THE REAL RANK ZERO PROPERTY OF CROSSED PRODUCT

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ABSTRACT. Let A be a unital C^* -algebra, and let (A, G, α) be a C^* -dynamical system with G abelian and discrete. In this paper, we introduce the continuous affine map R from the trace state space $T(A \times_\alpha G)$ of the crossed product $A \times_\alpha G$ to the α -invariant trace state space $T(A)_{\alpha^*}$ of A . If $A \times_\alpha G$ is of real rank zero and $\hat{G}$ is connected, we have proved that R is homeomorphic. Conversely, if R is homeomorphic, we also get some properties and real rank zero characterization of $A \times_\alpha G$. In particular, in that case, $A \times_\alpha G$ is of real rank zero if and only if each unitary element in $A \times_\alpha G$ with the form $u_A \prod_{i=1}^n x_i^* y_i^* x_i y_i$ can be approximated by the unitary elements in $A \times_\alpha G$ with finite spectrum, where $u_A \in U_0(A)$, $x_i, y_i \in C_c(G, A) \cap U_0(A \times_\alpha G)$, and if moreover A is a unital inductive limit of the direct sums of non-elementary simple C^* -algebras of real rank zero, then the u_A above can be cancelled.

1. INTRODUCTION

Let A be a unital C^* -algebra, and let A_{sa} be the set consists of all self-adjoint elements in A . The real rank of A is the smallest integer, $RR(A)$, such that for each n -tuple $(x_1, x_2, \dots, x_n)$ of elements in A_{sa} with $n \leq RR(A) + 1$, and every $\varepsilon > 0$, there is an n -tuple $(y_1, y_2, \dots, y_n)$ of elements in A_{sa} such that $\sum_{k=1}^n y_k^* y_k$ is invertible and $\|\sum_{k=1}^n (x_k - y_k)^2\| < \varepsilon$. In particular, $RR(A) = 0$ if and only if the invertible self-adjoint elements are dense in A_{sa} , which is equivalent to the fact that the self-adjoint elements with finite spectrum are dense in A_{sa} . For a non-unital C^* -algebra A , the real rank of A is $RR(\tilde{A})$, where $\tilde{A}$ is the unitization of A . The real rank zero property has been widely studied by many authors, for example L. Brown, H. Lin, G. K. Pedersen, S. Zhang and so on. Let A be a C^* -algebra, let G be a locally compact Hausdorff group, and let (A, G, α) be a C^* -dynamical system. There are also many efforts that have been made to understand the relation between the real ranks of A and of $A \times_\alpha G$, in particular, between the real rank zero properties of A and $A \times_\alpha G$. For example, in [4], it was guessed that $RR(A \times_\alpha G) \leq \dim \hat{G} + RR(A)$ for G finite. In [3], by using the Rokhlin property, some statements equivalent to $RR(A \times_\alpha \mathbf{Z}) = 0$ were given for $A = M_{2^\infty}$, which were generalized later in [9] for A to be a UHF algebra. For a C^* -algebra A , a bounded functional τ of A is called a trace state, if $\lim_\lambda \tau(e_\lambda) = 1$, where

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$\{e_\lambda\}$ is one approximate unit of A , and $\tau(ab) = \tau(ba)$ for any $a, b \in A$. The trace state space of A , which is notated by $T(A)$, is the set consisting of all trace states of A . For a C^* -dynamical system (A, G, α) , α induces a canonical action α^* on $T(A)$: $(\alpha_t^*(\tau))(a) = \tau(\alpha_t(a))$ ($\forall a \in A, \tau \in T(A), t \in G$). A trace state $\tau \in T(A)$ is called α -invariant if $\alpha_t^*(\tau) = \tau$ ($\forall t \in G$). Let $T(A)_{\alpha^*}$ be the set consisting of all α -invariant trace states. If moreover G is discrete and A is unital, then A can be viewed as a unital subalgebra of $A \times_\alpha G$. Therefore for any trace state of $A \times_\alpha G$, we get a trace state of A by restriction, which is moreover α -invariant by direct computation, i.e. there is a canonical mapping $R : T(A \times_\alpha G) \rightarrow T(A)_{\alpha^*}$. Since the trace state space is one of Elliott's classification invariants for unital amenable C^* -algebras, we want to find how much the mapping R reflects the relation of the real rank zero properties of A and $A \times_\alpha G$. In this paper, for G abelian, if $A \times_\alpha G$ is of real rank zero, and $\hat{G}$ is connected, we prove that R is homeomorphic. Conversely, if R is homeomorphic, we will get some properties and real rank zero characterization of $A \times_\alpha G$. In particular, in that case, $A \times_\alpha G$ is of real rank zero if and only if each unitary element in $A \times_\alpha G$ with the form $u_A \prod_{i=1}^n x_i^* y_i^* x_i y_i$ can be approximated by the unitary elements in $A \times_\alpha G$ with finite spectrum, where $u_A \in U_0(A)$, $x_i, y_i \in C_c(G, A) \cap U_0(A \times_\alpha G)$, and if moreover A is a unital inductive limit of the direct sums of non-elementary simple C^* -algebras of real rank zero, then $A \times_\alpha G$ is of real rank zero if and only if each unitary element in $A \times_\alpha G$ with the form $\prod_{i=1}^n x_i^* y_i^* x_i y_i$ can be approximated by the unitary elements in $A \times_\alpha G$ with finite spectrum.

