ON p -ADIC HERMITIAN EISENSTEIN SERIES

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ABSTRACT. In this paper we generalize the notion of p -adic modular form to the Hermitian modular case and prove a formula that shows a coincidence between certain p -adic Hermitian Eisenstein series and the genus theta series associated with Hermitian matrix with determinant p .

1. INTRODUCTION

In a previous study [8], the present author generalized the notion of p -adic Eisenstein series to the Siegel modular case and found a “curious formula” ([8], (8.6)) that shows a coincidence between p -adic Siegel-Eisenstein series and theta series. Strictly speaking, certain p -adic Siegel-Eisenstein series were shown to coincide with the genus theta series associated with a binary quadratic form of discriminant $-p$.

In another study [5], a more general type of p -adic Siegel-Eisenstein series was investigated. The results of both of these studies indicate that these kinds of p -adic Siegel-Eisenstein series become “true” modular forms, namely, ordinary modular forms of $\Gamma_0(p)$ -type.

In this note, we shall extend these findings in a different direction. Namely, we show that a similar phenomenon also occurs in the Hermitian modular case.

2. DEFINITION AND NOTATION

2.1. Hermitian modular forms. Let R be a subring of $\mathbb{C}$. We shall denote by $Her_m(R)$ the space of m by m Hermitian matrices over R with respect to the complex conjugate. Let $Her_m^+(R)$ be the subset of $Her_m(R)$ consisting of positive elements. The space

$$\mathbb{H}_n := \left\{ Z \in M_n(\mathbb{C}) \mid \frac{1}{\sqrt{-1}}(Z - {}^t\overline{Z}) \in Her_n^+(\mathbb{C}) \right\}$$

is called the *Hermitian upper-half space* of degree n , where ${}^t\overline{Z}$ denotes the complex conjugate, transpose matrix of Z . The group

$$G_n = U(n, n) := \left\{ M \in M_{2n}(\mathbb{C}) \mid {}^t\overline{M}J_nM = J_n, J_n = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \right\}$$

acts on $\mathbb{H}_n$ by the ordinary generalized linear fractional transformation.

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Let $\mathbb{K}$ be an imaginary quadratic field of discriminant $d_{\mathbb{K}}$ with the ring of integers $\mathcal{O}_{\mathbb{K}}$. The *Hermitian modular group* over $\mathbb{K}$ is defined by $\Gamma_n(\mathbb{K}) := G_n \cap M_{2n}(\mathcal{O}_{\mathbb{K}})$. We denote by $M_k(\Gamma_n(\mathbb{K}))$ the complex vector space consisting of Hermitian modular forms of weight k for $\Gamma_n(\mathbb{K})$.

Let $f(Z)$ be a Hermitian modular form in $M_k(\Gamma_n(\mathbb{K}))$. Then, $f(Z)$ has a Fourier expansion of the form

$$f(Z) = \sum_{0 \leq H \in \Lambda_n(\mathbb{K})} a_f(H) \exp[2\pi\sqrt{-1}\mathrm{tr}(HZ)], \quad Z \in \mathbb{H}_n,$$

where the index set $\Lambda_n(\mathbb{K})$ is defined by

$$\Lambda_n(\mathbb{K}) := \{ H = (h_{ij}) \in \mathrm{Her}_n(\mathbb{K}) \mid h_{ii} \in \mathbb{Z}, \sqrt{d_{\mathbb{K}}}h_{ij} \in \mathcal{O}_{\mathbb{K}} \ (i \neq j) \}.$$

2.2. Hermitian Eisenstein series. Define

$$\Gamma_n(\mathbb{K})_{\infty} := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n(\mathbb{K}) \mid C = O_n \right\}.$$

For an integer k such that $k > 2n$ and $w_{\mathbb{K}}|k$ ($w_{\mathbb{K}}$ is the order of the unit group of $\mathbb{K}$), define a series given by

$$G_k^{(n)}(Z) := \sum_{\begin{pmatrix} * & * \\ C & D \end{pmatrix} \in \Gamma_n(\mathbb{K})_{\infty} \setminus \Gamma_n(\mathbb{K})} \det(CZ + D)^{-k}, \quad Z \in \mathbb{H}_n.$$

This series belongs to $M_k(\Gamma_n(\mathbb{K}))$ and is called the *Hermitian Eisenstein series* of weight k for $\Gamma_n(\mathbb{K})$. It was shown by Braun [3] that all of the Fourier coefficients of $G_k^{(n)}$ are rational.

