstar_{j=1}^n \delta^{(j)}(u_j)(P_n) = 0 \end{aligned}$$

for arbitrary $P_n \in \mathcal{P}(^n X)$. Thus $\phi = \bigstar_{j=1}^{\infty} \delta^{(j)}(u_j)$ on $\mathcal{P}(X)$. Hence $\phi = \bigstar_{j=1}^{\infty} \delta^{(j)}(u_j)$ on $H_b(X)$. $\square$

Let us denote by E^{∞} the space of all finite sequences $(u_1, \dots, u_m, 0, \dots)$, $u_k \in E_k$. According to Theorem 5, every finite sequence $\mathbf{u} = (u_1, \dots, u_m, 0, \dots)$ defines a character $\phi_{\mathbf{u}} = \bigstar_{k=1}^m \delta^{(k)}(u_k) \in M_b$. Thus $E^{\infty} \subset M_b$ and for every $\mathbf{u}, \mathbf{v} \in E^{\infty}$, $\phi_{\mathbf{u}+\mathbf{v}} \in M_b$. Moreover, from the density of polynomials in $H_b(X)$ it follows that E^{∞} is dense in M_b with respect to the Gelfand topology. So we have proved the following theorem.

Theorem 6. M_b contains the dense linear subspace of all finite subsequences $(u_1, \dots, u_m, 0, \dots)$, $u_k \in E^k$.

According to [3, 7], the operation of sum on X may be discontinuous with respect to the Gelfand topology, induced from M_b . Hence, in general, E^{∞} is not a topological vector space. Thus, the density of E^{∞} in M_b does not imply that M_b is a linear space.

We need to have some properties of the radius function, proved by Aron, Cole and Gamelin in [3].

Proposition 7. (1) For each $r > 0$, the set of $\phi \in M_b$ satisfying $R(\phi) \leq r$ coincides with the spectrum of $H_{uc}^{\infty}(rB)$. In particular, $M_{uc} = \{\phi \in M_b : R(\phi) \leq 1\}$.

(2) For every $\phi, \psi \in H_b(x)'$, $R(\phi * \psi) \leq R(\phi) + R(\psi)$.

Example 8. 1. Let X be c_0 or Tsirelson's space. Then $E_k = \{0\}$ for $k > 1$ [4, 22].

2. Let $X = \ell_1$ and $\phi \in H_b(\ell_1)'$, $\|\phi\| = 1$. According to [3], $\phi \in M_b(\ell_1)$ if and only if for every $m = 1, 2, \dots$ there exists a symmetric measure on $\beta(\mathbb{N}^m)$, ν_m and a constant $c > 0$ such that $\|\nu_m\| \leq c^m$ and for each $P_m \in \mathcal{P}(^m \ell_1)$,

$$\phi(P_m) = \int_{\beta(\mathbb{N}^m)} \hat{P}_m d\nu_m,$$

where $\widehat{P}_m$ is just P_m regarded as a vector from $\ell^\infty(\mathbb{N}^m)$. By Theorem 5, $\phi \in M_b(\ell_1)$ if and only if there is a sequence of symmetric measures (μ_m) which are orthogonal to $\beta(\mathbb{N}^j) \times \beta(\mathbb{N}^k) \subset \beta(\mathbb{N}^m)$, for $m > 1$, $k + j = m$, $k, j > 0$, and functionals

$$u_m(P_m) = \int_{\beta(\mathbb{N}^m)} \widehat{P}_m d\mu_m$$

determine ϕ by formula (3).