2. MAIN RESULTS

We should first recall some basic notations and definitions in K -theory which will play roles later. Let A be a unital C^* -algebra. For each integer k , we denote the unitary group of $M_k(A)$ by $U^k(A)$, and the subgroup of $U^k(A)$ consisting of all elements connected to the unit of $M_k(A)$ by $U_0^k(A)$. Viewing $U^k(A)$ ($U_0^k(A)$) as a subgroup of $U^{k+1}(A)$ ($U_0^{k+1}(A)$) by identifying $\text{diag}(u, 1)$ with u for any $u \in U^k(A)$ ($U_0^k(A)$), we let $U^\infty(A) = \lim_{k \rightarrow \infty} U^k(A)$ as a topological group with the inductive limit topology coming from the inclusion $U^k(A) \subseteq U^{k+1}(A)$, and similarly let $U_0^\infty(A) = \lim_{k \rightarrow \infty} U_0^k(A)$ as a topological group with the inductive limit topology coming from the inclusion $U_0^k(A) \subseteq U_0^{k+1}(A)$. For any $n \in \mathbf{N} \cup \{\infty\}$, we denote the commutator subgroup of $U^n(A)$ and $U_0^n(A)$ by $DU^n(A)$ and $DU_0^n(A)$ respectively, and let $\overline{DU^n(A)}$ ($\overline{DU_0^n(A)}$) be the closure of $DU^n(A)$ ($DU_0^n(A)$). For any fixed $n \in \mathbf{N} \cup \{\infty\}$, we let q^0 be the quotient map from $U_0^n(A)$ to $U_0^n(A)/\overline{DU_0^n(A)}$.

Let $T(A)$ be the trace state space of A , which is a compact Hausdorff space with the w^* topology. (In this paper, we always assume that $T(A) \neq \emptyset$.) Let $AffT(A)$ denote the space of continuous affine real-valued function on $T(A)$, which is a real Banach space with the standard function norm. Let $\eta : [0, 1] \mapsto U_0^n(A)$ ($n \in \mathbf{N} \cup \{\infty\}$) be a piecewise smooth path of unitary from 1; we define $\Delta_n^1(\eta) \in AffT(A)$ by

$$\Delta_n^1(\eta)(\omega) = \frac{1}{2\pi i} \int_0^1 \omega(\eta'(t)\eta(t)^*) dt, \quad \omega \in T(A).$$

The key observations (see [1], Lemma 3) are:

- (1) $\Delta_n^1(\eta)$ depends only on η up to homotopy with fixed endpoints, and
- (2) $\Delta_n^1(\eta_1 \eta_2) = \Delta_n^1(\eta_1) + \Delta_n^1(\eta_2)$ for any piecewise smooth paths η_i with $\eta_i(1) = 1$ ($i = 1, 2$).

It follows that Δ_n^1 defines a group homomorphism $\Delta_n^0 : \pi_1(U_0^n(A)) \rightarrow AffT(A)$, where $\pi_1(U_0^n(A))$ is the fundamental group of $U_0^n(A)$. For $n = \infty$, $\pi_1(U_0^\infty(A)) = K_0(A)$ by Bott periodicity, and Δ_∞^0 is just the well-known canonical map $\rho : K_0(A) \rightarrow AffT(A)$. Let q be the quotient map from $AffT(A)$ to $AffT(A)/\overline{\Delta_n^0(\pi_1(U_0^n(A)))}$. Since for any unital C^* -algebra B , $U_0(B)$ is generated algebraically by $\{e^{ix} : x \in B_{sa}\}$, for each unitary $u \in U_0^n(A)$, there is a piecewise smooth path η_u in $U_0^n(A)$ from 1 to u . Therefore for each $n \in \mathbf{N} \cup \{\infty\}$, we can define a group homomorphism:

$$\Delta_n : U_0^n(A) \mapsto AffT(A)/\overline{\Delta_n^0(\pi_1(U_0^n(A)))}, \quad \Delta_n(u) = q(\Delta_n^1(\eta_u)).$$

This homomorphism makes it possible to get the information of the real rank zero property of $A \times_\alpha G$, which is relative to $U_0(A \times_\alpha G)$, from the mapping $R : T(A \times_\alpha G) \rightarrow T(A)_{\alpha^*}$.

Lemma 1. *Let (A, G, α) be a C^* -dynamical system with G abelian and discrete. There is a canonical faithful expectation ϕ of $A \times_\alpha G$ onto A with $\phi(f) = f(0)$, where $f \in C_c(G, A)$ and 0 is the unit of G .*

Proof. By duality, we have a C^* -dynamical system $(A \times_\alpha G, \hat{G}, \hat{\alpha})$, where $\hat{\alpha}_\sigma(f)(t) = (t, \sigma)f(t)$ for every $f \in C_c(G, A)$, $t \in G, \sigma \in \hat{G}$. Then let $\phi : A \times_\alpha G \rightarrow A$ be $\phi(x) = \int_{\hat{G}} \hat{\alpha}_\sigma(x) d\mu(\sigma)$ ($\forall x \in A \times_\alpha G$), where μ is the canonical Haar probability on the compact group $\hat{G}$. Since G is discrete, $A \subseteq A \times_\alpha G$, and $A = \{x \in A \times_\alpha G : \hat{\alpha}_\sigma(x) = x \text{ for } \forall \sigma \in \hat{G}\}$. So the image of ϕ is included in A , and for $\forall a, b \in A, x \in A \times_\alpha G$,

$$\phi(axb) = \int_{\hat{G}} \hat{\alpha}_\sigma(axb) d\mu(\sigma) = a\phi(x)b, \quad \phi(a) = \int_{\hat{G}} \hat{\alpha}_\sigma(a) d\mu(\sigma) = a.$$

It is easy to see that $\phi(f) = f(0)$ for $f \in C_c(G, A)$.

By the definition of ϕ , ϕ is a faithful completely positive map, and so a faithful expectation from $A \times_\alpha G$ onto A .

Lemma 2. *Let (A, G, α) be a C^* -dynamical system with G abelian and discrete and A unital, and let $T(A)_{\alpha^*} = \{\tau \in T(A) : \tau \circ \alpha_t = \tau, \forall t \in G\}$. Then there are continuous affine maps $\Phi : T(A)_{\alpha^*} \rightarrow T(A \times_\alpha G)$ and $R : T(A \times_\alpha G) \rightarrow T(A)_{\alpha^*}$ such that $R \circ \Phi = id$ and the image of Φ is*

$$T(A \times_\alpha G)_{\hat{\alpha}^*} = \{\tau \in T(A \times_\alpha G) : \tau \circ \hat{\alpha}_\sigma = \tau, \forall \sigma \in \hat{G}\}.$$

Therefore Φ is an affine homeomorphism from $T(A)_{\alpha^}$ onto $T(A \times_\alpha G)_{\hat{\alpha}^*}$.*

Proof. Let i be the canonical inclusion of A in $A \times_\alpha G$, and $R(\omega) = \omega \circ i$ for each $\omega \in T(A \times_\alpha G)$. It is easy to see that $R(\omega) \in T(A)_{\alpha^*}$. By Lemma 1, let ϕ be the canonical faithful expectation from $A \times_\alpha G$ onto A , and let $\Phi(\tau) = \tau \circ \phi$ for each $\tau \in T(A)_{\alpha^*}$.

