GAUGEABILITY FOR FEYNMAN-KAC FUNCTIONALS WITH APPLICATIONS TO SYMMETRIC α -STABLE PROCESSES

MASAYOSHI TAKEDA

(Communicated by Richard C. Bradley)

ABSTRACT. For symmetric α -stable processes, an analytic criterion for a measure being gaugeable was obtained by Z.-Q. Chen (2002), M. Takeda (2002) and M. Takeda and T. Uemura (2004). Applying it, we consider the ultracontractivity of Feynman-Kac semigroups and expectations of the number of branches hitting closed sets in branching symmetric α -stable processes.

1. INTRODUCTION

Let $\mathbb{M}^\alpha = (\mathbb{P}_x, X_t)$ ($0 < \alpha < 2$) be a symmetric α -stable process on $\mathbb{R}^d$. Let μ be a smooth measure and A_t^μ the positive continuous additive functional (PCAF) in the Revuz correspondence to μ . Then the measure μ is said to be *gaugeable* on an open set $D \subset \mathbb{R}^d$ if

$$(1.1) \quad \sup_{x \in D} \mathbb{E}_x [\exp(A_{\tau_D}^\mu)] < \infty,$$

where τ_D is the first exit time from D .

For a Brownian motion, Zhao [19] introduced a class of Green-tight measures and Chen [3] generalized the notion of Green-tightness for more general transient Markov processes. Let $\mathbb{M}^D$ be the absorbing process killed upon leaving D and assume that $\mathbb{M}^D$ is transient. Denote by $\mathcal{S}_\infty^D$ the extended class associated with $\mathbb{M}^D$ (see Definition 2.1 below). Chen [3] and Takeda [16] established an analytic condition for $\mu \in \mathcal{S}_\infty^D$ being gaugeable; define

$$\lambda(\mu; D) = \inf \left\{ \mathcal{E}^{(\alpha)}(u, u) : u \in C_0^\infty(D), \int_D u^2(x) \mu(dx) = 1 \right\}.$$

Then the gaugeability of μ is equivalent to that of $\lambda(\mu; D) > 1$, which is also equivalent to the subcriticality of Schrödinger operators. Applying these facts, we showed in [17] the differentiability of spectral functions. The objective of this paper is to give two other applications.

Received by the editors September 17, 2004 and, in revised form, March 30, 2005.

2000 *Mathematics Subject Classification*. Primary 60J45, 60J40, 35J10.

Key words and phrases. Symmetric stable process, gaugeability, subcriticality, ultracontractivity, branching process.

The author was supported in part by Grant-in-Aid for Scientific Research (No.15530229 (C)(2)), Japan Society for the Promotion of Science.

©2006 American Mathematical Society
 Reverts to public domain 28 years from publication

The first is relevant with the ultracontractivity of Schrödinger semigroups; let $p_t^\mu f(x) = \mathbb{E}_x[\exp(A_t^\mu)f(X_t)]$ and denote by $\|p_t^\mu\|_{1,\infty}$ the operator norm of p_t^μ from $L^1(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$. Then we have

Theorem 1.1. *Suppose that $d > \alpha$. Let $\mu \in \mathcal{S}_\infty^{\mathbb{R}^d}$ with $\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x-y|^{\alpha-d} d\mu(x) d\mu(y) < \infty$. Then*

$$(1.2) \quad \lambda(\mu; \mathbb{R}^d) > 1 \iff \|p_t^\mu\|_{1,\infty} \leq \frac{c}{t^{d/\alpha}}, \quad t > 0.$$

Theorem B.1.1 in [13] says that if 2μ is gaugeable on $\mathbb{R}^d$, the ultracontractivity in the right-hand side of (1.2) holds. In the proof, he used the Schwartz inequality in the Feynman-Kac formula and the duality arguments, which is the reason why the gaugeability of 2μ is required. Our argument is different; a class of Girsanov transforms treated in [5] plays a crucial role.

In the second application, motivated by [10], we consider a branching symmetric α -stable process; let $\mathbb{B}^\alpha = (\bar{X}_t, \bar{\mathbb{P}}_x)$ be the branching α -symmetric stable process with the branching rate k , a smooth measure of $\mathbb{M}^{(\alpha)}$, and the branching mechanism $\{p_n(x)\}_{n \geq 2}$, $\sum_{n \geq 2} p_n(x) = 1$. Set $Q(x) = \sum_{n \geq 2} n p_n(x)$, $\mu(dx) = (Q(x) - 1)k(dx)$. Denote by Cap the 0-capacity defined by the Dirichlet form generated by $\mathbb{M}^{(\alpha)}$. Then we have

Theorem 1.2. *Assume that $\sup_{x \in \mathbb{R}^d} Q(x) < \infty$. Then for a closed set K with $\text{Cap}(K) > 0$ and $\mu \in \mathcal{S}_\infty^{\mathbb{R}^d \setminus K}$,*

$$(1.3) \quad \lambda(\mu; \mathbb{R}^d \setminus K) > 1 \iff \bar{\mathbb{E}}_x[N_K] < \infty.$$

Here N_K is the number of branches of $\mathbb{B}^\alpha$ ever hitting K .