2.3. Genus theta series. Let $S \in \mathrm{Her}_m^+(\mathcal{O}_{\mathbb{K}})$. Define

$$\Theta^{(n)}(S; Z) := \sum_{X \in M_{m,n}(\mathcal{O}_{\mathbb{K}})} \exp[\pi\sqrt{-1}\mathrm{tr}(S[X]Z)], \quad Z \in \mathbb{H}_n,$$

where $S[X] := {}^t\overline{X}SX$. Let $\{S_1, \dots, S_h\}$ be a set of representatives of unimodular equivalence classes of the genus containing S . The *genus theta series* associated with S is defined by

$$\mathrm{genus} \Theta^{(n)}(S)(Z) := \left(\sum_{i=1}^h \frac{\Theta^{(n)}(S_i; Z)}{E(S_i)} \right) / \left(\sum_{i=1}^h \frac{1}{E(S_i)} \right),$$

where $E(S_i)$ is the order of the unit group of S_i .

2.4. p -adic Hermitian Eisenstein series. Put $\omega = (d_{\mathbb{K}} + \sqrt{d_{\mathbb{K}}})/2$ and define the matrices $\dot{Z} = (\dot{z}_{ij})$ and $\ddot{Z} = (\ddot{z}_{ij})$ by

$$\dot{Z} := \frac{\omega^t Z - \bar{\omega} Z}{\omega - \bar{\omega}}, \quad \ddot{Z} := \frac{Z - {}^t Z}{\omega - \bar{\omega}}.$$

Thus, any Hermitian modular form $f(Z)$ can be considered as a function of $n(n-1)/2$ complex variables $\ddot{z}_{ij} (i < j)$ in $\ddot{Z}$ and $n(n+1)/2$ complex variables $\dot{z}_{ij} (i \leq j)$ in $\dot{Z}$. If we put

$$\dot{q}_{ij} := \exp(2\pi\sqrt{-1}\dot{z}_{ij}) \ (i \leq j), \quad \ddot{q}_{ij} := \exp(2\pi\sqrt{-1}\ddot{z}_{ij}) \ (i < j),$$

then

$$q^H := \exp[2\pi\sqrt{-1}\mathrm{tr}(HZ)] = \prod_{i \leq j} \dot{q}_{ij}^{\xi_{ij}} \prod_{i < j} \ddot{q}_{ij}^{\eta_{ij}},$$

where $H = (h_{ij})$ and

$$\xi_{ii} = h_{ii}, \quad \xi_{ij} = h_{ij} + \bar{h}_{ij} \quad (i < j), \quad \eta_{ij} = \bar{\omega}h_{ij} + \omega\bar{h}_{ij} \quad (i < j)$$

are rational integers. Based on the semi-positivity of H , we may regard a Hermitian modular form $f(Z)$ as an element of a formal power series ring

$$\mathbb{C}[\dot{q}_{ij}^{\pm}, \ddot{q}_{ij}^{\pm}][[\dot{q}_{11}, \dots, \dot{q}_{nn}]].$$

Let $\{k_m\}_{m=1}^{\infty}$ be a sequence of increasing natural numbers. If the corresponding sequence of Hermitian Eisenstein series $\{G_{k_m}^{(n)}\} \subset \mathbb{Q}[\dot{q}_{ij}^{\pm}, \ddot{q}_{ij}^{\pm}][[\dot{q}_{11}, \dots, \dot{q}_{nn}]]$ converges p -adically to an element of $\mathbb{Q}_p[\dot{q}_{ij}^{\pm}, \ddot{q}_{ij}^{\pm}][[\dot{q}_{11}, \dots, \dot{q}_{nn}]]$, we shall call the limit $\lim_{m \rightarrow \infty} G_{k_m}^{(n)}$ a p -adic Hermitian Eisenstein series ([10] and [8], (5.3)).

3. MAIN RESULT

In the remainder of this note, we will restrict ourselves to the case in which

$$\boxed{\mathbb{K} = \mathbb{Q}(\sqrt{-1}) : \text{the Gaussian field}}.$$

In this case, we have $\mathcal{O}_{\mathbb{K}} = \mathbb{Z}[\sqrt{-1}]$ is the ring of Gaussian integers, $d_{\mathbb{K}} = -4$, and $w_{\mathbb{K}} = 4$. An integral Hermitian matrix $S \in \text{Her}_m(\mathcal{O}_{\mathbb{K}})$ is called *even* if $S[\mathbf{g}] \in 2\mathbb{Z}$ for all $\mathbf{g} \in M_{m,1}(\mathcal{O}_{\mathbb{K}})$.

