REGULARIZED PRODUCT EXPRESSIONS OF HIGHER RIEMANN ZETA FUNCTIONS

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ABSTRACT. As a generalization of recent work by Kurokawa, Matsuda, and Wakayama (2004) we introduce a higher Riemann zeta function for an abstract sequence. Then we explicitly determine its regularized product expression.

1. INTRODUCTION

It is known ([De]) that the Riemann zeta function $\zeta(s) = \prod_{p:\text{prime}} (1 - p^{-s})^{-1}$ has the regularized product expression

$$\prod_{\rho \in R} \left(\frac{s - \rho}{2\pi} \right) = 2^{-1/2} (2\pi)^{-2} \pi^{-s/2} s(s-1) \Gamma\left(\frac{s}{2}\right) \zeta(s),$$

where R is the set of the non-trivial zeros of $\zeta(s)$, and $\Gamma(s)$ is the classical gamma function.

On the other hand, a higher Riemann zeta function $\zeta_{l\infty}(s) := \prod_{n=1}^{\infty} \zeta(s + ln)$ has been introduced and studied in the paper [CL] for $l = 1$, [KMW] for $l \in \mathbb{Z}_{\geq 1}$. We now consider a generalized higher Riemann zeta function. Let $\Lambda = \{\lambda_k\}_{k \in I}$ be a sequence of complex numbers. Then we define a higher Riemann zeta function for the sequence Λ by

$$Z(s, \Lambda) := \prod_{\lambda \in \Lambda} \prod_{p:\text{prime}} (1 - p^{-s-\lambda})^{-1} = \prod_{\lambda \in \Lambda} \zeta(s + \lambda).$$

In this paper, we study several properties of the higher Riemann zeta function. For a “regularizable” sequence Λ , we see that $Z(s, \Lambda)$ has the regularized product expression

$$\prod_{\lambda \in \Lambda, \rho \in R} \left(\frac{s + \lambda - \rho}{2\pi} \right) = \text{“gamma factor”} \times Z(s, \Lambda).$$

Here $\prod$ denotes the “dotted” regularized product due to [KW3].

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Further, when Λ is given by a lattice $\Omega := \{n_1\omega_1 + \cdots + n_r\omega_r | n_j \in \mathbb{Z}_{\geq 0}\}$, we show that the higher Riemann zeta function

$$Z(s, \Omega) = \prod_{n_1, \dots, n_r=0}^{\infty} \prod_{p: \text{prime}} (1 - p^{-(s+n_1\omega_1+\cdots+n_r\omega_r)})^{-1}$$

satisfies a certain functional equation of the type $s \leftrightarrow 1 + \omega_1 + \cdots + \omega_r - s$. We remark that the form of $Z(s, \Omega)$ looks like the Selberg zeta function (when $r = 1$) defined for a discrete subgroup of a real rank 1 semi-simple Lie group (cf. [Go] [Ku2]).

2. NOTATION

2.1. Regularized product. Let $\Lambda = \{\lambda_k\}_{k \in I}$ be a sequence of complex numbers. We call the sequence Λ “regularizable” if Λ satisfies the following conditions: (i) $\operatorname{Re}(\lambda_k) \geq 0$ for all $k \in I$, and $\operatorname{Re}(\lambda_k) \rightarrow \infty$ as $k \rightarrow \infty$. (ii) The series $\sum_{k \in I, \lambda_k \neq 0} \lambda_k^{-s}$ converges for sufficiently large $\operatorname{Re}(s)$. (iii) The function $\Theta(x, \Lambda) := \sum_{k \in I} e^{-\lambda_k x}$ has an asymptotic expansion

$$\Theta(x, \Lambda) \sim \sum_{n=0}^{\infty} x^{t_n} T_n(\log x) \quad \text{as } x \rightarrow +0,$$

where $t_0 \leq 0$, $t_0 < t_1 < t_2 < \cdots \rightarrow +\infty$, and $T_n(z) \in \mathbb{C}[z]$ is a polynomial.

For a regularizable sequence, we can define the regularized product as follows.