Theorem 1.2 is an extension of [10, Theorem 4.1(iv)] to branching processes with jumps. For the proof of Theorem 1.2, we show that for $\mu \in \mathcal{S}_\infty^D$ with $\text{Cap}(\mathbb{R}^d \setminus D) > 0$

$$(1.4) \quad \sup_{x \in D} \mathbb{E}_x[\exp(A_{\tau_D}^\mu)] < \infty \iff \sup_{x \in D} \mathbb{E}_x[\exp(A_{\tau_D}^\mu); \tau_D < \infty] < \infty.$$

In [6], the gaugeability of μ on D is defined by the right-hand side of (1.4). The equation (1.4) tells us that for a measure in $\mathcal{S}_\infty^D$, two definitions of the gaugeability are equivalent.

2. NOTATIONS AND SOME FACTS

Let $\mathbb{M} = (\Omega, \mathcal{M}, \mathcal{M}_t, \theta_t, \mathbb{P}_x, X_t)$ be a symmetric α -stable process on $\mathbb{R}^d$. Here $\{\mathcal{M}_t\}_{t \geq 0}$ is the minimal (augmented) admissible filtration and θ_t , $t \geq 0$, are the shift operators satisfying $X_s(\theta_t) = X_{s+t}$ identically for $s, t \geq 0$. The Dirichlet form generated by $\mathbb{M}^\alpha$ is given by

$$\begin{aligned} \mathcal{E}^{(\alpha)}(u, v) &= K \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus d} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+\alpha}} dx dy, \\ \mathcal{D}(\mathcal{E}^{(\alpha)}) &= \left\{ u \in L^2(\mathbb{R}^d) : \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus d} \frac{(u(x) - u(y))^2}{|x - y|^{d+\alpha}} dx dy < \infty \right\} \end{aligned}$$

($K = \alpha 2^{\alpha-3} \pi^{-\frac{d+2}{2}} \sin(\frac{\alpha\pi}{2}) \Gamma(\frac{d+\alpha}{2}) \Gamma(\frac{\alpha}{2})$). Every function u in $\mathcal{D}(\mathcal{E}^{(\alpha)})$ admits a quasi-continuous version $\tilde{u}$ ([9, Theorem 2.1.3]). In the sequel we always assume that every function $u \in \mathcal{D}(\mathcal{E}^{(\alpha)})$ is represented by its quasi-continuous version.

Let $D \subset \mathbb{R}^d$ be an open set. Assume that the absorbing process $\mathbb{M}^D$ on D is transient and let $G_D(x, y)$ be the Green function of $\mathbb{M}^D$. Note that $\mathbb{M}^D$ is irreducible due to the strictly positivity of the Levy measure of $\mathbb{M}$. Following [3], we make the following definition.

Definition 2.1. A positive Radon measure μ on $\mathbb{R}^d$ is said to be in the class $\mu \in \mathcal{S}_\infty^D$, if for any $\epsilon > 0$ there exists a compact set $K \subset D$ and $\delta > 0$ such that

$$(2.1) \quad \sup_{(x,z) \in D \times D \setminus d} \int_{K^c} \frac{G_D(x, y) G_D(y, z)}{G_D(x, z)} \mu(dy) \leq \epsilon,$$

and for all measurable sets $B \subset K$ with $\mu(B) < \delta$,

$$(2.2) \quad \sup_{(x,z) \in D \times D \setminus d} \int_B \frac{G_D(x, y) G_D(y, z)}{G_D(x, z)} \mu(dy) \leq \epsilon.$$

When $D = \mathbb{R}^d$, we remove D in the notations; for example, we simply denote $\mathcal{S}_\infty$ for $\mathcal{S}_\infty^{\mathbb{R}^d}$. By the *3G-inequality*, a measure μ is in $\mathcal{S}_\infty$ if and only if it is Green-tight in the sense of Zhao [19] (cf. [3]). For $\mu \in \mathcal{S}_\infty^D$, let A_t^μ be the positive continuous additive functional of $\mathbb{M}$ in the Revus correspondence to the measures μ . It is known in [3, Proposition 2.2] that for $\mu \in \mathcal{S}_\infty^D$

$$(2.3) \quad \sup_{x \in D} \mathbb{E}_x[A_{\tau_D}^\mu] = \sup_{x \in D} \int_D G_D(x, y) \mu(dy) < \infty.$$

For a measure μ in $\mathcal{S}_\infty^D$, define

$$(2.4) \quad \lambda(\mu; D) = \inf \left\{ \mathcal{E}^{(\alpha)}(u, u) : u \in C_0^\infty(D), \int_D u^2(x) \mu(dx) = 1 \right\}.$$

By Lemma 3.1 in [15] and Theorem 4.1 below, we see that $\lambda(\mu; D)$ is the principal eigenvalue of the time changed process of $\mathbb{M}^D$ by $A_{\tau_D \wedge t}^\mu$. We abbreviate $\lambda(\mu; \mathbb{R}^d)$ as $\lambda(\mu)$.