For each $f, g \in C_c(G, A) \subseteq A \times_\alpha G$, since

$$\phi(f * g) = (f * g)(0) = \int f(s)\alpha_s(g(-s))ds,$$

it follows that

$$\begin{aligned}
 \Phi(\tau)(f * g) &= \tau \circ \phi(f * g) = \int \tau(f(s)\alpha_s(g(-s)))ds \\
 &= \int \tau(\alpha_s(\alpha_{-s}(f(s))g(-s)))ds \\
 &= \int \tau(\alpha_{-s}(f(s))g(-s))ds \\
 &= \int \tau(g(-s)\alpha_{-s}(f(s)))ds \\
 &= \int \tau(g(s)\alpha_s(f(-s)))ds \\
 &= \tau \circ \phi(g * f) = \Phi(\tau)(g * f).
 \end{aligned}$$

So $\Phi(\tau) \in T(A \times_{\alpha} G)$, and it is easy to see $R \circ \Phi(\tau) = \tau$ ($\forall \tau \in T(A)_{\alpha_*}$). Since for each $\sigma_0 \in \hat{G}$, $\Phi(\tau) \circ \hat{\alpha}_{\sigma_0} = \tau \circ \phi \circ \hat{\alpha}_{\sigma_0}$ and $\phi \circ \hat{\alpha}_{\sigma_0}(x) = \int_{\hat{G}} \hat{\alpha}_{\sigma} \hat{\alpha}_{\sigma_0}(x) d\sigma = \phi(x)$ ($\forall x \in A \times_{\alpha} G$), it follows that $\Phi(\tau) \circ \hat{\alpha}_{\sigma} = \Phi(\tau)$ ($\forall \sigma \in \hat{G}$), i.e. $\Phi(\tau) \in T(A \times_{\alpha} G)_{\hat{\alpha}_*}$.

If $\omega \in T(A \times_{\alpha} G)$ with $\omega \circ \hat{\alpha}_{\sigma} = \omega$ ($\forall \sigma \in \hat{G}$), then $\Phi(R(\omega)) = \omega \circ i \circ \phi$. For any $t \in G$, $a \in A$, let $a\delta(t, \cdot)$ be the element in $C_c(G, A)$ with value a at t , 0 at any other $s \in G$ which is not t , then for every $f \in C_c(G, A)$, $\omega \circ i \circ \phi(f) = \omega(f(0)\delta(0, \cdot))$. Since for any $\sigma \in \hat{G}$,

$$\omega(a\delta(t, \cdot)) = \omega(\hat{\alpha}_{\sigma}(a\delta(t, \cdot))) = (t, \sigma)\omega(a\delta(t, \cdot)),$$

$\omega(a\delta(t, \cdot)) = 0$, if $t \neq 0$, and so $\omega(f(\cdot)) = \omega(f(0)\delta(0, \cdot)) = \omega \circ i \circ \phi(f(\cdot))$.

Therefore $\omega = \omega \circ i \circ \phi = \Phi(R(\omega)) \in \text{image of } \Phi$, i.e. the image of Φ is $\{\tau \in T(A \times_{\alpha} G) : \tau \circ \hat{\alpha}_{\sigma} = \tau \text{ for } \forall \sigma \in \hat{G}\}$.

Proposition 1. *Let (B, H, β) be a C^* -dynamical system with H connected and B of real rank zero. Then for any $\tau \in T(B)$, $h \in H$, $\tau \circ \beta_h = \tau$, therefore $T(B)_{\beta_*} = T(B)$.*

Proof. First we assume that B is unital, let $\omega \in T(B)$, and let p be a projection in B . Then the mapping $h \rightarrow \omega(\beta_h(p))$, $\forall h \in H$, is continuous on H . For any $h_0 \in H$, there is a neighborhood V_0 of h_0 such that $\|\beta_h(p) - p\| < 1$ for any $h \in V_0$, and so there is a unitary element u_h in B such that $\beta_h(p) = u_h p u_h^*$. Since ω is a trace, the mapping $h \rightarrow \omega(\beta_h(p))$ is locally constant on H . Let $U = \{h \in H \mid \omega(\beta_h(p)) = \omega(p)\}$; then, by the discussion above U is a non-empty clopen set. Therefore $U = H$ by the connectedness of H , i.e. for any $\omega \in T(B)$, projection $p \in B$, $\omega(\beta_h(p)) = \omega(p)$ ($\forall h \in H$). Since B is real rank zero, for any $b \in B$, b can be approximated by the linear combination of projections in B , and so $\omega(\beta_h(b)) = \omega(b)$ ($\forall h \in H$). This completes the proof in the unital case.

Now if B is non-unital, and we let $\tilde{B} = B + \mathbf{C}1$ be the unitization of B , then $\tilde{B}$ is of real rank zero by definition. For any $h \in H$, let $\tilde{\beta}_h$ be the automorphism of $\tilde{B}$ such that $\tilde{\beta}_h(b + \gamma 1) = \beta_h(b) + \gamma 1$. Then $\tilde{\beta} : H \rightarrow \text{Aut}(\tilde{B})$, $h \rightarrow \tilde{\beta}_h$, define a C^* -dynamical system $(\tilde{B}, H, \tilde{\beta})$. Therefore $T(\tilde{B})_{\tilde{\beta}_*} = T(\tilde{B})$. For any $\tau \in T(B)$ (the trace state of B), we define $\tilde{\tau}$ by $\tilde{\tau}(b + \gamma 1) = \tau(b) + \gamma$. It is easy to see $\tilde{\tau} \in T(\tilde{B})$ with $\tilde{\tau}|_B = \tau$. So $\tau \circ \beta_h = (\tilde{\tau}|_B) \circ \beta_h = (\tilde{\tau} \circ \tilde{\beta}_h)|_B = \tilde{\tau}|_B = \tau$, and so $T(B)_{\beta_*} = T(B)$.