Lemma 2.1 ([Ill], [KW3]). *Assume that the sequence $\Lambda = \{\lambda_k\}_{k \in I}$ is regularizable. Then, (1) for $\operatorname{Re}(z) > 0$, the function*

$$\zeta(s, z, \Lambda) := \sum_{k \in I} (z + \lambda_k)^{-s}$$

has an analytic continuation as a meromorphic function on $s \in \mathbb{C}$. (2) The “dotted” regularized product

$$\prod_{k \in I} (z + \lambda_k) := \exp \left\{ -\operatorname{CT}_{s=0} \frac{\zeta(s, z, \Lambda)}{s} \right\}$$

exists, and this is an entire function on $z \in \mathbb{C}$ with zeros at $z = -\lambda_k$ ($k \in I$). Here $\operatorname{CT}_{s=0} f(s)$ denotes the constant term in the Laurent expansion of $f(s)$ at $s = 0$.

We remark that the dotted regularized product $\prod_{m \in I, n \in J} (z + a_m + b_n)$ exists if $\{a_m\}_{m \in I}$ and $\{b_n\}_{n \in J}$ are regularizable sequences. This follows at once from the relation $\Theta(x, \{a_m + b_n\}_{m \in I, n \in J}) = \Theta(x, \{a_m\}_{m \in I}) \times \Theta(x, \{b_n\}_{n \in J})$.

2.2. Multiple gamma function and multiple sine function. Let $\omega = (\omega_1, \dots, \omega_r)$, where $\omega_j \in \mathbb{C}$, $\operatorname{Re}(\omega_j) > 0$, and $\Omega := \{n_1\omega_1 + \cdots + n_r\omega_r | n_j \in \mathbb{Z}_{\geq 0}\}$. We fix $\operatorname{Re}(z) > 0$. Then the multiple zeta function of Barnes ([Ba]) is defined by

$$\zeta(s, z, \omega) := \zeta(s, z, \Omega) = \sum_{n_1, \dots, n_r=0}^{\infty} (z + n_1\omega_1 + \cdots + n_r\omega_r)^{-s}.$$

This sum converges absolutely for $\operatorname{Re}(s) > r$. It is seen that the sequence Ω is regularizable, and $\zeta(s, z, \omega)$ has a meromorphic continuation to $s \in \mathbb{C}$. Further, for

$m \in \mathbb{Z}_{\geq 0}$, it is known that

$$\zeta(-m, z, \omega) = \frac{(-1)^m \cdot m! \cdot B_{m+r}(z, \omega)}{(m+r)!}.$$

Here $B_n(z, \omega)$ is the multiple Bernoulli polynomials given by

$$\frac{x^r e^{-zx}}{(1 - e^{-\omega_1 x}) \cdots (1 - e^{-\omega_r x})} = \sum_{n=0}^{\infty} \frac{B_n(z, \omega)}{n!} x^n,$$

for $|x| < \min_{1 \leq j \leq r} |2\pi/\omega_j|$. We remark that $\zeta(s) = \zeta(s, 1, (1))$ is the Riemann zeta function and $\zeta(-m) = (-1)^m B_{m+1}/(m+1)$, where $B_m = B_m(1, (1))$ is the usual Bernoulli number.

Now we define the multiple gamma function $\Gamma(z, \omega)$ and the multiple sine function $S(z, \omega)$ by using the regularized product as follows ([Ku1]):

$$\Gamma(z, \omega) := \exp \left\{ \frac{d}{ds} \zeta(s, z, \omega) \Big|_{s=0} \right\} = \prod_{n_1, \dots, n_r=0}^{\infty} (z + n_1 \omega_1 + \cdots + n_r \omega_r)^{-1},$$

$$S(z, \omega) := \Gamma(z, \omega)^{-1} \cdot \Gamma(\omega_1 + \cdots + \omega_r - z, \omega)^{(-1)^r}.$$

It is seen that these functions satisfy the conditions

$$\Gamma(z, (\omega_1, \dots, \omega_{r-1}, \omega_r)) = \Gamma(z, (\omega_1, \dots, \omega_{r-1})) \cdot \Gamma(z + \omega_r, (\omega_1, \dots, \omega_{r-1}, \omega_r)),$$

$$S(z, (\omega_1, \dots, \omega_{r-1}, \omega_r)) = S(z, (\omega_1, \dots, \omega_{r-1})) \cdot S(z + \omega_r, (\omega_1, \dots, \omega_{r-1}, \omega_r)).$$

We also note that $\Gamma(z, (1)) = (2\pi)^{-1/2} \Gamma(z)$ and $S(z, (1)) = 2 \sin(\pi z)$.