Let $p_t^{\mu, D}(x, y)$ be the integral kernel of the Feynman-Kac semigroup,

$$\mathbb{E}_x[\exp(A_t^\mu) f(X_t); t < \tau_D] = \int_D p_t^{\mu, D}(x, y) f(y) dy,$$

and $G^{\mu, D}(x, y)$ its Green function, $G^{\mu, D}(x, y) = \int_0^\infty p_t^{\mu, D}(x, y) dt$. We then have

Theorem 2.2 ([3], [16]). *Let $\mu \in \mathcal{S}_\infty^D$. Then the following conditions are equivalent:*

- (i) $\sup_{x \in D} \mathbb{E}_x[e^{A_{\tau_D}^\mu}] < \infty$;
- (ii) $G^{\mu, D}(x, y) < \infty$ for $x, y \in D, x \neq y$;
- (iii) $\lambda(\mu; D) > 1$.

Remark 2.3. The equivalence between (i) and (iii) holds for any $\mu \in \mathcal{S}_\infty$ (see [16, Theorem 2.4]).

3. PROOF OF THEOREM 1.1

Proof of Theorem 1.1, the ($\Leftarrow$) part. Let $p(t, x, y)$ be the transition probability density. Then it satisfies the upper estimate,

$$p(t, x, y) \leq C \frac{1}{t^{d/\alpha}} \left(\frac{1}{1 + \frac{|x-y|}{t^{1/\alpha}}} \right)^{d+\alpha},$$

where C is a positive constant (see [2]). Thus by the same argument as in [1, Theorem 8.1], the integral kernel $p^\mu(t, x, y)$ of P_t^μ satisfies, for any $\lambda > 1$,

$$\begin{aligned} p^\mu(t, x, y) &\leq C' \frac{1}{t^{d/\alpha}} \left(\frac{1}{1 + \frac{|x-y|}{t^{1/\alpha}}} \right)^{\frac{d+\alpha}{\lambda}} \\ &\leq C' \frac{t^{(\frac{d}{\alpha}+1)\frac{1}{\lambda} - \frac{d}{\alpha}}}{|x-y|^{\frac{d+\alpha}{\lambda}}}, \quad 0 < t \leq 1, \end{aligned}$$

where C' is a positive constant depending on λ . In particular, for any x, y with $x \neq y$, $\sup_{0 < t \leq 1} p^\mu(t, x, y) < \infty$. Moreover, the assumption implies that $p^\mu(t, x, y) \leq \frac{c}{t^{d/\alpha}}$. Therefore we have

$$G^\mu(x, y) \leq \int_0^1 p^\mu(t, x, y) dt + c \int_1^\infty \frac{1}{t^{d/\alpha}} dt < \infty.$$

Now it follows from Theorem 2.2 that $\lambda(\mu) > 1$. $\square$

To prove the converse, we need some results. Let $\mathcal{D}_e(\mathcal{E}^{(\alpha)})$ be the *extended Dirichlet space*, that is, the family of measurable function u on $\mathbb{R}^d$ such that $|u| < \infty$ a.e. and there exists an $\mathcal{E}^{(\alpha)}$ -Cauchy sequence $\{u_n\}$ of functions in $\mathcal{D}(\mathcal{E}^{(\alpha)})$ such that $\lim_{n \rightarrow \infty} u_n = u$ a.e. (see [9]). We denote $G\mu(x) = \int_{\mathbb{R}^d} G(x, y) d\mu(y)$.

Lemma 3.1. *If $\mu \in \mathcal{S}_\infty$ satisfies $\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(x, y) d\mu(x) d\mu(y) < \infty$, then $G\mu$ belongs to $\mathcal{D}_e(\mathcal{E}^{(\alpha)})$.*

Proof. First note that for $\mu \in \mathcal{S}_\infty$

$$(3.1) \quad \int_{\mathbb{R}^d} u^2 d\mu \leq \|G\mu\|_\infty \mathcal{E}^{(\alpha)}(u, u), \quad u \in \mathcal{D}_e(\mathcal{E}^{(\alpha)})$$

(cf. [14]). Then by applying (3.1) to $\mu_K(\cdot) = \mu(K \cap \cdot)$, we have

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi d\mu_K &\leq (\mu(K))^{1/2} \left(\int_{\mathbb{R}^d} \varphi^2 d\mu_K \right)^{1/2} \\ &\leq (\mu(K))^{1/2} \|G\mu_K\|_\infty^{1/2} \mathcal{E}^{(\alpha)}(\varphi, \varphi)^{1/2}. \end{aligned}$$

Hence equation (2.3) says that the measure μ_K is of finite energy integral, and thus

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi d\mu_K &\leq \mathcal{E}^{(\alpha)}(G\mu_K, G\mu_K)^{1/2} \mathcal{E}^{(\alpha)}(\varphi, \varphi)^{1/2} \\ &\leq \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(x, y) d\mu_K(x) d\mu_K(y) \right)^{1/2} \mathcal{E}^{(\alpha)}(\varphi, \varphi)^{1/2}. \end{aligned}$$

By letting K increase to $\mathbb{R}^d$, we find that μ is of finite energy integral, and thus $G\mu$ is in $\mathcal{D}_e(\mathcal{E}^{(\alpha)})$. $\square$

Assume that $\lambda(\mu) > 1$ and set

$$h(x) = \mathbb{E}_x[e^{A_\infty^\mu}].$$

Then by Theorem 2.2, $1 \leq h(x) \leq \sup_x \mathbb{E}_x[e^{A_\infty^\mu}] < \infty$.