A main result of this note is as follows.

Theorem 3.1. *Let p be a prime number such that $p \equiv 3 \pmod{4}$, and let $S \in \text{Her}_2^+(\mathcal{O}_{\mathbb{K}})$ be an even integral matrix such that $\det(S) = p$ (actually such a matrix S exists for each prime p satisfying $p \equiv 3 \pmod{4}$). If we define a sequence*

$$k_m = k_m(p) := 2 + (p-1)p^{m-1},$$

then the corresponding sequence $G_{k_m}^{(n)}$ defines a p -adic Hermitian Eisenstein series. Moreover, we have

$$(3.1) \quad \lim_{m \rightarrow \infty} G_{k_m}^{(n)} = \text{genus } \Theta^{(n)}(S).$$

Namely, the p -adic Hermitian Eisenstein series $\lim_{m \rightarrow \infty} G_{k_m}^{(n)}$ becomes an ordinary Hermitian modular form of weight 2 and level p .

4. PROOF OF THE MAIN THEOREM

We shall prove (3.1) by showing the coincidence of each Fourier coefficient of both sides. Let

$$\lim_{m \rightarrow \infty} G_{k_m}^{(n)} = \sum \tilde{a}^{(n)}(H) q^H$$

and

$$\text{genus } \Theta^{(n)}(S) = \sum b^{(n)}(H) q^H$$

be the Fourier expansions. Our goal is to prove the following identity:

$$(4.1) \quad \tilde{a}^{(n)}(H) = b^{(n)}(H).$$

However, before proving this identity, we must investigate the coefficients $\tilde{a}^{(n)}(H)$ and $b^{(n)}(H)$. The coefficient $\tilde{a}^{(n)}(H)$ is the p -adic limit of $a_{k_m}^{(n)}(H)$ ($m \rightarrow \infty$). On

the other hand, by the definition of the genus theta series, the coefficient $b^{(n)}(H)$ is given by

$$\left(\sum_{i=1}^h \frac{A(S_i, 2H)}{E(S_i)} \right) / \left(\sum_{i=1}^h \frac{1}{E(S_i)} \right),$$

where

$$A(S_i, 2H) := \sharp \{ X \in M_{2,n}(\mathcal{O}_{\mathbb{K}}) \mid {}^t \overline{X} S_i X = 2H \}.$$

First, we shall prove the following lemma.

Lemma 4.1. *If $r := \text{rank}(H) > 2$, then*

$$(4.2) \quad \tilde{a}^{(n)}(H) = b^{(n)}(H) = 0.$$

Proof. Since $\text{rank}(H) > \text{rank}(S_i) = 2$, the numbers $A(S_i, 2H)$ vanish. This implies $b^{(n)}(H) = 0$. Next, we prove that $\tilde{a}^{(n)}(H) = \lim_{m \rightarrow \infty} a_{k_m}(H) = 0$. For this purpose, we set

$$\mathbb{B}_k = \begin{cases} \frac{B_k}{k} & \text{if } k \text{ is even,} \\ \frac{B_{k, \chi-4}}{k} & \text{if } k \text{ is odd,} \end{cases}$$

where B_k (resp. $B_{k, \chi}$) is the k -th Bernoulli number (resp. the k -th generalized Bernoulli number). If we pursue an argument similar to that of Boecherer [1] in the Hermitian modular case, then we see that the coefficient $a_{k_m}(H)$ can be written as

$$a_{k_m}^{(n)}(H) = \prod_{i=0}^{r-1} \mathbb{B}_{k_m-i}^{-1} \cdot (\text{rational integer}).$$

(Refer also to [11], Proposition 4.7 (*Case SU*).)