3. HIGHER RIEMANN ZETA FUNCTION

3.1. Definition and analytic continuation. First of all, we introduce a higher Riemann zeta function as follows.

Definition 3.1. Define a higher Riemann zeta function for $\Lambda = \{\lambda_k\}_{k=0}^{\infty}$ by

$$Z(s, \Lambda) = Z(s, \{\lambda_k\}_{k=0}^{\infty}) := \prod_{k=0}^{\infty} \zeta(s + \lambda_k) = \prod_{k=0}^{\infty} \prod_{p:\text{prime}} (1 - p^{-(s+\lambda_k)})^{-1}.$$

To show the absolute convergence of $Z(s, \Lambda)$, we prepare the following lemma.

Lemma 3.2. Fix a positive number $x \geq 2$. If the sequence $\Lambda = \{\lambda_k\}_{k=0}^{\infty}$ satisfies (i) $0 \leq \operatorname{Re}(\lambda_0) \leq \operatorname{Re}(\lambda_1) \leq \cdots \rightarrow \infty$, and (ii) the sum $\sum_{k \in I, \lambda_k \neq 0} \lambda_k^{-s}$ converges absolutely for large $\operatorname{Re}(s)$, then we have

$$\left| \sum_{k=j}^{\infty} x^{-\lambda_k} \right| \ll_j x^{-\operatorname{Re}(\lambda_j)}, \quad \text{for all } j \geq 0.$$

Proof. We observe that

$$\begin{aligned} \left| \sum_{k=j}^{\infty} x^{-\lambda_k} \right| &= \left| x^{-\lambda_j} \cdot \sum_{k=j}^{\infty} x^{-(\lambda_k - \lambda_j)} \right| \\ &\leq x^{-\operatorname{Re}(\lambda_j)} \cdot \sum_{k=j}^{\infty} 2^{-(\operatorname{Re}(\lambda_k) - \operatorname{Re}(\lambda_j))}. \end{aligned}$$

Under condition (ii), the sum of the second factor is bounded. $\square$

Theorem 3.3. Assume that the sequence $\Lambda = \{\lambda_k\}_{k=0}^\infty$ satisfies conditions (i) and (ii) in the previous lemma. Then $Z(s, \Lambda)$ is defined for $\operatorname{Re}(s) > 1 - \operatorname{Re}(\lambda_0)$. Moreover $Z(s, \Lambda)$ has an analytic continuation as a meromorphic function on $s \in \mathbb{C}$.

Proof. We see from the lemma that

$$\left| \sum_{k=0}^{\infty} \sum_p p^{-(s+\lambda_k)} \right| \ll \sum_p p^{-\operatorname{Re}(s) - \operatorname{Re}(\lambda_0)}.$$

The right side converges in $\operatorname{Re}(s) > 1 - \operatorname{Re}(\lambda_0)$. Hence $Z(s, \Lambda)$ is holomorphic on this domain. Further, we have

$$Z(s, \{\lambda_k\}_{k=0}^\infty) = \prod_{k=0}^{j-1} \zeta(s + \lambda_k) \cdot \prod_{k=j}^{\infty} \zeta(s + \lambda_k) = \prod_{k=0}^{j-1} \zeta(s + \lambda_k) \times Z(s, \{\lambda_k\}_{k=j}^\infty)$$

for any j . From the previous lemma, we see that the second factor is now defined for $\operatorname{Re}(s) > 1 - \operatorname{Re}(\lambda_j)$. This provides a meromorphic continuation to $\mathbb{C}$. $\square$

We remark that the higher Riemann zeta function has the Dirichlet series expression $Z(s, \Lambda) = \sum_{n=1}^{\infty} g_\Lambda(n) n^{-s}$, where

$$g_\Lambda(n) = \sum_{\substack{n_0 \cdot n_1 \cdot n_2 \cdots = n \\ n_0 \geq 1, n_1 \geq 1, \dots}} n_0^{-\lambda_0} \cdot n_1^{-\lambda_1} \cdot n_2^{-\lambda_2} \cdots.$$

From this expression, we see that $g_\Lambda(n)$ is multiplicative and $Z(s, \Lambda)$ has the Euler product

$$Z(s, \Lambda) = \prod_{p:\text{prime}} \sum_{m=0}^{\infty} \frac{g_\Lambda(p^m)}{p^{ms}}, \quad \text{for } \operatorname{Re}(s) > 1 - \operatorname{Re}(\lambda_0).$$

Moreover, using the Tauberian theorem (cf. [Mu]), we obtain the behavior of $g_\Lambda(n)$ as follows.