Lemma 3.2. *Assume that $\lambda(\mu) > 1$. Then it holds that*

$$h(x) = G(h\mu)(x) + 1.$$

Proof. Define $M_t := \mathbb{E}[\exp(A_\infty^\mu) | \mathcal{M}_t]$. Then by the Markov property

$$\begin{aligned} h(X_t) &= \mathbb{E}_{X_t}[\exp(A_\infty^\mu)] = \mathbb{E}_x[\exp(A_\infty^\mu(\theta_t)) | \mathcal{M}_t] \\ &= \mathbb{E}_x[\exp(A_\infty^\mu - A_t^\mu) | \mathcal{M}_t] = \exp(-A_t^\mu) M_t, \end{aligned}$$

and thus

$$\begin{aligned} (3.2) \quad \mathbb{E}_x \left[\int_0^t h(X_s) dA_s^\mu \right] &= \mathbb{E}_x \left[\int_0^t \exp(-A_s^\mu) M_s dA_s^\mu \right] \\ &= \mathbb{E}_x[M_0] - \mathbb{E}_x[\exp(-A_t^\mu) M_t] - \mathbb{E}_x \left[\int_0^t \exp(-A_s^\mu) dM_s \right] \\ &= h(x) - \mathbb{E}_x[h(X_t)]. \end{aligned}$$

Noting that

$$\lim_{t \rightarrow \infty} h(X_t) = \lim_{t \rightarrow \infty} \exp(-A_t^\mu) M_t = \exp(-A_\infty^\mu) \exp(A_\infty^\mu) = 1,$$

we have the lemma by letting t to ∞ in (3.2). $\square$

Suppose that $\mu \in \mathcal{S}_\infty$ satisfies $\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(x, y) d\mu(x) d\mu(y) < \infty$. Then it follows from Lemma 3.1 that $G(h\mu) \in \mathcal{D}_e(\mathcal{E}^{(\alpha)})$. Thus we have the Fukushima decomposition for $G(h\mu)$ ([9, Theorem 5.2.2]):

$$G(h\mu)(X_t) - G(h\mu)(X_0) = M_t^{[G(h\mu)]} + N_t^{[G(h\mu)]}.$$

Since the left-hand side of the above equals $h(X_t) - h(X_0)$ by Lemma 3.3, we may denote $M_t^{[h]} = M_t^{[G(h\mu)]}$. Furthermore, it follows from [9, Lemma 5.4.1] that $N_t^{[G(h\mu)]} = -\int_0^t h(X_s) dA_s^\mu$. Therefore we have

$$(3.3) \quad h(X_t) - h(X_0) = M_t^{[h]} - \int_0^t h(X_s) dA_s^\mu.$$

Define a martingale by

$$M_t = \int_0^t \frac{1}{h(X_{s-})} dM_s^h$$

and denote by L_t the unique solution of the Doleans-Dade equation:

$$(3.4) \quad Z_t = 1 + \int_0^t Z_{s-} dM_s.$$

Then we see from the Doleans-Dade formula that L_t is expressed by

$$\begin{aligned} L_t &= \exp \left(M_t - \frac{1}{2} \langle M^c \rangle_t \right) \prod_{0 < s \leq t} (1 + \Delta M_s) \exp(-\Delta M_s) \\ &= \exp \left(M_t - \frac{1}{2} \langle M^c \rangle_t \right) \prod_{0 < s \leq t} \frac{h(X_s)}{h(X_{s-})} \exp \left(1 - \frac{h(X_s)}{h(X_{s-})} \right). \end{aligned}$$

Here M_t^c is the continuous part of M_t and $\Delta M_s = M_s - M_{s-}$. By Itô's formula applied to the semi-martingale $h(X_t)$ with the function $\log x$, we see that L_t has the following expression:

$$(3.5) \quad L_t = \frac{h(X_t)}{h(X_0)} \exp(A_t^\mu).$$

Denote by $\mathbb{M}^L = (\Omega, \mathbb{P}_x^L, X_t)$ the transformed process of $\mathbb{M}^\alpha$ by L_t , $d\mathbb{P}_x^L(\omega) = L_t(\omega) d\mathbb{P}_x(\omega)$. Lemma 3.1 says that $u := \log(G(h\mu) + 1) \in \mathcal{D}_e(\mathcal{E}^{(\alpha)})$ and $h = \exp(u)$.

Thus the transform by L_t belongs to the class of Girsanov transforms considered in [5]. In particular, the Dirichlet form generated by $\mathbb{M}^L$ is identified as follows.