Theorem 1. *Let (A, G, α) be a C^* -dynamical system, with A unital, G abelian and discrete, $\hat{G}$ connected. If $A \times_\alpha G$ is of real rank zero, then $T(A \times_\alpha G) = T(A \times_\alpha G)_{\hat{\alpha}^*}$, and the mapping $R : T(A \times_\alpha G) \rightarrow T(A)_{\alpha^*}$ defined in Lemma 2 is an affine homeomorphism with $R^{-1} = \Phi$.*

Proof. Let $(A \times_\alpha G, \hat{G}, \hat{\alpha})$ be the dual C^* -dynamical system of (A, G, α) . Then $T(A \times_\alpha G) = T(A \times_\alpha G)_{\hat{\alpha}^*}$ by Proposition 1. So, keeping the notations as in Lemma 2, Φ is an affine continuous bijection with $\Phi^{-1} = R$. This completes the proof of the theorem.

Note. Conversely, for an abelian group G and a C^* -dynamical system (A, G, α) with $R : T(A \times_\alpha G) \rightarrow T(A)_{\alpha^*}$ homomorphic, we may not get that $A \times_\alpha G$ is of real rank zero. The simplest trivial example is that $G = \{e\}$ and A is not of real rank zero. The following is another example for G non-trivial.

Example 1. Let G be a finite abelian group with unit e , let B be a unital C^* -algebra acting on a Hilbert space $\mathbf{H}$, let $A = C(G, B) = C(G) \otimes B$, and let (A, G, α) be a C^* -dynamical system with $(\alpha_s y)(t) = y(t - s)$, $\forall s, t \in G, y \in A$. Then $A \times_\alpha G = C(G, B) \times_\alpha G$. Let $\pi : B \rightarrow \mathbf{B}(l^2(G, \mathbf{H}))$, $(\pi(b)\xi)(t) = b(\xi(t))$; then π is faithful. Let $\{\delta_s | s \in G\}$ be the canonical basis of $l^2(G)$, and let $M_{|G|}$ be the matrix algebra with size $|G|$, where δ_s is the characterization function on G of $\{s\} \subseteq G$. We have the map $\rho : M_{|G|} \rightarrow \mathbf{B}(l^2(G))$, for $(x_{st}) \in M_{|G|}$,

$$\rho((x_{st}))(\sum_{s \in G} \beta_s \delta_s) = \sum_{s \in G} (\sum_{t \in G} x_{st} \beta_t) \delta_s, \text{ i.e. } \rho((x_{st}))(\delta_s) = \sum_{t \in G} x_{ts} \delta_t.$$

It is easy to see ρ is an isomorphism, and $\pi \otimes \rho : B \otimes M_{|G|} = M_{|G|}(B) \rightarrow \mathbf{B}(l^2(G, \mathbf{H})) \otimes \mathbf{B}(l^2(G)) = \mathbf{B}(l^2(G \times G, \mathbf{H}))$ is an injective homomorphism. By [12], 7.7.12, there is an injective homomorphism $\Phi : A \times_\alpha G \rightarrow \mathbf{B}(l^2(G, \mathbf{H})) \otimes \mathbf{B}(l^2(G)) = \mathbf{B}(l^2(G \times G, \mathbf{H}))$ such that $\text{Image}(\Phi) = \text{Image}(\pi \otimes \rho)$, and Φ is defined as follows: For every $b \in B, f, g \in C(G)$, and $Z(r, s) = bf(s-r)g(s) \in A \times_\alpha G = C(G, B) \times_\alpha G$, $\Phi(z) = \pi(b) \otimes v_{fg}$, where $v_{fg} \in \mathbf{B}(l^2(G))$, $v_{fg}(\eta) = \langle \eta, f \rangle g, \forall \eta \in l^2(G)$. Therefore, we have isomorphism $\Psi = (\pi \otimes \rho)^{-1} \circ \Phi : A \times_\alpha G \rightarrow M_{|G|}(B)$, and so $RR(A \times_\alpha G) = RR(B)$.

Let $z_t = \delta_e \otimes \delta_t \otimes b \in C(G \times G, B) \subseteq A \times_\alpha G$. Then $z_t(r, s) = b\delta_e(r)\delta_t(s) = b\delta_t(s-r)\delta_t(s)$, and then $\Phi(z_t) = \pi(b) \otimes v_{\delta_t \delta_t}$. Let $v_{\delta_t \delta_t} = \rho((x_{rs}))$ with $(x_{rs}) \in M_{|G|}$. Since $v_{\delta_t \delta_t}(\delta_s) = \langle \delta_s, \delta_t \rangle \delta_t = \delta_s(t)\delta_t$ and $\rho((x_{rs}))(\delta_s) = \sum_{r \in G} x_{rs} \delta_r, x_{rs} = \delta_r(t)\delta_s(t)$. So $(x_{rs}) = E_{tt}$, i.e. $\rho(E_{tt}) = v_{\delta_t \delta_t}$, and $\Phi(z_t) = (\pi \otimes \rho)(b \otimes E_{tt})$, where $E_{tt} \in M_{|G|}$ is the matrix with 1 at (t, t) position, and 0 at other positions. Therefore $\Psi(z_t) = b \otimes E_{tt} \in B \otimes M_{|G|} = M_{|G|}(B)$. Let $\phi : T(B) = T(M_{|G|}(B)) \rightarrow T(A)_{\alpha^*} = T(C(G) \otimes B)_{\alpha^*}$, for every $\tau \in T(B)$, $\delta_t \otimes b \in C(G) \otimes B = A$, $\phi(\tau)(\delta_t \otimes b) = \frac{\tau(b)}{|G|}$. Then ϕ is an affine homeomorphism.