Corollary 3.4. If $\Lambda = \{\lambda_k\}_{k=0}^\infty$ satisfies (i) and (ii), and $0 \leq \lambda_0 = \cdots = \lambda_{K-1} < \lambda_K \leq \cdots$, then we have

$$\sum_{n \leq x} g_\Lambda(n) = (c_\Lambda + o(1)) x^{1-\lambda_0} \cdot \log^{K-1} x \quad \text{as } x \rightarrow \infty,$$

where c_Λ is a constant which is given by

$$c_\Lambda := \frac{1}{(K-1)!} \lim_{s \rightarrow 1-\lambda_0} (s-1+\lambda_0)^K Z(s, \Lambda) = \frac{1}{(K-1)!} Z(1-\lambda_0, \{\lambda_k\}_{k=K}^\infty). \quad \square$$

3.2. Regularized product expression. We first recall the regularized product expression of the Riemann zeta function.

Lemma 3.5 ([De]). For $\operatorname{Re}(z) > 1$ and $\operatorname{Re}(s) > 1$, we have

$$(3.1) \quad \sum_{\rho \in R} (z - \rho)^{-s} = z^{-s} + (z-1)^{-s} - \sum_{n=0}^{\infty} (z+2n)^{-s} - \frac{1}{\Gamma(s)} \sum_{k=0}^{\infty} \sum_{p:\text{prime}} \frac{\log p}{p^{nz}} (\log p^n)^{s-1},$$

where R denotes the set of the non-trivial zeros of the Riemann zeta function. $\square$

This relation follows by applying Weil's explicit formula. For a proof, see [De]. From the lemma we see that the function $\sum_{\rho \in R} (z - \rho)^{-s}$ has a meromorphic continuation to $s \in \mathbb{C}$ and the Riemann zeta function has the regularized product expression

$$\prod_{\rho \in R} \left(\frac{s - \rho}{2\pi} \right) = \frac{s}{2\pi} \cdot \frac{s-1}{2\pi} \cdot \prod_{n=0}^{\infty} \left(\frac{s+2n}{2\pi} \right)^{-1} \cdot \zeta(s).$$

Further, using properties of the Hurwitz zeta function and the gamma function, we have

$$\prod_{\rho \in R} \left(\frac{s - \rho}{2\pi} \right) = 2^{-1/2} (2\pi)^{-2} \cdot s(s-1) \cdot \pi^{-s/2} \cdot \Gamma\left(\frac{s}{2}\right) \cdot \zeta(s).$$

We remark that this function is invariant under $s \leftrightarrow 1-s$.

Next we describe the regularized product expression of the higher Riemann zeta function as follows.

Theorem 3.6. *Let Λ be a regularizable sequence. Then the higher Riemann zeta function $Z(s, \Lambda)$ has the following regularized product expression:*

$$(3.2) \quad \prod_{\lambda \in \Lambda, \rho \in R} \left(\frac{s + \lambda - \rho}{2\pi} \right) = \prod_{\lambda \in \Lambda} \left(\frac{s + \lambda}{2\pi} \right) \cdot \prod_{\lambda \in \Lambda} \left(\frac{s - 1 + \lambda}{2\pi} \right) \\ \cdot \prod_{\lambda \in \Lambda, n \geq 0} \left(\frac{s + \lambda + 2n}{2\pi} \right)^{-1} \cdot Z(s, \Lambda).$$

Proof. From (3.1), we see that

$$\sum_{\lambda \in \Lambda} \sum_{\rho \in R} (z + \lambda - \rho)^{-s} = \sum_{\lambda \in \Lambda} (z + \lambda)^{-s} + \sum_{\lambda \in \Lambda} (z + \lambda - 1)^{-s} - \sum_{\lambda \in \Lambda} \sum_{n=0}^{\infty} (z + \lambda + 2n)^{-s} \\ - \frac{1}{\Gamma(s)} \sum_{\lambda \in \Lambda} \sum_{n=1}^{\infty} \sum_p \frac{\log p}{p^{n(z+\lambda)}} (\log p^n)^{s-1},$$

for $\text{Re}(z) > 1$ and large $\text{Re}(s)$. We observe that the right-hand side is now meromorphic for any $s \in \mathbb{C}$. Therefore the dotted product $\prod_{\lambda, \rho} \{(z + \lambda - \rho)/2\pi\}$ exists. Further we obtain the equation (3.2) by using the relation

$$\log Z(z, \Lambda) = \sum_{\lambda \in \Lambda} \sum_p \sum_{n=1}^{\infty} \frac{1}{np^{n(z+\lambda)}} = \text{CT}_{s=0} \frac{1}{s} \left\{ \frac{(2\pi)^s}{\Gamma(s)} \sum_{\lambda \in \Lambda} \sum_{n=1}^{\infty} \sum_p \frac{\log p}{p^{n(z+\lambda)}} (\log p^n)^{s-1} \right\}.$$