Theorem 3.3 ([5]). *The transformed process $\mathbb{M}^L$ is an $h^2 dx$ -symmetric Hunt process and its Dirichlet form $(\mathcal{E}^h, \mathcal{F}^h)$ on $L^2(\mathbb{R}^d, h^2 dx)$ is written as*

$$\begin{cases} \mathcal{E}^h(u, v) = K \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus d} \frac{(u(x) - u(y))(v(x) - v(y))h(x)h(y)}{|x - y|^{d+\alpha}} dx dy, \\ \mathcal{D}(\mathcal{E}^h) = \mathcal{D}(\mathcal{E}^{(\alpha)}). \end{cases}$$

Proof of Theorem 1.1, the $(\implies)$ part. Noting that $(\mathcal{E}^h, \mathcal{D}(\mathcal{E}^h))$ is equivalent to $(\mathcal{E}^{(\alpha)}, \mathcal{D}(\mathcal{E}^{(\alpha)}))$, we have the Sobolev inequality

$$\left(\int_{\mathbb{R}^d} |f|^{\frac{2d}{d-\alpha}} h^2 dx \right)^{\frac{d-\alpha}{2d}} \leq c \sqrt{\mathcal{E}^h(f, f)}, \quad f \in \mathcal{F}^h,$$

and so we have by [7, Theorem 2.4.2] that

$$\|p_t^h f\|_\infty \leq \frac{c}{t^{\frac{d}{\alpha}}} \int_{\mathbb{R}^d} |f| h^2 dx, \quad t > 0.$$

Here p_t^h is the semigroup of $\mathbb{M}^L$. We see from (3.5) that

$$p_t^h f(x) = \frac{1}{h(x)} p_t^\mu(hf)(x).$$

Hence

$$\begin{aligned} \|p_t^\mu f\|_\infty &= \|h p_t^h \left(\frac{|f|}{h} \right)\|_\infty \leq \frac{c \|h\|_\infty}{t^{\frac{d}{\alpha}}} \int_{\mathbb{R}^d} \frac{|f|}{h} h^2 dx \\ &\leq \frac{c \|h\|_\infty^2}{t^{\frac{d}{\alpha}}} \int_{\mathbb{R}^d} |f| dx, \end{aligned}$$

which implies

$$\|p_t^\mu\|_{1,\infty} \leq \frac{c'}{t^{\frac{d}{\alpha}}}, \quad t > 0.$$

□

Example 3.4. Let σ_r be the surface measure of the sphere $\partial B_r = \{|x| = r\}$. Since the symmetric α -stable process hits the sphere ∂B_r if $1 < \alpha$, the measure σ_r is in $\mathcal{S}_\infty$. Combining Theorem 1.1 with [18, Example 4.1], we find that

$$\begin{aligned} \|p_t^{\sigma_r}\|_{1,\infty} \leq \frac{c}{t^{\frac{d}{\alpha}}}, \quad t > 0 &\iff \lambda(\sigma_r) > 1 \\ &\iff \left\{ \frac{\sqrt{\pi} \Gamma\left(\frac{d+\alpha}{2} - 1\right) \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{\alpha-1}{2}\right) \Gamma\left(\frac{d-\alpha}{2}\right)} \right\}^{\frac{1}{\alpha-1}} > r. \end{aligned}$$

Here $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$.

4. PROOF OF THEOREM 1.2

Let K be a closed set with $\text{Cap}(K) > 0$ and denote $D = \mathbb{R}^d \setminus K$. Let $(\mathcal{E}_D^{(\alpha)}, \mathcal{D}(\mathcal{E}_D^{(\alpha)}))$ be the Dirichlet form generated by the absorbing process of $\mathbb{M}^{(\alpha)}$ on D (cf. Section 4.4 in [9]). Let $\mathcal{D}_e(\mathcal{E}_D^{(\alpha)})$ be the extended Dirichlet space of $(\mathcal{E}_D^{(\alpha)}, \mathcal{D}(\mathcal{E}_D^{(\alpha)}))$. Because of the remark just below Definition 3.1 in [3] and Proposition 2.3 in [3], we have the next theorem in the same manner as in [17, Theorem 3.3].

Theorem 4.1. *Let $\mu \in \mathcal{S}_\infty^D$. Then the embedding of $(\mathcal{E}_D^{(\alpha)}, \mathcal{D}_e(\mathcal{E}_D^{(\alpha)}))$ to $L^2(D; \mu)$ is compact.*

Moreover, by the same argument as in [17, Section 4], we have

Lemma 4.2. *If $\lambda(\mu; D) = 1$, then there exists a positive continuous bounded function u_0 on D such that*

$$(4.1) \quad u_0(x) = \mathbb{E}_x \left[\int_0^{\tau_D} u_0(X_t) dA_t^\mu \right]$$

and so equation (4.1) says that the function u_0 is $p_t^{\mu, D}$ -excessive, $p_t^{\mu, D} u_0(x) \leq u_0(x)$ for every $t > 0$ and $x \in D$.