Let Ψ^* be the affine map from $T(B) = T(M_{|G|}(B))$ to $T(A \times_\alpha G)$ induced by Ψ . Then Ψ^* is an affine homeomorphism, and for every $\tau \in T(B)$, $\delta_t \otimes b \in A$,

$$(R \circ \Psi^*)(\tau)(\delta_t \otimes b) = \Psi^*(\tau)(\delta_e \otimes \delta_t \otimes b) = \tau_{|G|}(\Psi(z_t)) = \tau_{|G|}(b \otimes E_{tt}) = \frac{\tau(b)}{|G|},$$

where $\tau_{|G|} \in T(M_{|G|}(B))$ is defined by $\tau \in T(B)$. So $R \circ \Psi^* = \phi$, i.e. we have the following commutative graph:

$$\begin{array}{ccc} T(B) = T(M_{|G|}(B)) & \xrightarrow{\Psi^*} & T(A \times_{\alpha} G) \\ \phi \downarrow & & R \downarrow \\ T(C(G) \otimes B)_{\alpha^*} & \xrightarrow{id} & T(A)_{\alpha^*} \end{array}$$

Therefore R is an affine homeomorphism, but if $RR(B)$ is not 0, then $RR(A \times_{\alpha} G) = RR(B)$ is also not 0.

Proposition 2. *Let (A, G, α) be a C^* -dynamical system with A unital. For every $\phi \in \text{Aff}(T(A)_{\alpha_*})$, $\psi \in \text{Aff}T(A)$, there are $a_{\phi}, a_{\psi} \in A_{sa}$, such that $\phi(\tau) = \tau(a_{\phi})$ ($\forall \tau \in T(A)_{\alpha_*}$), $\psi(\tau) = \tau(a_{\psi})$ ($\forall \tau \in T(A)$). Therefore the restriction mapping R_G from $\text{Aff}T(A)$ to $\text{Aff}(T(A)_{\alpha_*})$ is surjective.*

Proof. Let V, V_{α} be the real linear subspace of $(A_{sa})^* = (A^*)_{sa}$ which are linearly spanned by $T(A)$ and $T(A)_{\alpha_*}$ respectively; then $V_{\alpha} \subseteq V \subseteq (A_{sa})^*$. Let

$$\begin{aligned} W &= \{\tau \in (A^*)_{sa} : \tau(u^*au) = \tau(a), \forall a \in A, u \in U(A)\}, \\ W_{\alpha} &= \{\tau \in W : \tau(\alpha_t(a)) = \tau(a), \forall a \in A, t \in G\}. \end{aligned}$$

It is clear that $V \subseteq W$, $V_{\alpha} \subseteq W_{\alpha}$, and W, W_{α} are w^* -closed in $(A_{sa})^* = (A^*)_{sa}$. Let $\tau \in W$, $\rho \in W_{\alpha}$, $\tau = \tau_+ - \tau_-$, $\rho = \rho_+ - \rho_-$ be the Jordan decomposition of τ and ρ respectively; then

$$\tau_+, \tau_-, \rho_+, \rho_- \in A_+^*, \|\tau\| = \|\tau_+\| + \|\tau_-\|, \|\rho\| = \|\rho_+\| + \|\rho_-\|.$$

For any $u \in U(A)$, $t \in G$, it is easy to see that

$$\tau = \tau_+(u^* \cdot u) - \tau_-(u^* \cdot u), \quad \rho = \rho_+(u^* \cdot u) - \rho_-(u^* \cdot u),$$

and

$$\rho = \rho_+(\alpha_t(\cdot)) - \rho_-(\alpha_t(\cdot)).$$

Since $\|\tau_+\| = \|\tau_+(u^* \cdot u)\|$, $\|\tau_-\| = \|\tau_-(u^* \cdot u)\|$, $\|\rho_+\| = \|\rho_+(u^* \cdot u)\|$, $\|\rho_-\| = \|\rho_-(u^* \cdot u)\|$, and $\|\rho_+(\alpha_t(\cdot))\| = \|\rho_+\|$, $\|\rho_-\| = \|\rho_-(\alpha_t(\cdot))\|$, by the uniqueness of the Jordan decomposition, $\tau_+ = \tau_+(u^* \cdot u)$, $\tau_- = \tau_-(u^* \cdot u)$, $\rho_+ = \rho_+(u^* \cdot u)$, $\rho_- = \rho_-(u^* \cdot u)$, $\rho_+ = \rho_+ \circ \alpha_t$, $\rho_- = \rho_- \circ \alpha_t$. Therefore

$$\frac{\tau_+}{\|\tau_+\|}, \frac{\tau_-}{\|\tau_-\|} \in T(A), \quad \frac{\rho_+}{\|\rho_+\|}, \frac{\rho_-}{\|\rho_-\|} \in T(A)_{\alpha_*},$$

and so

$$\tau = \|\tau_+\| \frac{\tau_+}{\|\tau_+\|} - \|\tau_-\| \frac{\tau_-}{\|\tau_-\|} \in V, \quad \rho = \|\rho_+\| \frac{\rho_+}{\|\rho_+\|} - \|\rho_-\| \frac{\rho_-}{\|\rho_-\|} \in V_{\alpha},$$

i.e. $W \subseteq V$ and $W_{\alpha} \subseteq V_{\alpha}$. Therefore $W = V$ and $W_{\alpha} = V_{\alpha}$; as a consequence, V and V_{α} are w^* -closed in $(A^*)_{sa} = (A_{sa})^*$.

