ON GENERALIZED HYPERINTERPOLATION ON THE SPHERE

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(Communicated by Andreas Seeger)

ABSTRACT. It is shown that second-order results can be attained by the generalized hyperinterpolation operators on the sphere, which gives an affirmative answer to a question raised by Reimer in *Constr. Approx.* **18**(2002), no. 2, 183–203.

1. INTRODUCTION

Let $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$ ($d \geq 3$) be the unit sphere of $\mathbb{R}^d$ endowed with the usual rotation invariant measure $d\sigma(x)$. We denote by $\mathcal{H}_k^d$ the space of all spherical harmonics of degree k on $\mathbb{S}^{d-1}$ and Π_N^d the space of all spherical polynomials of degree at most N . The spaces $\mathcal{H}_k^d$ are mutually orthogonal with respect to the inner product

$$\langle f, g \rangle = \int_{\mathbb{S}^{d-1}} f(x)g(x) d\sigma(x),$$

and the space Π_N^d can be expressed as a direct sum

$$(1.1) \quad \Pi_N^d = \mathcal{H}_0^d \oplus \mathcal{H}_1^d \oplus \cdots \oplus \mathcal{H}_N^d.$$

Also, it is known that for $f \in \mathcal{H}_k^d$,

$$f(x) = \int_{\mathbb{S}^{d-1}} f(y)G_k(x \cdot y) d\sigma(y), \quad x \in \mathbb{S}^{d-1},$$

where here and throughout,

$$G_k(t) = \frac{2k + d - 2}{(d - 2)|\mathbb{S}^{d-1}|} P_k^{\frac{d-2}{2}}(t), \quad t \in [-1, 1],$$

$|\mathbb{S}^{d-1}|$ is the surface area of $\mathbb{S}^{d-1}$, and $P_k^{\frac{d-2}{2}}(t)$ is the usual ultraspherical polynomial of degree k and index $\frac{d-2}{2}$, as defined in [Sz, p. 80]. Thus, by (1.1), we have, for $f \in \Pi_N^d$,

$$(1.2) \quad f(x) = \int_{\mathbb{S}^{d-1}} f(y)E_N(x \cdot y) d\sigma(y), \quad x \in \mathbb{S}^{d-1},$$

Received by the editors April 23, 2005.

2000 *Mathematics Subject Classification.* Primary 41A15, 41A17; Secondary 41A05, 46E22.

Key words and phrases. Spherical polynomials, generalized hyperinterpolation, second-order moduli of smoothness, unit sphere.

The author was supported in part by the NSERC Canada under grant G121211001.

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where

$$(1.3) \quad E_N(x \cdot y) = \sum_{k=0}^N G_k(x \cdot y).$$

We denote by $d(x, y)$ the geodesic distance $\arccos x \cdot y$ between x and y on $\mathbb{S}^{d-1}$, $B(x, r)$ the spherical cap $\{y \in \mathbb{S}^{d-1} : d(x, y) \leq r\}$ centered at $x \in \mathbb{S}^{d-1}$ of radius $r > 0$, and $B(x, \alpha, \alpha + \beta)$ the spherical collar $\{y \in \mathbb{S}^{d-1} : \alpha \leq d(x, y) \leq \alpha + \beta\}$ centered at $x \in \mathbb{S}^{d-1}$ of spherical height $\beta > 0$. We shall use the notation $|E|$ to denote the measure $\sigma(E)$ of a subset $E \subset \mathbb{S}^{d-1}$, and write $A \sim B$ for the statement $C^{-1}A \leq B \leq CA$, where $C > 0$ is called the constant of equivalence.

Throughout this paper, we assume that $\{Q_N\}_{N=0}^\infty$ is a sequence of positive cubature formulas on $\mathbb{S}^{d-1}$:

$$\int_{\mathbb{S}^{d-1}} f(x) d\sigma(x) = Q_N f := \sum_{\omega \in \Lambda_N} \lambda_{N,\omega} f(\omega), \quad \text{for all } f \in \Pi_{N+1}^d,$$

where Λ_N is a finite subset of $\mathbb{S}^{d-1}$, $\lambda_{N,\omega} > 0$ for all $\omega \in \Lambda_N$. A cubature formula like Q_N is said to be of degree $N + 1$. It was shown by Reimer [Re1] that a positive cubature formula Q_N of degree $N + 1$ on the sphere must satisfy the following regularity condition, introduced by Sloan and Womersley [SW]:

$$(1.4) \quad \max_{x \in \mathbb{S}^{d-1}} \sum_{\omega \in \Lambda_N \cap B(x, N^{-1})} \lambda_{N,\omega} \leq CN^{-(d-1)},$$

where $C > 0$ depends only on d .

The *hyperinterpolation operators* L_N on $\mathbb{S}^{d-1}$, introduced by Sloan [Sl] (see also [SW]), are defined as

$$L_N f(x) = \sum_{\omega \in \Lambda_{2N-1}} \lambda_{2N-1,\omega} f(\omega) E_N(x \cdot \omega), \quad x \in \mathbb{S}^{d-1}, \quad f \in C(\mathbb{S}^{d-1}), \quad N = 1, 2, \dots,$$

where the E_N 's are defined by (1.3). The following remarkable result was proved by [SW] ($d = 3$, under the regularity assumption (1.4)), [LS] (d arbitrary, under the regularity assumption (1.4)), and by [Re1] (d arbitrary, no additional regularity assumption):

$$(1.5) \quad \|L_N\|_\infty \sim N^{\frac{d-2}{2}},$$

where the constant of equivalence depends only on d , and

$$\|L_N\|_\infty := \sup\{\|L_N f\|_\infty : f \in C(\mathbb{S}^{d-1})\}.$$

We refer to [SW], [LS], [Re1] and [Re2] for the background information of hyperinterpolation on the sphere.