Set

$$h_D(x) = \mathbb{E}_x[\exp(A_{\tau_D}^\mu); \tau_D < \infty].$$

Lemma 4.3. *If $\sup_{x \in D} h_D(x) < \infty$, then h_D is a $p_t^{\mu, D}$ -excessive function. Moreover, it satisfies*

$$(4.2) \quad h_D(x) = \mathbb{P}_x[\tau_D < \infty] + \mathbb{E}_x \left[\int_0^{\tau_D} h_D(X_t) dA_t^\mu \right].$$

Proof. The $p_t^{\mu, D}$ -excessiveness follows from the Markov property

$$\begin{aligned} p_t^{\mu, D} h_D(x) &= \mathbb{E}_x[\exp(A_t^\mu) \mathbb{E}_{X_t}[\exp(A_{\tau_D}^\mu); \tau_D < \infty]; t < \tau_D] \\ &= \mathbb{E}_x[\mathbb{E}_x[\exp(A_t^\mu + A_{\tau_D}^\mu(\theta_t)); \tau_D(\theta_t) < \infty, t < \tau_D | \mathcal{M}_t]] \\ &= \mathbb{E}_x[\exp(A_{\tau_D}^\mu); t < \tau_D < \infty] \leq h_D(x). \end{aligned}$$

For the proof of (4.2), note that

$$\begin{aligned} (4.3) \quad \mathbb{E}_x \left[\int_0^{\tau_D} h_D(X_t) dA_t^\mu \right] &= \mathbb{E}_x \left[\int_0^{\tau_D} \mathbb{E}_{X_t}[\exp(A_{\tau_D}^\mu); \tau_D < \infty] dA_t^\mu \right] \\ &= \mathbb{E}_x \left[\int_0^{\tau_D} \mathbb{E}_x[\exp(A_{\tau_D}^\mu(\theta_t)); \tau_D(\theta_t) < \infty | \mathcal{M}_t] dA_t^\mu \right] \\ &= \mathbb{E}_x \left[\int_0^{\tau_D} \exp(-A_t^\mu) \mathbb{E}_x[\exp(A_{\tau_D}^\mu); t < \tau_D < \infty | \mathcal{M}_t] dA_t^\mu \right]. \end{aligned}$$

Set

$$\begin{aligned} X_t &= \exp(-A_t^\mu) \exp(A_{\tau_D}^\mu) I_{\{t < \tau_D < \infty\}}, \\ Y_t &= \exp(-A_t^\mu) \mathbb{E}_x[\exp(A_{\tau_D}^\mu) | \mathcal{M}_t] I_{\{t < \tau_D < \infty\}}. \end{aligned}$$

Then applying Exercise (1.13) of Chap. V in [11], we have

$$\mathbb{E}_x \left[\int_0^\infty X_t dA_t^\mu \right] = \mathbb{E}_x \left[\int_0^\infty Y_t dA_t^\mu \right].$$

Namely, the right-hand side of (4.3) equals

$$\begin{aligned} & \mathbb{E}_x \left[\exp(A_{\tau_D}^\mu) \int_0^{\tau_D} \exp(-A_t^\mu) dA_t^\mu \right] \\ &= \mathbb{E}_x [\exp(A_{\tau_D}^\mu); \tau_D < \infty] - \mathbb{P}_x [\tau_D < \infty], \end{aligned}$$

which implies (4.2). $\square$

Lemma 4.4. *If $\sup_{x \in D} h_D(x) < \infty$, then $\lambda(\mu; D) \geq 1$.*

Proof. Suppose that $\lambda(\mu; D) < 1$. Then there is $0 < \theta < 1$ such that $\lambda(\theta\mu; D) = 1$. It follows from Lemma 4.3 that h -transformation $p_t(x, y) := \frac{1}{h_D(x)} p_t^{\mu, D}(x, y) h_D(y)$ is a transition probability density. Denote by $\mathbb{M}$ the Markov processes generated by $p_t(x, y)$. The kernel $p_t^\theta(x, y) := \frac{1}{h_D(x)} p_t^{\theta\mu, D}(x, y) h_D(y)$ is also a transition probability density. In fact, $p_t^\theta(x, y)$ is the transition density of the subprocess of $\mathbb{M}$ by the multiplicative functional $\exp(-(1-\theta)A_t^\mu)$;

$$\frac{1}{h_D(x)} p_t^{\theta\mu, D}(h_D f)(x) = \frac{1}{h_D(x)} \mathbb{E}_x [\exp(-(1-\theta)A_t^\mu) \exp(A_t^\mu) h_D(X_t) f(X_t); t < \tau_D].$$

Hence $p_t^\theta(x, y)$ defines a transient Markov process, which is contradictory to Theorem 2.2. $\square$

Lemma 4.5. *If $\sup_{x \in D} h_D(x) < \infty$, then $\lambda(\mu; D) > 1$.*

Proof. Suppose that $\lambda(\mu; D) = 1$. Let u_0 be the function in Lemma 4.2. Then Theorem 2.2 says that $\frac{1}{u_0(x)} p_t^{\mu, D}(x, y) u_0(y)$ is a transition probability density generating a recurrent Markov process. Moreover, since

$$\frac{1}{u_0(x)} \mathbb{E}_x \left[\exp(A_t^\mu) u_0(X_t) \left(\frac{h_D}{u_0} \right) (X_t); t < \tau_D \right] \leq \frac{h_D}{u_0}(x),$$

that is, h_D/u_0 is excessive with respect to $\frac{1}{u_0(x)} p_t^{\mu, D}(x, y) u_0(y)$, there exists a positive constant c such that $h_D = cu_0$. However, this is impossible by Lemma 4.2 and Lemma 4.3 because $\mathbb{P}_x[\tau_D < \infty] > 0$ for all $x \in D$. Therefore, Lemma 4.4 leads us to this lemma. $\square$