Let $\phi \in \text{Aff}(T(A)_{\alpha_*})$, and we define g to be the linear functional on V_{α} : $g(\tau) = \lambda\phi(\tau_1) - \mu\phi(\tau_2)$, if $\tau = \lambda\tau_1 - \mu\tau_2 \in V_{\alpha}$ with $\lambda, \mu \in \mathbf{R}_+ \cup \{0\}$, $\tau_1, \tau_2 \in T(A)_{\alpha_*}$. If $\tau = \lambda'\tau'_1 - \mu'\tau'_2$ is another decomposition with $\lambda', \mu' \in \mathbf{R}_+ \cup \{0\}$, $\tau'_1, \tau'_2 \in T(A)_{\alpha_*}$, then $\lambda\tau_1 + \mu'\tau'_2 = \lambda'\tau'_1 + \mu\tau_2$. Since $\lambda' + \mu = \lambda'\tau'_1(1) + \mu\tau_2(1) = \lambda\tau_1(1) + \mu'\tau'_2(1) = \lambda + \mu'$, $\frac{\lambda'}{\lambda' + \mu}\tau'_1 + \frac{\mu}{\lambda' + \mu}\tau_2 = \frac{\lambda}{\lambda + \mu'}\tau_1 + \frac{\mu'}{\lambda + \mu'}\tau'_2 \in T(A)_{\alpha_*}$. So

$$\frac{\lambda'}{\lambda' + \mu}\phi(\tau'_1) + \frac{\mu}{\lambda' + \mu}\phi(\tau_2) = \frac{\lambda}{\lambda + \mu'}\phi(\tau_1) + \frac{\mu'}{\lambda + \mu'}\phi(\tau'_2).$$

Therefore $\lambda\phi(\tau_1) - \mu\phi(\tau_2) = \lambda'\phi(\tau'_1) - \mu'\phi(\tau'_2)$, and so the definition of g is well defined. It is also easy to see g is linear on V_α . Let $\{\tau_i\}_{i \in \Lambda}$ be a bounded net in V_α with τ_i w^* -convergent to τ in V_α . If $g(\tau_i)$ is not convergent to $g(\tau)$, then there is a $\varepsilon > 0$ such that $\{i \in \Lambda : |g(\tau_i) - g(\tau)| \geq \varepsilon\} = s_\Lambda$ is a directed subset of Λ with the order of Λ . So $\{\tau_j\}_{j \in s_\Lambda}$ is a subnet of $\{\tau_i\}_{i \in \Lambda}$.

Let $\tau_i = (\tau_i)_+ - (\tau_i)_-$ be the Jordan decomposition of τ_i ; then

$$\|\tau_i\| = \|(\tau_i)_+\| + \|(\tau_i)_-\|.$$

Since $\{\|\tau_i\|, i \in \Lambda\}$ is bounded, both $\{\|(\tau_i)_+\|, i \in \Lambda\}$ and $\{\|(\tau_i)_-\|, i \in \Lambda\}$ are bounded, too. Since the bounded closed ball of $(A_{sa})^*$ is w^* compact, there are subnets $\{(\tau_k)_+\} \subseteq \{(\tau_j)_+ : j \in s_\Lambda\}$ and $\{(\tau_k)_-\} \subseteq \{(\tau_j)_- : j \in s_\Lambda\}$ with $(\tau_k)_+$ and $(\tau_k)_-$ w^* convergent to ρ_1 and ρ_2 in V_α , respectively. So $\tau = \rho_1 - \rho_2$, and $(\tau_k)_+(1) \rightarrow \rho_1(1)$, $(\tau_k)_-(1) \rightarrow \rho_2(1)$. Therefore $\frac{(\tau_k)_+}{(\tau_k)_+(1)} \in T(A)_{\alpha_*}$ is w^* -convergent to $\frac{\rho_1}{\rho_1(1)} \in T(A)_{\alpha_*}$, and $\frac{(\tau_k)_-}{(\tau_k)_-(1)} \in T(A)_{\alpha_*}$ is w^* -convergent to $\frac{\rho_2}{\rho_2(1)} \in T(A)_{\alpha_*}$ (here, without loss of generality, we assume that all the denominators are not zero). Then

$$\begin{aligned} g(\tau_k) &= (\tau_k)_+(1)\phi\left(\frac{(\tau_k)_+}{(\tau_k)_+(1)}\right) - (\tau_k)_-(1)\phi\left(\frac{(\tau_k)_-}{(\tau_k)_-(1)}\right) \\ &\longmapsto \rho_1(1)\phi\left(\frac{\rho_1}{\rho_1(1)}\right) - \rho_2(1)\phi\left(\frac{\rho_2}{\rho_2(1)}\right) = g(\rho_1 - \rho_2) = g(\tau), \end{aligned}$$

i.e.

$$g(\tau_k) \rightarrow g(\tau).$$

This contradicts the fact that $|g(\tau_k) - g(\tau)| \geq \varepsilon$, and so $g(\tau_i) \rightarrow g(\tau)$. By the Krein-Smulian Theorem, we see that g is a w^* -continuous linear functional on V_α with $g|_{T(A)_{\alpha_*}} = \phi$. By the Hahn-Banach extension theorem, there is a w^* -continuous linear functional on $(A_{sa})^*$ which is an extension of g . By the well-known dual theorem, there is an element $a_\phi \in A_{sa}$ such that $g(\tau) = \tau(a_\phi)(\forall \tau \in V_\alpha)$, and so $\phi(\tau) = \tau(a_\phi)(\forall \tau \in T(A)_{\alpha_*})$.

Similar discussion says, for any $\psi \in \text{Aff}T(A)$, there is also $a_\psi \in A_{sa}$ such that $\psi(\tau) = \tau(a_\psi)(\forall \tau \in T(A))$.

Lemma 3 ([16], Theorem 3.2). *For $n \in \mathbf{N} \cup \{\infty\}$, Δ_n induces a homeomorphic group isomorphism*

$$\Delta_n : U_0^n(A)/\overline{DU_0^n(A)} \mapsto \text{Aff}T(A)/\overline{\Delta_n^0(\pi_1(U_0^n(A)))}$$

with $\Phi_n = \Delta_n^{-1}$ defined as follows: By the duality theorem, for any element ξ in $\text{Aff}T(A)$, we have $a \in A_{sa}$ with $\xi(\tau) = \tau(a)$ for any $\tau \in T(A)$, and so denote ξ by $\hat{a}$. Then define $\Phi_n(q(\xi)) = q^0(e^{2\pi i a})$. In particular,

$$U_0^\infty(A)/\overline{DU_0^\infty(A)} \cong \text{Aff}T(A)/\overline{\rho(K_0(A))}.$$