If $i \equiv 2 \pmod{p-1}$, then

$$\text{ord}_p(\mathbb{B}_{k_m-i}^{-1}) = \text{ord}_p\left(\frac{k_m-i}{B_{k_m-i}}\right) \geq 1$$

(based on the classical theorem of von Staudt and Clausen). In particular, if $i = 2$, then

$$\text{ord}_p(\mathbb{B}_{k_m-2}^{-1}) = \text{ord}_p\left(\frac{p^{m-1}(p-1)}{B_{p^{m-1}(p-1)}}\right) = m.$$

If $i \not\equiv 2 \pmod{p-1}$, then there exists a p -adic number $d_i \in \mathbb{Q}_p$ satisfying

$$\lim_{m \rightarrow \infty} \frac{k_m - i}{B_{k_m-i, \chi-4}} = d_i.$$

Therefore, we have

$$\text{ord}_p(\mathbb{B}_{k_m-i}) = \text{ord}_p\left(\frac{k_m-i}{B_{k_m-i, \chi-4}}\right) = \text{ord}_p(d_i) \quad (\text{for sufficiently large } m).$$

Combining these results, we have $\lim_{m \rightarrow \infty} \prod_{i=0}^{r-1} \mathbb{B}_{k_m-i}^{-1} = 0$. This implies

$$\lim_{m \rightarrow \infty} a_{k_m}^{(n)}(H) = 0,$$

and completes the proof. $\square$

From this result, for the purpose of proving (3.1), it suffices to prove that the identity (4.1) holds for the case $n = 2$. Namely, our goal is reduced to the proof of the identity

$$(4.3) \quad \tilde{a}^{(2)}(H) = b^{(2)}(H)$$

for any $H \in \Lambda_2(\mathbb{K})$.

Lemma 4.2. *If $\text{rank}(H) = 2$, then*

$$\tilde{a}^{(2)}(H) = b^{(2)}(H).$$

Proof. We shall first calculate $\tilde{a}^{(2)}(H)$. For this purpose, we introduce an explicit formula for $a_k^{(2)}(H)$ given by Krieg ([6], p. 678, Theorem and p. 679, Corollary):

$$a_k^{(2)}(H) = \frac{4k(k-1)}{B_k \cdot B_{k-1, \chi_{-4}}} \sum_{0 < d | \varepsilon(H)} d^{k-1} G_{\chi_{-4}} \left(k-2, \frac{\det(2H)}{d^2} \right),$$

where

$$\begin{aligned} G_{\chi_{-4}}(s, N) &:= \frac{1}{1 + |\chi_{-4}(N)|} (\sigma_{s, \chi_{-4}}(N) - \sigma_{s, \chi_{-4}}^*(N)), \quad (s, N) \in \mathbb{Z}^2, \\ \sigma_{s, \chi_{-4}}(N) &:= \sum_{0 < d | N} \chi_{-4}(d) d^s, \quad \sigma_{s, \chi_{-4}}^*(N) := \sum_{0 < d | N} \chi_{-4}(N/d) d^s, \\ \varepsilon(H) &:= \max\{l \in \mathbb{N} \mid l^{-1}H \in \Lambda_2(\mathbb{K})\}. \end{aligned}$$

From this formula, we obtain

$$\begin{aligned} \tilde{a}^{(2)}(H) &= \frac{8}{B_{2, \chi_{-4}^2} \cdot B_{1, \chi_{-4}^2 \chi_{-4}}} \sum_{\substack{0 < d | \varepsilon(H) \\ (d, p)=1}} d \tilde{G}_{\chi_{-4}} \left(\frac{\det(2H)}{d^2} \right) \\ &= \frac{48}{p-1} \sum_{\substack{0 < d | \varepsilon(H) \\ (d, p)=1}} d \tilde{G}_{\chi_{-4}} \left(\frac{\det(2H)}{d^2} \right), \end{aligned}$$

where

$$\begin{aligned} \tilde{G}_{\chi_{-4}}(N) &:= \frac{1}{1 + |\chi_{-4}(N)|} (\tilde{\sigma}_{\chi_{-4}}(N) - \tilde{\sigma}_{\chi_{-4}}^*(N)), \quad N \in \mathbb{Z}, \\ \tilde{\sigma}_{\chi_{-4}}(N) &:= \sum_{\substack{0 < d | N \\ (d, p)=1}} \chi_{-4}(d), \quad \tilde{\sigma}_{\chi_{-4}}^*(N) := \sum_{\substack{0 < d | N \\ (d, p)=1}} \chi_{-4}(N/d). \end{aligned}$$

