A THEOREM ON REFLEXIVE LARGE RANK OPERATOR SPACES

LIFENG DING

(Communicated by David R. Larson)

ABSTRACT. If every nonzero operator in an n -dimensional operator space $\mathbb{S}$ has rank $\geq 2n$, then $\mathbb{S}$ is reflexive.

Let $\mathbb{V}$ be a vector space over a field $\mathbb{F}$, and let $\mathcal{L}(\mathbb{V})$ be the space of all linear operators on $\mathbb{V}$. Suppose $\mathbb{S}$ is a finite-dimensional subspace of $\mathcal{L}(\mathbb{V})$. The *reflexive closure* of $\mathbb{S}$ is given by

$$\text{ref } \mathbb{S} = \{t \in \mathcal{L}(\mathbb{V}) : tx \in \mathbb{S}x \text{ for all } x \in \mathbb{V}\},$$

where $\mathbb{S}x = \{sx : s \in \mathbb{S}\}$, which is called the *cyclic subspace*. We note that every element t of $\text{ref } \mathbb{S}$ interpolates $\mathbb{S}$, or t is locally in $\mathbb{S}$. It is obvious that $\mathbb{S} \subseteq \text{ref } \mathbb{S}$. If $\mathbb{S} = \text{ref } \mathbb{S}$, then $\mathbb{S}$ is said to be *reflexive*. The reflexivity problem was originally introduced as a topological notion by P. Halmos to describe those operator algebras that are determined by their closed invariant subspace lattices of a Hilbert space. Loginov and Sulman extended reflexivity to include operator subspaces which are not necessarily algebras. The purely algebraic version of reflexivity and a unified approach was introduced by Hadwin [4, 5]. When the operator subspace is finite dimensional the topological version and the algebraic version of reflexivity coincide. The reflexivity problem is far from being complete. A historical account can be found in [1, 2, 4, 5, 6, 8].

A well-known result of Larson [6] asserts that for the finite-dimensional $\mathbb{S}$, $\text{ref } \mathbb{S} = \mathbb{S} + \text{ref } \mathbb{S}_F$, where $\mathbb{S}_F$ is the collection of all finite rank operators in $\mathbb{S}$. Hence if $\mathbb{S}_F = \{0\}$, then $\mathbb{S}$ is reflexive. In [2, 3] we further find that if the rank of each nonzero operator in $\mathbb{S}$ is large enough, then $\mathbb{S}$ is reflexive. For example, in [2] we proved that if the dimension of $\mathbb{S}$ is n , and every nonzero operator of $\mathbb{S}$ has rank $> 2n^2 - n$, then $\mathbb{S}$ is reflexive. This result was improved to that if every nonzero operator of $\mathbb{S}$ has rank $> n^2$, then $\mathbb{S}$ is reflexive [3]. In this paper we lower the rank to $2n$. This is achieved by making full use of Proposition 1.1 in [3] to prove a generalized version of Theorem 2.1 in [3] and using Theorem 1.2 in [3].

A vector x is a *separating vector* for $\mathbb{S}$ if the map $\mathbb{S} \rightarrow \mathbb{S}x$ by sending s to sx is injective. A linear subspace $\mathbb{M}$ of $\mathbb{V}$ is said to be a *separating subspace* for $\mathbb{S}$ if the only member s of $\mathbb{S}$ satisfying $s(\mathbb{M}) = \{0\}$ is $s = 0$. It is easy to see that if $\mathbb{S}$ has a separating vector x , then any subspace containing x is a separating subspace for $\mathbb{S}$. However, if the dimension of a separating subspace is greater than 1, then $\mathbb{S}$ may

---

Received by the editors May 2, 2001 and, in revised form, November 8, 2004.

2000 *Mathematics Subject Classification*. Primary 47L05; Secondary 15A04.

Key words and phrases. Reflexive operator space, separating vector.

©2006 American Mathematical Society
 Reverts to public domain 28 years from publication

not have a separating vector. Trivially, $\mathbb{V}$ is a separating subspace for $\mathbb{S}$. Of course, if $\mathbb{M}$ is a separating subspace for $\mathbb{S}$, then any subspace containing $\mathbb{M}$ is a separating subspace for $\mathbb{S}$. In [3] we proved the following theorem:

Theorem 1 (Theorem 1.2, [3]). *Suppose the linear subspace $\mathbb{S} \subseteq \mathcal{L}(\mathbb{V})$ has a separating vector x and a separating subspace $\mathbb{M}$ satisfying*

- (1) $\dim \mathbb{S} < \text{cardinality of the field } \mathbb{F}$,
- (2) $\mathbb{S}x \cap \mathbb{S}(\mathbb{M}) = \{0\}$.