Lemma 4. *Let (A, G, α) be a C^* -dynamical system with A unital and G abelian and discrete, let R_G be the restriction map from $\text{Aff}T(A)$ to $\text{Aff}(T(A)_{\alpha_*})$, let R^* be the homomorphism from $\text{Aff}(T(A)_{\alpha_*})$ to $\text{Aff}T(A \times_\alpha G)$ induced by the affine map $R : T(A \times_\alpha G) \rightarrow T(A)_{\alpha_*}$ defined in Lemma 2, and let $\Psi = R^* \circ R_G$. Then $\Psi : \text{Aff}T(A) \rightarrow \text{Aff}T(A \times_\alpha G)$ maps $\overline{\Delta_n^0(\pi_1(U_0^n(A)))}$ into $\overline{\Delta_n^0(\pi_1(U_0^n(A \times_\alpha G)))}$, and induces a group homomorphism from $\text{Aff}T(A)/\overline{\Delta_n^0(\pi_1(U_0^n(A)))}$ to*

$AffT(A \times_\alpha G)/\overline{\Delta_n^0(\pi_1(U_0^n(A \times_\alpha G)))}$, which is denoted by ψ , such that the diagram

$$\begin{array}{ccc} U_0^n(A)/\overline{DU_0^n(A)} & \xrightarrow{\Delta_n} & AffT(A)/\overline{\Delta_n^0(\pi_1(U_0^n(A)))} \\ \phi \downarrow & & \psi \downarrow \\ U_0^n(A \times_\alpha G)/\overline{DU_0^n(A \times_\alpha G)} & \xrightarrow{\Delta_n} & AffT(A \times_\alpha G)/\overline{\Delta_n^0(\pi_1(U_0^n(A \times_\alpha G)))} \end{array}$$

commutes, where ϕ is induced by the inclusion map from $U_0^n(A)$ to $U_0^n(A \times_\alpha G)$.

Proof. Let γ be a closed piecewise smooth path based at the identity of $U_0^n(A)$, and let $[\gamma]_A$ be the element in $\pi_1(U_0^n(A))$ with representative γ . Then γ can also be viewed as a closed piecewise smooth path in $U_0^n(A \times_\alpha G)$, and let $[\gamma]_{A \times_\alpha G}$ be the element in $\pi_1(U_0^n(A \times_\alpha G))$ with representative γ . Let $\omega \in T(A \times_\alpha G)$; then

$$\begin{aligned} \Psi(\Delta_n^0([\gamma]_A))(\omega) &= R_G(\Delta_n^0([\gamma]_A))(R(\omega)) \\ &= \frac{1}{2\pi i} \int_0^1 R(\omega)(\gamma'(t)\gamma(t)^*)dt = \Delta_n^0([\gamma]_{A \times_\alpha G})(\omega). \end{aligned}$$

So $\Psi(\Delta_n^0([\gamma]_A)) = \Delta_n^0([\gamma]_{A \times_\alpha G})$. Since R^* and R_G are continuous in norm topology, so is Ψ . Therefore Ψ maps $\overline{\Delta_n^0(\pi_1(U_0^n(A)))}$ into $\overline{\Delta_n^0(\pi_1(U_0^n(A \times_\alpha G)))}$. Let $a \in A_{sa}$, $\hat{a} \in AffT(A)$, $\hat{a}(\tau) = \tau(a)$, $\forall \tau \in T(A)$. Then $\Delta_n^{-1}(q(\hat{a})) = q^0(e^{2\pi i a})$, and so $\phi \circ \Delta_n^{-1}(q(\hat{a})) = q_G^0(e^{2\pi i a})$, where q_G^0 is the quotient map from $U_0^n(A \times_\alpha G)$ to $U_0^n(A \times_\alpha G)/\overline{DU_0^n(A \times_\alpha G)}$. Since for any $\omega \in T(A \times_\alpha G)$,

$$\Psi(\hat{a})(\omega) = R_G(\hat{a})(R(\omega)) = R(\omega)(a) = \omega(a) = \hat{a}(\omega),$$

$\psi(q(\hat{a})) = q_G(\Psi(\hat{a})) = q_G(\hat{a})$, where q_G is the quotient map from $AffT(A \times_\alpha G)$ to $AffT(A \times_\alpha G)/\overline{\Delta_n^0(\pi_1(U_0^n(A \times_\alpha G)))}$. Therefore $(\Delta_n^{-1} \circ \psi)(q(\hat{a})) = \Delta_n^{-1}(q_G(\hat{a})) = q_G^0(e^{2\pi i a}) = \phi \circ \Delta_n^{-1}(q(\hat{a}))$, i.e. $\phi \circ \Delta_n^{-1} = \Delta_n^{-1} \circ \psi$.

Theorem 2. Let (A, G, α) be a C^* -dynamical system with A unital, G abelian and discrete, and with $R: T(A \times_\alpha G) \rightarrow T(A)_{\alpha^*}$ homeomorphic. Then $A \times_\alpha G$ is of real rank zero if and only if each unitary element in $A \times_\alpha G$ with the form $u_A \prod_{i=1}^n x_i^* y_i^* x_i y_i$ can be approximated by the unitary elements in $A \times_\alpha G$ with finite spectrum, where $u_A \in U_0(A)$, $x_i, y_i \in C_c(G, A) \cap U_0(A \times_\alpha G)$.