By combining Lemma 4.5 and Theorem 2.2, we have

Theorem 4.6. *Let $\mu \in \mathcal{S}_\infty^D$. Then*

$$\begin{aligned} \lambda(\mu; D) > 1 &\iff \sup_{x \in D} \mathbb{E}_x [\exp(A_{\tau_D}^\mu)] < \infty \\ &\iff \sup_{x \in D} \mathbb{E}_x [\exp(A_{\tau_D}^\mu); \tau_D < \infty] < \infty. \end{aligned}$$

Proof of Theorem 1.2. Let $\bar{X}_t^D$ be the process that stops when $\bar{X}_t$ reaches K . Then, N_K is the number of the offsprings of $\bar{X}_t^D$ that reach K . Let N_t be the number of the offspring of $\bar{X}_t^D$ at t and N_t^D the number of the offspring of $\bar{X}_t^D$ that stay in D at t . Then by the strong Markov property, $m(t, x) := \mathbb{E}_x[N_t]$ and $m^D(t, x) := \mathbb{E}_x[N_t^D]$ satisfy

$$m(t, x) = \mathbb{E}_x \left[\int_0^{\sigma_K \wedge t} e^{-A_s^k} m(t-s, X_s) Q(X_s) dA_s^k \right] + \mathbb{E}_x [e^{-A_{\sigma_K}^k} \wedge t]$$

and

$$m^D(t, x) = \mathbb{E}_x \left[\int_0^{\sigma_K \wedge t} e^{-A_s^k} m^D(t-s, X_s) Q(X_s) dA_s^k \right] + \mathbb{E}_x [e^{-A_t^k}; t < \sigma_K],$$

respectively. Moreover, functions

$$v(t, x) := \mathbb{E}_x [\exp(A_{\sigma_K \wedge t}^\mu)], \quad v^D(t, x) := \mathbb{E}_x [\exp(A_t^\mu); t < \sigma_K]$$

also satisfy the equations above, respectively. Since the bounded solution of each equation above is unique by [8, Lemma 4.4.2, Lemma 4.3.1], we see that

$$\bar{\mathbb{E}}_x[N_t] = \mathbb{E}_x[\exp(A_{\sigma_K \wedge t}^\mu)], \quad \bar{\mathbb{E}}_x[N_t^D] = \mathbb{E}_x[\exp(A_t^\mu); t < \sigma_K]$$

and

$$\begin{aligned} \bar{\mathbb{E}}_x[N_K] &= \lim_{t \rightarrow \infty} (\bar{\mathbb{E}}_x[N_t] - \bar{\mathbb{E}}_x[N_t^D]) \\ &= \mathbb{E}_x[\exp(A_{\sigma_K}^\mu); \sigma_K < \infty]. \end{aligned}$$

Now Theorem 4.6 leads us to Theorem 1.2. $\square$

Lemma 4.7. *Let K be a closed set with $\text{Cap}(K) > 0$. Then for $\mu \in \mathcal{S}_\infty^D$, $\lambda(\mu; D) > \lambda(\mu)$.*

Proof. Let F be the topological support of μ and denote by $(\check{\mathcal{E}}^{(\alpha)}, \mathcal{D}(\check{\mathcal{E}}^{(\alpha)}))$ the Dirichlet form on $L^2(F, \mu)$ generated by the time changed process by A_t^μ . By (3.1) the restriction of $u \in \mathcal{D}_e(\mathcal{E}^{(\alpha)})$ to F , $u|_F$, belongs to $\mathcal{D}(\check{\mathcal{E}}^{(\alpha)})$ ([9, Lemma 6.2.2]), and

$$\mathcal{E}^{(\alpha)}(u, u) = \check{\mathcal{E}}^{(\alpha)}(u|_F, u|_F).$$

As a result of Theorem 4.1, the embedding of $(\mathcal{E}_D^{(\alpha)}, \mathcal{D}_e(\mathcal{E}_D^{(\alpha)}))$ to $L^2(D; \mu)$ is compact. Hence there exists the function which attains the infimum in the definition of $\lambda(\mu; D)$. Consequently, $\lambda(\mu; D) \geq \lambda(\mu)$. Indeed, suppose that $\lambda(\mu; D) = \lambda(\mu)$ and let u_0 and u_D be functions attaining $\lambda(\mu)$ and $\lambda(\mu; D)$, respectively. Then $u_0|_F = u_D|_F$ μ -a.e. because $u_0|_F$ and $u_D|_F$ are normalized principal eigenfunctions of the time changed process by A_t^μ and it is irreducible.