3. (Cf. [1, Example 3.1].) Let $X = \ell_p$ for some integer p , $1 < p < \infty$. For every n , put

$$v_n = \frac{1}{n^{1/p}}(e_1 + \cdots + e_n),$$

where (e_k) is the standard basis in ℓ_p . Since $\|v_n\| = 1$, $R(\delta(v_n)) = 1$ and so $\delta(v_n) \in M_{uc} \subset M_b$. By compactness of M_{uc} there is an accumulation point $\phi \in M_{uc}$ of the sequence $(\delta(v_n))$. If $0 < k < p$, then by Pitt's Theorem (see [16, Theorem 5.1]) every polynomial $P \in \mathcal{P}({}^k\ell_p)$ is weakly continuous on bounded sets. Since v_n is weakly null in ℓ_p , $\phi(P) = 0$. On the other hand, $\phi(Q) = 1$ for the polynomial $Q(x) = \sum_{n=1}^\infty x_n^p$. Thus $\phi \in \Phi_{p-1}$ and $\phi \neq 0$. In other words, if $\phi = \bigast_{k=1}^\infty \delta^{(k)}(u_k)$ is the representation of ϕ by Theorem 5, then $u_k = 0$ for $k < p$ and $u_p \neq 0$.

Proposition 9. *Let $\phi \in M_b$ and let $\phi = \bigast_{k=1}^\infty \delta^{(k)}(u_k)$, $u_k \in E_k$, be its representation. Then*

$$\limsup_{k \rightarrow \infty} \|u_k\|_k^{1/k} \leq R(\phi) \leq \sum_{k=1}^\infty \|u_k\|_k^{1/k}.$$

Proof. The first inequality holds because $\|u_k\|_k \leq \|\phi_k\|$ and by the definition of the radius function. The second inequality follows from Proposition 7 and the following calculation:

$$R(\delta^{(k)}(u_k)) = \limsup_{m \rightarrow \infty} \|\pi_{km}(\delta^{(k)}(u_k))\|^{1/km} = \|\delta^{(k)}(u_k)\|^{m/km} = \|u_k\|^{1/k}.$$

□

Let F be an analytic map from ℓ_1 to ℓ_1 defined by

$$F(x) = F\left(\sum_{n=1}^\infty x_n e_n\right) = \sum_{n=1}^\infty x_n^n e_n.$$

We denote by $F(\ell_1)$ the range of F and by $F(B_{\ell_1})$ the range of F restricted to the unit ball $B_{\ell_1} \subset \ell_1$.

Given a sequence of Banach spaces $(E_n, \|\cdot\|_n)_{n=1}^\infty$ and $0 < \rho \leq \infty$ the Köthe sequence space $\lambda^1(K_\rho; (E_n))$ (where $K_\rho = \{(r^n)_{n=1}^\infty : 0 < r < \rho\}$) is the Fréchet space

$$\left\{ (x_n)_{n=1}^\infty \in \prod_{n=1}^\infty E_n : p_r((x_n)_{n=1}^\infty) = \sum_{n=1}^\infty \|x_n\|_n r^n < \infty \forall r, 0 < r < \rho \right\},$$

endowed with the topology given by the seminorms $\{p_r\}_{0 < r < \rho}$. By Cauchy-Hadamard's formula,

$$\lambda^1(K_\rho; (E_n)) = \left\{ (x_n)_{n=1}^\infty \in \prod_{n=1}^\infty E_n : \limsup_{n \rightarrow \infty} \|x_n\|_n^{1/n} \leq \frac{1}{\rho} \right\}.$$

- Corollary 10.** (1) M_b contains every sequence $\mathbf{u} = (u_k)_{k=1}^\infty$, $u_k \in E_k$, such that the sequence $(\|u_k\|)_{k=1}^\infty$ is in $F(\ell_1)$.
 (2) M_{uc} contains every sequence $\mathbf{u} = (u_k)_{k=1}^\infty$, $u_k \in E_k$, such that the sequence $(\|u_k\|)_{k=1}^\infty$ is in $F(B_{\ell_1})$.
 (3) Every complex homomorphism $\phi \in M_b$ is contained in a Köthe sequence space $\lambda^1(K_\rho; (E_n))$ for $\rho = 1/R(\phi)$.
 (4) M_{uc} is contained in $\lambda^1(K_1; (E_n))$.