Proof. Keeping the notations as above, by Proposition 2, $R_G: AffT(A) \rightarrow AffT(T(A)_{\alpha^*})$ is surjective. Since R is homeomorphic, Ψ is surjective, and so is ψ . Then $\phi: U_0(A)/\overline{DU_0(A)} \rightarrow U_0(A \times_\alpha G)/\overline{DU_0(A \times_\alpha G)}$ is surjective by Lemma 4. For any $u \in U_0(A \times_\alpha G)$, there exists $u_A \in U_0(A)$ such that $\phi(q^0(u_A)) = q_G^0(u)$, and so there exists $v \in \overline{DU_0(A \times_\alpha G)}$ such that $u = u_A v$. By definition, v can be approximated by the unitary elements with the form $\prod_{i=1}^n \bar{x}_i^* \bar{y}_i^* \bar{x}_i \bar{y}_i$, where $\bar{x}_i, \bar{y}_i \in U_0(A \times_\alpha G)$. For arbitrary $\bar{x} \in U_0(A \times_\alpha G)$, it is well known that $\bar{x}$ can be approximated by the unitary elements with the form $e^{ih_1} e^{ih_2} \dots e^{ih_k}$ for some $h_1, h_2, \dots, h_k \in (A \times_\alpha G)_{sa}$. Since for each i ($1 \leq i \leq k$), h_i can be approximated by the self-adjoint elements in $C_c(G, A) \subseteq A \times_\alpha G$, $\bar{x}$ can be approximated by the elements in $C_c(G, A) \cap U_0(A \times_\alpha G)$, and so we can choose the $\bar{x}_i$ and $\bar{y}_i$ above in $C_c(G, A) \cap U_0(A \times_\alpha G)$. Then the remainder of the proof is from [11, 4.2.8]

Proposition 3. Let A be a unital inductive limit of the direct sums of non-elementary simple C^* -algebras of real rank zero. Then $U_0^n(A) = \overline{DU_0^n(A)}$.

Proof. By Lemma 3, it is enough to prove that $AffT(A) = \overline{\Delta_1^0(\pi_1(U_0(A)))}$. Let $A = \lim_{n \rightarrow \infty} (A_n, \phi_n)$, where $A_n = \bigoplus_{i=1}^{k_n} A_{n,i}$, and all $A_{n,i}$ are non-elementary simple C^* -algebras of real rank zero. Since $A_{n,i}$ is simple, we may assume all $\phi_n : A_n \rightarrow A$ are injective and view A_n as a subalgebra of A . Let $a \in (A_n)_{sa}$; then $a = \bigoplus_{i=1}^{k_n} a_i$ and $\hat{a} = \bigoplus_{i=1}^{k_n} \hat{a}_i \in AffT(A)$. Since $A_{n,i}$ is of real rank zero, $\hat{a}_i \in \overline{span}_{\mathbf{R}}\{\hat{p} : p = p^* = p^2 \in A_{n,i}\}$.

Let $p \in A_{n,i}$ be a projection, and $r \in \mathbf{R}$. Then for any $\varepsilon > 0$, we can choose $n, m \in \mathbf{N}$ such that $|r - \frac{m}{2^n}| < \varepsilon$, $|\frac{r}{2^n}| < \varepsilon$. By [18, Theorem 2.1], there are equivalent orthogonal subprojections $p_1, p_2, \dots, p_{2^n}$ of p and subprojection q of p such that $p \sim \sum_{i=1}^{2^n} p_i \oplus q$ and $q \preceq p_1$, and so $\hat{p} = 2^n \hat{p}_1 + \hat{q}$, $\hat{q} \leq \hat{p}_1$. Then

$$\begin{aligned} \|r\hat{p} - m\hat{p}_1\| &\leq \|r\hat{p} - \frac{m}{2^n}\hat{p}\| + \|\frac{m}{2^n}\hat{p} - m\hat{p}_1\| \\ &\leq \varepsilon + \frac{m\|\hat{q}\|}{2^n} \leq \varepsilon + \frac{m\|\hat{p}_1\|}{2^n} \leq \varepsilon + \frac{m}{2^n} \frac{\|\hat{p}\|}{2^n} \\ &\leq 2\varepsilon + \frac{r}{2^n} \leq 3\varepsilon. \end{aligned}$$

Therefore $\overline{span}_{\mathbf{R}}\{\hat{p} : p = p^* = p^2 \in A_{n,i}\} = \overline{span}_{\mathbf{Z}}\{\hat{p} : p = p^* = p^2 \in A_{n,i}\}$.

By the discussion above, $\hat{a} \in \overline{span}_{\mathbf{Z}}\{\hat{p} : p = p^* = p^2 \in A_n\} \subseteq \overline{span}_{\mathbf{Z}}\{\hat{p} : p = p^* = p^2 \in A\}$. For a projection p in A , we have $\eta : [0, 1] \rightarrow U_0(A)$, $\eta(t) = e^{2\pi i t p}$. It is easy to see that $\Delta_1^1(\eta) = \hat{p}$, so $\hat{p} \in \Delta_1^0(\pi_1(U_0(A)))$. Therefore $\hat{a} \in \overline{span}_{\mathbf{Z}}\{\hat{p} : p = p^* = p^2 \in A\} \subseteq \overline{\Delta_1^0(\pi_1(U_0(A)))}$. This completes the proof of $AffT(A) = \overline{\Delta_1^0(\pi_1(U_0(A)))}$.

Note. The assumption is necessary that the direct sums of the building blocks are non-elementary. For example let $A = M_n(\mathbf{C})$ ($n \geq 1$). Then $U_0^n(A) = U^n(A)$ and $\overline{DU^n(A)} \neq U_0^n(A)$, since for any $a \in \overline{DU^n(A)}$, the determinant $|a|$ of a must be 1.

Theorem 3. *Let A be a unital inductive limit of the direct sums of non-elementary simple C^* -algebras of real rank zero, let (A, G, α) be a C^* -dynamical system with G abelian and discrete, and let $R : T(A \times_{\alpha} G) \rightarrow T(A)_{\alpha_*}$ be homeomorphic. Then*

(1) $U_0^n(A \times_{\alpha} G) = \overline{DU_0^n(A \times_{\alpha} G)}$.

(2) $A \times_{\alpha} G$ is of real rank zero if and only if each unitary element in $A \times_{\alpha} G$ with the form $\prod_{i=1}^n x_i^* y_i^* x_i y_i$ can be approximated by the unitary elements in $A \times_{\alpha} G$ with finite spectrum, where $x_i, y_i \in C_c(G, A) \cap U_0(A \times_{\alpha} G)$.

Proof. From the discussion in the proof of Theorem 2, we know that ϕ defined in Lemma 4 is surjective. Then (1) is from Lemma 4 and Proposition 3 directly. The proof of (2) is similar to that of Theorem 2 by use of (1).

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