Then $\mathbb{S}$ is reflexive.

Proposition 2 (Proposition 1.1, [3]). *Suppose $\mathbb{S}$ is a linear subspace of $\mathcal{L}(\mathbb{V})$, and the dimension of $\mathbb{S}$ is less than the cardinality of $\mathbb{F}$. Let x be a separating vector for $\mathbb{S}$, and let $\mathbb{W}$ be a linear subspace of $\mathbb{V}$ satisfying $\mathbb{S}x \cap \mathbb{W} = \{0\}$. Then for each vector $y \in \mathbb{V}$ there is a scalar $\lambda \in \mathbb{F}$ such that $y + \lambda x$ is a separating vector for $\mathbb{S}$ and $\mathbb{S}(y + \lambda x) \cap \mathbb{W} = \{0\}$.*

The proof of the proposition is obtained by using the mapping $\phi = E_x^{-1} \cdot P \cdot E_y$ on $\mathbb{S}$ where E_y is an evaluation map, and P is a projection with range $\mathbb{S}x$, and the fact that the number of different eigenvalues of ϕ is less than the cardinality of the field $\mathbb{F}$. The next Theorem 3 is a generalized version of Theorem 2.1 in [3].

Theorem 3. *Suppose $\mathbb{V}$ is a vector space over a field $\mathbb{F}$. Let $\mathbb{W}$ be a finite-dimensional linear subspace of $\mathbb{V}$, and let $\mathbb{S}$ be an n -dimensional linear subspace of $\mathcal{L}(\mathbb{V})$, where n is less than the cardinality of the field $\mathbb{F}$. If every nonzero operator in $\mathbb{S}$ has rank greater than or equal to $n + \dim \mathbb{W}$, then $\mathbb{S}$ has a separating vector x such that $\mathbb{S}x \cap \mathbb{W} = \{0\}$.*

Proof. We proceed by induction on the dimension n . For $n = 1$, $\mathbb{S} = \text{span}\{s\}$, and rank of $s \geq 1 + \dim \mathbb{W}$. Choose a vector $y \in \text{range of } s$ and $y \notin \mathbb{W}$. Let $sx = y$. Then x separates $\mathbb{S}$ and $\mathbb{S}x \cap \mathbb{W} = \{0\}$.

Assume the statement is true for each $(n-1)$ -dimensional subspace of $\mathcal{L}(\mathbb{V})$. We consider n -dimensional $\mathbb{S}$. Let $s_1, \dots, s_n$ be a basis of $\mathbb{S}$. Denote by $\mathbb{S}'$ the linear span of $\{s_1, \dots, s_{n-1}\}$. By the induction assumption we get a separating vector x for $\mathbb{S}'$ and $\mathbb{S}'x \cap \mathbb{W} = \{0\}$.

If $s_n x \notin \text{linear span}\{\mathbb{S}'x, \mathbb{W}\}$, then for any scalars $\alpha_1, \dots, \alpha_{n-1}, \alpha_n \in \mathbb{F}$, with $\alpha_1 s_1 x + \dots + \alpha_{n-1} s_{n-1} x + \alpha_n s_n x \in \mathbb{W}$, we have $\alpha_n s_n x = z - (\alpha_1 s_1 x + \dots + \alpha_{n-1} s_{n-1} x)$ for some $z \in \mathbb{W}$. Since $s_n x \notin \text{linear span}\{\mathbb{S}'x, \mathbb{W}\}$, $\alpha_n s_n x = 0$, and $z - (\alpha_1 s_1 x + \dots + \alpha_{n-1} s_{n-1} x) = 0$. It follows that $\alpha_n = 0$, and $z = 0$ and $\alpha_1 s_1 x + \dots + \alpha_{n-1} s_{n-1} x = 0$, since $\mathbb{S}'x \cap \mathbb{W} = \{0\}$. Then $\alpha_1 = \dots = \alpha_{n-1} = 0$. This proves that $\mathbb{S}x \cap \mathbb{W} = \{0\}$, and $s_1 x, \dots, s_{n-1} x, s_n x$ are linearly independent, and hence x is a separating vector for $\mathbb{S}$. The proof of the theorem is done in this case. So we need only consider the case that $s_n x \in \text{linear span}\{\mathbb{S}'x, \mathbb{W}\}$.