Proof. Since $F^{-1}((\|u_k\|)_{k=1}^\infty) \in \ell_1$, $\sum_{k=1}^\infty \|u_k\|^{1/k} \leq \infty$ and by Proposition 9, $R(\phi_u) < \infty$. Thus $\phi_u \in M_b$. Moreover, if $F^{-1}((\|u_k\|)_{k=1}^\infty) \in F(B_{\ell_1})$, then $R(\phi_u) \leq 1$ and $\phi_u \in M_{uc}$.

Suppose that $\phi_u \in M_b$ for some $\mathbf{u} = (u_k)_{k=1}^\infty$. Then $R(\phi_u) < \infty$ and by Proposition 9, $\limsup_{k \rightarrow \infty} \|u_k\|^{1/k} \leq R(\phi_u)$. Hence $\phi_u \in \lambda^1(K_{1/R(\phi_u)}; (E_n))$. In particular, if $R(\phi_u) \leq 1$, then $\phi_u \in \lambda^1(K_1; (E_n))$. $\square$

Dixon [14] has given an example of an algebra of polynomials of infinitely many variables which admits discontinuous scalar-valued homomorphisms. In [15] a construction is given of a discontinuous scalar-valued homomorphism of an algebra of polynomials on an arbitrary infinite-dimensional Banach space. The next corollary shows that the restriction of a discontinuous complex homomorphism on $A_n(X) \cap \mathcal{P}(X)$ can be continuous for every n . Note that the problem of existence of discontinuous complex homomorphisms on $H_b(X)$ for an infinite-dimensional Banach space X is still open and equivalent to the famous Michael Problem [20], [21, p. 240].

Corollary 11. *If the sequence of algebras $A_n(X)$ does not stabilize, then there is a discontinuous complex homomorphism ζ on $\mathcal{P}(X)$ such that the restriction of ζ on $A_n(X) \cap \mathcal{P}(X)$ is a continuous complex homomorphism for every n .*

Proof. By Corollary 2 and Theorem 5 there exists an infinite sequence $(u_k)_{k=1}^\infty$, $u_k \in E_k$, $u_k \neq 0$. Since each E_k is a linear space, we can choose u_k such that $\limsup_{k \rightarrow \infty} \|u_k\|_k^{1/k} = \infty$. Put $\zeta = \bigstar_{k=1}^\infty \delta^{(k)}(u_k)$. Evidently,

$$\zeta(f) = \bigstar_{k=1}^n \delta^{(k)}(u_k)(f)$$

for every $f \in A_n(X)$. So ζ is well defined and continuous on $A_n(X) \cap \mathcal{P}(X)$. If ζ is continuous on $\mathcal{P}(X)$, then it can be extended to a continuous complex homomorphism on $H_b(X)$. But this contradicts Proposition 9. $\square$

In [11] Deghoul, using Borsuk's theorem, shows that there is an "exceptional" character ϕ on $H_b(\ell_2)$ such that ϕ vanishes on odd degree homogeneous polynomials and is different from the evaluation at 0. The next proposition delivers the existence of exceptional characters on $H_b(X)$ for a large number of X .

Proposition 12. *Suppose that $A_m(X) \neq A_k(X)$ for some $m > 1$ and all $k < m$. Then there exists a nontrivial character $\psi_0 \in M_b$ such that $\psi_0(P) = 0$ for every homogeneous polynomial P , $\deg P \neq nm$, $n = 1, \dots, \infty$.*

Proof. By Corollary 2 there exists a nontrivial character $\psi \in M_b$ which vanishes on all k -homogeneous polynomials for $k < m$. From Theorem 5 it follows that E_m contains a nonzero vector u_m . Put $\psi_0 = \delta^{(m)}(u_m)$. Then ψ_0 vanishes on all homogeneous polynomials excepting nm -homogeneous polynomials, $n = 1, 2, \dots$ $\square$

ACKNOWLEDGEMENTS

The author is greatly indebted to the referee for his attention to the paper and numerous helpful comments. The author wishes to thank the University of Saskatchewan, where the paper was written, for the invitation and financial support.

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