Next, we shall calculate $b^{(2)}(H)$. By the Siegel formula in the case of Hermitian forms (cf. Braun [2], Otremba [9]), we have

$$b^{(2)}(H) = \frac{1}{2} \prod_{q \leq \infty} \alpha_q(S, 2H),$$

where

$$\begin{aligned} \alpha_q(S, 2H) &= \lim_{a \rightarrow \infty} q^{-4a} A_{q^a}(S, 2H), \\ A_{q^a}(S, 2H) &:= \#\{X \in M_2(\mathcal{O}_{\mathbb{K}}) \bmod q^a \mid {}^t \overline{X} S X \equiv 2H \bmod q^a\} \end{aligned}$$

and

$$\alpha_\infty(S, 2H) = \det(S)^{-2} \cdot \prod_{j=1}^2 \frac{(2\pi)^j}{\sqrt{|d_{\mathbb{K}}|}^j \cdot (j-1)!} = \frac{\pi^3}{p^2}.$$

In order to calculate $\alpha_q(S, 2H)$ more explicitly, we introduce the following notation: let $f(H)$ be the integer defined by $\det(2H) = \varepsilon(H)^2 \cdot f(H)$ and put

$$\det(2H) = \prod_{q:\text{prime}} q^{e_q}, \quad \varepsilon(H) = \prod_{q:\text{prime}} q^{\varepsilon_q}, \quad f(H) = \prod_{q:\text{prime}} q^{f_q}.$$

We proceed the calculation of $\alpha_q(S, 2H)$ by dividing the following three cases:

- (i) q is a prime number such that $q \neq 2, p$,
- (ii) $q = p$,
- (iii) $q = 2$ (ramified prime).

For case (i), we have

$$\alpha_q(S, 2H) = (1 - q^{-2})(1 - \chi_{-4}(q)q^{-1}) \sum_{l=0}^{\varepsilon_q} q^l \left(\sum_{m=0}^{e_q-2l} \chi_{-4}(q)^m \right).$$

This follows from the main theorem in [7] (p. 235) (unramified case).

For case (ii), we obtain

$$\alpha_q(S, 2H) = \alpha_p(S, 2H) = \begin{cases} \frac{(p+1)^2}{p} & \text{if } f_p \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

This formula is derived from a formula by Hironaka [4] by setting $m = n = 2$, $\lambda = (\varepsilon_p + f_p, \varepsilon_p)$, and $\mu = (1, 0)$ in the main formula in Theorem 4 ([4], p. 61).

For case (iii), we have

$$\begin{aligned} \alpha_2(S, 2H) \\ = 2^2 \cdot (1 - 2^{-2}) \sum_{l=0}^{\varepsilon_2} 2^l \{1 - \chi_{-4}(\det(2H)/2^{\varepsilon_2})(1 - \chi_{-4}(\det(2H)/2^{2l})^2)\}. \end{aligned}$$

As in case (i), this formula can be obtained from the main theorem in [7] (p. 235) (ramified case).

Now we assume that $\text{ord}_p(\det(2H))$ is odd, that is, the number f_p is odd. In this case, by the above formulas, we have

$$\begin{aligned} b^{(2)}(H) &= \frac{1}{2} \cdot \prod_{q \leq \infty} \alpha_q(S, 2H) \\ &= 2 \cdot \frac{1}{\zeta(2)L(1; \chi_{-4})} \cdot \frac{1}{(1 - p^{-2})(1 - \chi_{-4}(p)p^{-1})} \cdot \frac{(p+1)^2}{p} \cdot \frac{\pi^3}{p^2} \\ &\quad \cdot \sum_{\substack{0 < d | \varepsilon(H) \\ (d, p) = 1}} dF\left(\frac{\det(2H)}{d^2}\right) \\ &= \frac{48}{p-1} \sum_{\substack{0 < d | \varepsilon(H) \\ (d, p) = 1}} dF\left(\frac{\det(2H)}{d^2}\right), \end{aligned}$$

where

$$F(N) := \prod_{\substack{q:\text{prime} \\ q \neq p, 2}} \left(\sum_{m=0}^{\beta_q} \chi_{-4}(q)^m \right) \cdot \frac{1 - \chi_{-4}(N')}{1 + |\chi_{-4}(N)|}, \quad N = \prod q^{\beta_q} = 2^{\beta_2} \cdot N'.$$

It is easy to see that $G_{\chi_{-4}}(s, N)$ has the following expression:

$$G_{\chi_{-4}}(s, N) = \frac{1}{1 + |\chi_{-4}(N)|} \prod_{q \neq 2} \left(\sum_{m=0}^{\beta_q} \chi_{-4}(q)^m q^{ms} \right) (1 - \chi_{-4}(N') 2^{\beta_2 s}).$$