Let $s_n x = (\alpha_1 s_1 x + \dots + \alpha_{n-1} s_{n-1} x) + w$, for some scalars $\alpha_1, \dots, \alpha_{n-1} \in \mathbb{F}$, and $w \in \mathbb{W}$. Replacing s_n by $s'_n = s_n - \alpha_1 s_1 - \dots - \alpha_{n-1} s_{n-1}$, we may assume $s_n x \in \mathbb{W}$.

Since rank of $s_n \geq n + \dim \mathbb{W}$, there is $y \in \mathbb{V}$ such that $s_n y \notin \text{linear span}\{\mathbb{S}'x, \mathbb{W}\}$. Note that $\mathbb{S}'x \cap \mathbb{W} = \{0\}$, so this also implies that $\mathbb{S}'x \cap \text{linear span}\{s_n y, \mathbb{W}\} = \{0\}$. By Proposition 2, there is scalar λ so that $y + \lambda x$ separates $\mathbb{S}'$, and $\mathbb{S}'(y + \lambda x) \cap \text{linear span}\{s_n y, \mathbb{W}\} = \{0\}$. We next claim $y + \lambda x$ is a separating vector for $\mathbb{S}$.

Let $t \in \mathbb{S}'$ and $\alpha \in \mathbb{F}$ with $(t + \alpha s_n)(y + \lambda x) = 0$. Then

$$t(y + \lambda x) = -\alpha s_n(y + \lambda x) = -\alpha s_n y - \alpha \lambda s_n x,$$

where $\alpha \lambda s_n x \in \mathbb{W}$. By the disjointness of $\mathbb{S}'(y + \lambda x)$ and linear span $\{s_n y, \mathbb{W}\}$, we have $t(y + \lambda x) = 0$, and $\alpha s_n y = -\alpha \lambda s_n x$. Hence $t = 0$, and $\alpha = 0$, since $s_n y \notin \mathbb{W}$. This completes the proof of the claim.

Finally, we show that $\mathbb{S}(y + \lambda x) \cap \mathbb{W} = \{0\}$. Indeed, suppose there are scalars $\alpha_1, \dots, \alpha_n$ satisfying

$$\alpha_1 s_1(y + \lambda x) + \dots + \alpha_{n-1} s_{n-1}(y + \lambda x) + \alpha_n s_n(y + \lambda x) \in \mathbb{W}.$$

Then $\alpha_1 s_1(y + \lambda x) + \dots + \alpha_{n-1} s_{n-1}(y + \lambda x) + \alpha_n s_n(y + \lambda x) = z$ for some $z \in \mathbb{W}$. It follows that

$$-\alpha_n s_n y = (\alpha_1 s_1(y + \lambda x) + \dots + \alpha_{n-1} s_{n-1}(y + \lambda x)) + (\alpha_n \lambda s_n x - z).$$

Since $\alpha_n \lambda s_n x - z \in \mathbb{W}$ and $s_n y \notin$ linear span of $\{\mathbb{S}'x, \mathbb{W}\}$, we conclude that $\alpha_n s_n y = 0$. It follows that $\alpha_n = 0$. Again, the disjointness of $\mathbb{S}'(y + \lambda x) \cap \mathbb{W} = \{0\}$ implies that $z = 0$, and $\alpha_1 s_1(y + \lambda x) + \dots + \alpha_{n-1} s_{n-1}(y + \lambda x) = 0$. Therefore,

$$\alpha_1 = \dots = \alpha_{n-1} = 0.$$

□

Before stating our next result we set the notation below. If $\mathbb{M}$ is a subspace of $\mathbb{V}$, we write $\mathbb{S}(\mathbb{M})$ for the linear span of $\{s(\mathbb{M}): s \in \mathbb{S}\}$. We also assume that the cardinality of the field $\mathbb{F}$ is greater than the dimension of $\mathbb{S}$. Combining Theorem 3 and Theorem 1, we immediately have the following conclusion.

Theorem 4. *Let $\mathbb{V}$ be a vector space over a field $\mathbb{F}$ and $\mathbb{S}$ an n -dimensional subspace of $\mathcal{L}(\mathbb{V})$. Suppose $\mathbb{M}$ is a separating subspace for $\mathbb{S}$ and $\mathbb{S}(\mathbb{M})$ is finite dimensional. If every nonzero operator in $\mathbb{S}$ has rank greater than or equal to $n + \dim \mathbb{S}(\mathbb{M})$, then $\mathbb{S}$ is reflexive.*

Proof. By Theorem 3 there is a separating vector x for $\mathbb{S}$ such that $\mathbb{S}x \cap \mathbb{S}(\mathbb{M}) = \{0\}$. The conclusion follows immediately from Theorem 1. □

In [2, 3] we actually obtained several separating subspaces. For example, in Theorem 2.3 [3] we assume every nonzero operator in $\mathbb{S}$ has rank greater than n^2 . Then $\mathbb{S}$ has a separating vector x . Let $\mathbb{W}$ be a vector space complement of $\mathbb{S}x$ in $\mathbb{V}$. Then $\mathbb{U} = \{u \in \mathbb{V} : \mathbb{S}u \subseteq \mathbb{W}\}$ is a separating subspace for $\mathbb{S}$. Theorem 4 opens a door to lower the rank requirement. A much better improvement of lowering the rank is ready now. Before we state the improved result we give a corollary of Theorem 3.