Since for any $\varphi \in \mathcal{D}_e(\mathcal{E}^{(\alpha)})$

$$\mathcal{E}^{(\alpha)}(u_0, \varphi) = \int_{\mathbb{R}^d} u_0 \varphi d\mu, \quad \mathcal{E}^{(\alpha)}(u_D, \varphi) = \int_{\mathbb{R}^d} u_D \varphi d\mu,$$

we have $\mathcal{E}^{(\alpha)}(u_0 - u_D, \varphi) = 0$ for any $\varphi \in \mathcal{D}_e(\mathcal{E}^{(\alpha)})$, Therefore $u_0 = u_D$, which contradicts that $\text{Cap}(K) > 0$. $\square$

Corollary 4.8. *Let K be a closed set with $\text{Cap}(K) > 0$. If $\lambda(\mu) \geq 1$ and $\mu \in \mathcal{S}_\infty^D$, then $\bar{\mathbb{E}}_x[N_K] < \infty$.*

Example 4.9. Suppose that $d = 1$ and $1 < \alpha < 2$. Let $k = \delta_a$, $a \neq 0$, the Dirac measure at a and $p_2(x) = 1$. Then it is known in [12] that

$$\lambda(\delta_a; \mathbb{R} \setminus \{0\}) = -\frac{\Gamma(\alpha) \cos\left(\frac{\pi\alpha}{2}\right)}{2|a|^{\alpha-1}}.$$

Hence

$$\sup_{x \in \mathbb{R} \setminus \{0\}} \bar{\mathbb{E}}[N_{\{0\}}] < \infty \iff 0 < |a| < \left(-\frac{\Gamma(\alpha) \cos\left(\frac{\pi\alpha}{2}\right)}{2} \right)^{1/(\alpha-1)}.$$

REFERENCES

1. S. Albeverio, P. Blanchard and Z.M. Ma, *Feynman-Kac semigroups in terms of signed smooth measures*, Random Partial Differential Equations, eds: U. Hornung et al., Birkhäuser, 1991 MR1185735 (93i:60140)
2. A. Bendikov, *Asymptotic formulas for symmetric stable semigroups*, Expo. Math., **12** (1994), 381-384. MR1297844 (95j:60029)
3. Z.-Q. Chen, *Gaugeability and Conditional Gaugeability*, Trans. Amer. Math. Soc., **354** (2002), 4639-4679. MR1926893 (2003i:60127)
4. Z.-Q. Chen and R.M. Song, *General gauge and conditional gauge theorems*, Ann. Probab., **30** (2002), 1313-1339. MR1920109 (2003f:60135)
5. Z.-Q. Chen and S.T. Zhang, *Girsanov and Feynman-Kac type transformations for symmetric Markov processes*, Ann. Inst. Henri Poincaré, **38** (2002), 475-505. MR1914937 (2004e:60128)
6. K.L. Chung and Z. Zhao, *From Brownian Motion to Schrödinger's Equation*, Springer (1995). MR1329992 (96f:60140)
7. E.B. Davies, *Heat Kernels and Spectral Theory*, Cambridge Univ. Press, Cambridge, U.K. (1989). MR0990239 (90e:35123)
8. D.A. Dawson, *Measure-valued Markov processes*, Lectures Notes in Math., **1541**, Springer, (1993), 1-260. MR1242575 (94m:60101)
9. M. Fukushima, Y. Oshima and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, De Gruyter, 1994. MR1303354 (96f:60126)
10. A. Grigor'yan and M. Kelbert, *Recurrence and transience of branching diffusion processes on Riemannian manifolds*, Ann. Probab., **31** (2003), 244-284. MR1959793 (2003k:60211)
11. D. Revuz and M. Yor, *Continuous martingales and Brownian motion*, Third edition, Grundlehren der Mathematischen Wissenschaften, **293**, Springer, 1999. MR1725357 (2000h:60050)
12. Y. Shiozawa, *Principal eigenvalues for time changed processes of one-dimensional α -stable processes*, Probability and Mathematical Statistics, **24** (2004), 111-122.
13. B. Simon, *Schrödinger semigroups*, Bull. Amer. Math. Soc., **7**, 447-536, (1982). MR0670130 (86b:81001a)
14. P. Stollmann and J. Voigt, *Perturbation of Dirichlet forms by measures*, Potential Analysis, **5** (1996) 109-138. MR1378151 (97e:47065)
15. M. Takeda, *Exponential decay of lifetime and a Theorem of Kac on total occupation times*, Potential Analysis, **11** (1999), 235-247. MR1717103 (2000i:60084)
16. M. Takeda, *Subcriticality and conditional gaugeability of generalized Schrödinger operators*, J. Funct. Anal., **191** (2002), 343-376. MR1911190 (2003e:60176)
17. M. Takeda and K. Tsuchida, *Differentiability of spectral functions for symmetric α -stable process*, to appear in Trans. Amer. Math. Soc.
18. M. Takeda and T. Uemura, *Subcriticality and gaugeability for symmetric α -stable processes*, Forum Math. **16** (2004), 505-517. MR2044025 (2005d:60124)
19. Z. Zhao, *Subcriticality and gaugeability of the Schrödinger operator*, Trans. Amer. Math. Soc., **334** (1992), 75-96. MR1068934 (93a:81041)

MATHEMATICAL INSTITUTE, TOHOKU UNIVERSITY, AOBA, SENDAI, 980-8578, JAPAN

E-mail address: `takeda@math.tohoku.ac.jp`