Hence, we obtain

$$F(N) = \tilde{G}_{\chi_{-4}}(N).$$

Therefore, if $\text{ord}_p(\det(2H))$ is odd, then $\tilde{a}^{(2)}(H) = b^{(2)}(H)$. On the other hand, if $\text{ord}_p(\det(2H))$ is even, we have $\tilde{a}^{(2)}(H) = b^{(2)}(H) = 0$. This completes the proof of Lemma 4.2. $\square$

Lemma 4.3. *If $\text{rank}(H) = 1$, then*

$$\tilde{a}^{(2)}(H) = b^{(2)}(H).$$

Proof. In the case where $\text{rank}(H) = 1$, we have

$$a_k^{(2)}(H) = a_k^{(1)}(\varepsilon(H)) = -\frac{2k}{B_k} \sigma_{k-1}(\varepsilon(H)).$$

Thus, we obtain

$$\tilde{a}^{(2)}(H) = -\frac{4}{B_{2, \chi_{-p}^2}} \sigma_{1, \chi_{-p}^2}(\varepsilon(H)) = -\frac{4}{(1-p)B_2} \sum_{\substack{0 < d | \varepsilon(H) \\ (d, p) = 1}} d = \frac{24}{p-1} \sum_{\substack{0 < d | \varepsilon(H) \\ (d, p) = 1}} d.$$

Next, we shall calculate $b^{(2)}(H)$ for the case in which $\text{rank}(H) = 1$. In this case, we have $2H \sim \begin{pmatrix} 2\varepsilon(H) & 0 \\ 0 & 0 \end{pmatrix}$ (unimodular equivalence). As in the case of $\text{rank}(H) = 2$, we can apply the Siegel formula and obtain

$$b^{(2)}(H) = \prod_{q \leq \infty} \alpha_q(S, 2\varepsilon(H)),$$

where

$$\alpha_q(S, 2\varepsilon(H)) = \lim_{a \rightarrow \infty} q^{-3a} A_{q^a}(S, 2\varepsilon(H)),$$

$$A_{q^a}(S, 2\varepsilon(H)) := \sharp \{ X \in M_{2,1}(\mathcal{O}_{\mathbb{K}}) \bmod q^a \mid {}^t \overline{X} S X \equiv 2\varepsilon(H) \bmod q^a \}$$

and

$$\alpha_{\infty}(S, 2\varepsilon(H)) = \det(S)^{-1} \cdot 2\varepsilon(H) \cdot \pi^2 = \frac{2\varepsilon(H)\pi^2}{p}.$$

The value $\alpha_q(S, 2\varepsilon(H))$ is given as follows.

If q is a prime number satisfying $q \neq 2, p$ (which corresponds to case (i) in the proof of Lemma 4.2), then

$$\alpha_q(S, 2\varepsilon(H)) = (1 - q^{-2}) \sum_{l=0}^{\varepsilon_q} q^{-l}.$$

If $q = p$ (case (ii)), then

$$\alpha_p(S, 2\varepsilon(H)) = \frac{1+p}{p^{1+\varepsilon_p}}.$$

If $q = 2$ (case (iii)), then

$$\alpha_2(S, 2\varepsilon(H)) = 2 \cdot (1 - 2^{-2}) \sum_{l=0}^{\varepsilon_2} 2^{-l}.$$

Combining these formulas, we have

$$\begin{aligned} b^{(2)}(H) &= \prod_{q \leq \infty} \alpha_q(S, 2\varepsilon(H)) \\ &= \frac{1}{\zeta(2) \cdot (1 - p^{-2})} \cdot \frac{1+p}{p^{1+\varepsilon_p}} \cdot 2 \prod_{q \neq p} \left(\sum_{l=0}^{\varepsilon_q} q^{-l} \right) \cdot \frac{2\varepsilon(H)\pi^2}{p} \\ &= \frac{24}{p-1} \sum_{\substack{0 < d | \varepsilon(H) \\ (d, p) = 1}} d \\ &= \tilde{a}^{(2)}(H). \end{aligned}$$

This completes the proof of Lemma 4.3. $\square$

Since $\tilde{a}^{(2)}(O_2) = b^{(2)}(O_2) = 1$, we conclude the proof of the desired formula (4.3).

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