Corollary 5. *Let $\mathbb{V}$ be a vector space over a field $\mathbb{F}$ and suppose $\mathbb{S}$ is an n -dimensional linear subspace of $\mathcal{L}(\mathbb{V})$, where n is less than the cardinality of the field $\mathbb{F}$. Then if every nonzero operator in $\mathbb{S}$ has rank $\geq n$, then $\mathbb{S}$ has a separating vector.*

Proof. In Theorem 3 we choose $\mathbb{W} = \{0\}$. The proof follows. □

Remark 6. This corollary recovers Theorem 2.1 of [3].

Theorem 7. *Let $\mathbb{V}$ be a vector space over a field $\mathbb{F}$. Suppose $\mathbb{S}$ is an n -dimensional linear subspace of $\mathcal{L}(\mathbb{V})$, where n is less than the cardinality of the field $\mathbb{F}$. Then, if every nonzero operator in $\mathbb{S}$ has rank $\geq 2n$, then $\mathbb{S}$ is reflexive.*

Proof. By Corollary 5 there is a separating vector x for $\mathbb{S}$. Then the cyclic subspace $\mathbb{S}x$ has dimension n . Note that the separating vector x generates a one-dimensional separating subspace for $\mathbb{S}$, and this matches the assumption of $\mathbb{S}(\mathbb{M})$ of Theorem 4. Hence the proof is completed. $\square$

Remark 8. In the proof we see that if the rank $\geq 2n$, then we obtain a disjoint pair of separating vectors which provides an example for the main result Theorem 2.1 in [2].

Remark 9. The contrapositive of Corollary 5 states that for n linear operators $s_1, \dots, s_n$ on $\mathbb{V}$, if $s_1u, \dots, s_nu$ are linearly dependent for every vector $u \in \mathbb{V}$, then there are scalars $\alpha_1, \dots, \alpha_n$, not all zero, such that $s = \alpha_1s_1 + \dots + \alpha_ns_n$ has rank $\leq n - 1$. As pointed out in Remark 2.2 of [3] this is Aupetit's well-known improvement of Kaplansky's lemma which states that if a linear operator on a linear space is locally algebraic, and the degrees of the related polynomials form a bounded set, then the operator is algebraic.

Remark 10. After this paper was accepted, J. Li and Z. Pan [7] showed that $2n$ can be reduced to $2n - 1$ in Theorem 6 when the scalar field is $\mathbb{C}$.

REFERENCES

- [1] E. A. Azoff, *On finite rank operators and preannihilators*, Memoirs Amer. Math. Soc. **357** (1986). MR0858467 (88a:47041)
- [2] L. Ding, *Separating vectors and reflexivity*, Lin. Alg. Appl. **174** (1992), 37-52. MR1176449 (94a:47075)
- [3] L. Ding, *On a pattern of reflexive operator spaces*, Proc. Amer. Math. Soc. **124** (1996), 3101-3108. MR1343689 (97h:47039)
- [4] D. Hadwin, *Algebraically reflexive linear transformations*, Lin. Multilin. Alg. **14** (1983), 225-233. MR0718951 (85e:47003)
- [5] D. Hadwin, *A general view of reflexivity*, Trans. Amer. Math. Soc. **344** (1994), 325-360. MR1239639 (95f:47071)
- [6] D. R. Larson, *Reflexivity, algebraic reflexivity, and linear interpolation*, Amer. J. Math. **110** (1988), 283-299. MR0935008 (89d:47096)
- [7] J. Li and Z. Pan, *Reflexivity and hyperreflexivity of operator spaces*, Math. Anal. Appl. **279** (2003), 210-215. MR1970501 (2004a:47001)
- [8] H. Radjavi and P. Rosenthal, *Invariant Subspaces*, Springer-Verlag, 1973. MR0367682 (51:3924)

DEPARTMENT OF MATHEMATICS AND STATISTICS, GEORGIA STATE UNIVERSITY, ATLANTA, GEORGIA 30303-3083

E-mail address: matlfd@panther.gsu.edu