This completes the proof. $\square$

3.3. Semi-lattice and functional equation. Let $\Omega := \{n_1\omega_1 + \cdots + n_r\omega_r \mid n_j \in \mathbb{Z}_{\geq 0}\}$ with $\text{Re}(\omega_j) > 0$. Then it is seen that Ω is a regularizable sequence. Now, we consider the higher Riemann zeta function for the semi-lattice Ω .

Definition 3.7. Define the higher Riemann zeta function of the weight $\omega = (\omega_1 \cdots \omega_r)$ by

$$Z(s, \omega) := Z(s, \Omega) = \prod_{n_1, \dots, n_r=0}^{\infty} \prod_{p: \text{prime}} (1 - p^{-(s+n_1\omega_1+\cdots+n_r\omega_r)})^{-1}.$$

We see that this product absolutely for $\operatorname{Re}(s) > 1$ and $Z(s, \omega)$ has an analytic continuation as a meromorphic function on $s \in \mathbb{C}$. Obviously, this is invariant under the arrangement of ω_j 's.

Proposition 3.8. *The higher Riemann zeta function $Z(s, \omega)$ defined for $\operatorname{Re}(s) > 1$ has a meromorphic continuation to the whole complex plane and satisfies*

(3.3)

$$\begin{aligned} Z(s, (\omega_1, \dots, \omega_{r-1}, \omega_r)) &= Z(s, (\omega_1, \dots, \omega_{r-1})) \cdot Z(s + \omega_r, (\omega_1, \dots, \omega_{r-1}, \omega_r)), \\ Z(s, \omega) \times \prod_{\substack{1 \leq k_1 < \dots < k_j \leq r \\ 1 \leq j \leq r}} Z(s + \omega_{k_1} + \dots + \omega_{k_j}, \omega)^{(-1)^j} &= \zeta(s). \end{aligned}$$

Proof. We observe that

$$\begin{aligned} Z(s, (\omega_1, \dots, \omega_{r-1}, \omega_r)) &= \prod_{n=0}^{\infty} Z(s + n\omega_r, (\omega_1, \dots, \omega_{r-1})) \\ &= Z(s, (\omega_1, \dots, \omega_{r-1})) \cdot \sum_{n=0}^{\infty} Z(s + n\omega_r + \omega_r, (\omega_1, \dots, \omega_{r-1})) \\ &= Z(s, (\omega_1, \dots, \omega_{r-1})) \cdot Z(s + \omega_r, (\omega_1, \dots, \omega_{r-1}, \omega_r)). \end{aligned}$$

Thus the relation (3.3) holds. By using (3.3) repeatedly, we have

$$\begin{aligned} \zeta(s) &= Z(s, (\omega_1)) \cdot Z(s + \omega_1, (\omega_1))^{-1} \\ &= Z(s, (\omega_1, \omega_2)) Z(s + \omega_2, (\omega_1, \omega_2))^{-1} \\ &\quad \cdot Z(s + \omega_1, (\omega_1, \omega_2))^{-1} Z(s + \omega_1 + \omega_2, (\omega_1, \omega_2)) \\ &= \dots = Z(s, \omega) \times \prod_{\substack{1 \leq k_1 < \dots < k_j \leq r \\ 1 \leq j \leq r}} Z(s + \omega_{k_1} + \dots + \omega_{k_j}, \omega)^{(-1)^j}. \end{aligned}$$

This completes the proof. $\square$

Next we show the regularized product expression of $Z(s, \omega)$ as follows.

Theorem 3.9. *We have*

$$\begin{aligned} &\prod_{n_1, \dots, n_r \geq 0, \rho \in R} \left(\frac{s + n_1\omega_1 + \dots + n_r\omega_r - \rho}{2\pi} \right) \\ &= \exp \left\{ \left(-\frac{B_r(s, \omega)}{r!} - \frac{B_r(s-1, \omega)}{r!} + \frac{B_{r+1}(s, (2, \omega_1, \dots, \omega_r))}{(r+1)!} \right) \log(2\pi) \right\} \\ &\quad \times \Gamma(s, \omega)^{-1} \cdot \Gamma(s-1, \omega)^{-1} \cdot \Gamma(s, (2, \omega_1, \dots, \omega_r)) \cdot Z(s, \omega). \end{aligned}$$

Here $B_n(s, \omega)$ is the multiple Bernoulli polynomial and $\Gamma(s, \omega)$ is the multiple gamma function.

Proof. We note that

$$\begin{aligned} &\prod_{n_1, \dots, n_r=0}^{\infty} \left(\frac{z + n_1\omega_1 + \dots + n_r\omega_r}{2\pi} \right) \\ &= (2\pi)^{-\zeta(0, z, \omega)} \prod_{n_1, \dots, n_r=0}^{\infty} (z + n_1\omega_1 + \dots + n_r\omega_r). \end{aligned}$$

Then, the result follows from Theorem 3.6. $\square$

From this theorem, we see that the function

$$\hat{Z}(s, \omega) := \exp \left\{ \left(-\frac{B_r(s, \omega)}{r!} - \frac{B_r(s-1, \omega)}{r!} + \frac{B_{r+1}(s, (2, \omega_1, \dots, \omega_r))}{(r+1)!} \right) \log(2\pi) \right\} \\ \times \Gamma(s, \omega)^{-1} \cdot \Gamma(s-1, \omega)^{-1} \cdot \Gamma(s, (2, \omega_1, \dots, \omega_r)) \cdot Z(s, \omega)$$

is an entire function of order $r+1$ with zeros at $s = \rho - n_1\omega_1 - \dots - n_r\omega_r$. Further $\hat{Z}(s, \omega)$ satisfies the relation

$$\hat{Z}(s, (\omega_1, \dots, \omega_{r-1}, \omega_r)) = \hat{Z}(s, (\omega_1, \dots, \omega_{r-1})) \cdot \hat{Z}(s + \omega_r, (\omega_1, \dots, \omega_{r-1}, \omega_r)).$$

Finally, we give the functional equation of the higher Riemann zeta function $Z(s, \omega)$ as follows.

Theorem 3.10. *Define the function $\Lambda(s, \omega)$ by*

$$\Lambda(s, \omega) := \hat{Z}(s, \omega) \cdot \hat{Z}(1 + \omega_1 + \dots + \omega_r - s, \omega)^{(-1)^{r+1}}.$$

Then we have

(3.4)

$$\Lambda(s, (\omega_1, \dots, \omega_{r-1}, \omega_r)) = \Lambda(s, (\omega_1, \dots, \omega_{r-1})) \cdot \Lambda(s + \omega_r, (\omega_1, \dots, \omega_{r-1}, \omega_r)).$$

Moreover, $\Lambda(s, \omega)$ satisfies the functional equation

$$(3.5) \quad \Lambda(s, \omega) \times \prod_{\substack{1 \leq k_1 < \dots < k_j \leq r \\ 1 \leq j \leq r}} \Lambda(s + \omega_{k_1} + \dots + \omega_{k_j}, \omega)^{(-1)^j} = 1.$$

Proof. We observe (3.3) and

$$\frac{\hat{Z}(1 + \omega_1 + \dots + \omega_{r-1} + \omega_r - s, (\omega_1, \dots, \omega_{r-1}, \omega_r))}{\hat{Z}(1 + \omega_1 + \dots + \omega_{r-1} + \omega_r - (s + \omega_r), (\omega_1, \dots, \omega_{r-1}, \omega_r))} = \frac{\hat{Z}(1 + \omega_1 + \dots + \omega_{r-1} - s, (\omega_1, \dots, \omega_{r-1}))}{\hat{Z}(1 + \omega_1 + \dots + \omega_{r-1} - s, (\omega_1, \dots, \omega_{r-1}))}.$$

Then the equation (3.4) follows immediately.

Next, using (3.4) repeatedly, we have

$$\Lambda(s, \omega) \times \prod_{\substack{1 \leq k_1 < \dots < k_j \leq r \\ 1 \leq j \leq r}} \Lambda(s + \omega_{k_1} + \dots + \omega_{k_j}, \omega)^{(-1)^j} = \hat{\zeta}(s) \cdot \hat{\zeta}(1 - s)^{-1} = 1,$$

where

$$\hat{\zeta}(s) := \prod_{\rho \in R} \left(\frac{s - \rho}{2\pi} \right) = 2^{-1/2} (2\pi)^{-2} \cdot s(s-1) \cdot \pi^{-s/2} \cdot \Gamma\left(\frac{s}{2}\right) \cdot \zeta(s).$$

Hence the theorem follows. $\square$

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