In spite of the best-order result (1.5), pointwise convergence (for arbitrary $f \in C(\mathbb{S}^{d-1})$) cannot be attained by hyperinterpolation. For this reason, Reimer [Re2] introduced the concept of *generalized hyperinterpolation*, whose definition is as follows. Suppose that

$$D_N(t) = \sum_{k=0}^N a_{N,k} G_k(t), \quad N = 0, 1, \dots,$$

is a sequence of nonnegative polynomials on $[-1, 1]$ satisfying

$$\lim_{N \rightarrow \infty} a_{N,0} = \lim_{N \rightarrow \infty} a_{N,1} = 1.$$

Then, the *generalized hyperinterpolation operators* associated to $\{D_N(t)\}_{N=0}^\infty$ are defined as

$$(1.6) \quad GL_N(f)(x) \equiv GL_{N,D_N}(f)(x) := \sum_{\omega \in \Lambda_N} \lambda_{N,\omega} D_N(x \cdot \omega) f(\omega), \quad f \in C(\mathbb{S}^{d-1}), \quad x \in \mathbb{S}^{d-1}.$$

Note that in the definition of GL_N one assumes that the positive cubature formula involved is of degree $N+1$ rather than $2N$. The point of generalized hyperinterpolation is that it can achieve a uniform convergence result for arbitrary $f \in C(\mathbb{S}^{d-1})$ without increasing the cost of evaluation, as was shown by Reimer [Re2, Theorem 2].

An important example arises in generalized hyperinterpolation when one considers the Newman-Shapiro operators on $\mathbb{S}^{d-1}$, whose definition is given as follows. Let

$$(1.7) \quad B_{2N+1}(t) = B_{2N}(t) = \gamma_{N+1} \left(\frac{G_{N+1}(t)}{t - \xi_{N+1}} \right)^2, \quad t \in [-1, 1],$$

where ξ_{N+1} is the largest root of G_{N+1} , and the constant γ_{N+1} is chosen so that

$$\int_{\mathbb{S}^{d-1}} B_{2N}(x \cdot y) d\sigma(y) = 1, \quad x \in \mathbb{S}^{d-1}.$$

The *Newman-Shapiro operators* are then defined by

$$T_N f(x) = \int_{\mathbb{S}^{d-1}} f(y) B_N(x \cdot y) d\sigma(y), \quad x \in \mathbb{S}^{d-1}, \quad N = 1, 2, \dots$$

We denote by GL_{N,B_N} the generalized hyperinterpolation operators associated to the kernels B_N . For the Newman-Shapiro operators T_N , Reimer [Re2, Theorem 7] proved that

$$\|f - T_N f\|_\infty \leq C\omega_2(f, \frac{1}{N}), \quad f \in C(\mathbb{S}^{d-1}),$$

where

$$(1.8) \quad \omega_2(f, t) = \max \left\{ |f_{u,v}(\alpha + t) - 2f_{u,v}(\alpha) + f_{u,v}(\alpha - t)| : \right. \\ \left. \alpha \in \mathbb{R}, \quad u, v \in \mathbb{S}^{d-1} \text{ with } u \perp v \right\},$$

is the modulus of smoothness of second order, and

$$f_{u,v}(\varphi) = f(u \cos \varphi + v \sin \varphi), \quad \text{for } \varphi \in \mathbb{R} \text{ and } u, v \in \mathbb{S}^{d-1} \text{ with } u \perp v;$$

while for the discrete operators GL_{N,B_N} , he proved [Re2, Theorem 6] that

$$\|f - GL_{N,B_N} f\|_\infty \leq C\omega_1(f, N^{-1}), \quad f \in C(\mathbb{S}^{d-1}),$$

where

$$\omega_1(f, t) = \sup \{ |f(x) - f(y)| : x, y \in \mathbb{S}^{d-1} \text{ and } d(x, y) \leq t \}$$

is the first-order modulus of smoothness. It was asked by Reimer [Re2] whether second-order results can be attained by generalized hyperinterpolation operators. Our main result (Theorem 1.1 below) in this paper gives an affirmative answer to this question.

Theorem 1.1. *Suppose that*

$$D_N(t) = \sum_{k=0}^N a_{N,k} G_k(t), \quad N = 1, 2, \dots,$$

is a sequence of polynomials on $[-1, 1]$ satisfying

$$(1.9) \quad \int_{\mathbb{S}^{d-1}} D_N(x \cdot y) d\sigma(y) = 1, \quad x \in \mathbb{S}^{d-1}, \quad N = 1, 2, \dots,$$

and

$$(1.10) \quad \sup_N \int_0^\pi (1 + N\theta)^2 |D_N(\cos \theta)| \sin^{d-2} \theta d\theta \leq K < \infty.$$

Let $GL_N \equiv GL_{N,D_N}$ denote the generalized hyperinterpolation operators defined by (1.6). Then for all $f \in C(\mathbb{S}^{d-1})$, we have

$$(1.11) \quad \|f - GL_N f\|_\infty \leq CK \omega_2(f, \frac{1}{N}),$$

where $C > 0$ depends only on d , and $\omega_2(f, t)$ is defined by (1.8).

Remark 1.1. It was shown in [Re2, pp. 197–199] that the kernels B_N defined by (1.7) satisfy both of the conditions (1.9) and (1.10). Thus, by Theorem 1.1, (1.11) is true for the discrete operators GL_{N,B_N} .

Remark 1.2. The point of Theorem 1.1 is that we do not assume the positivity of the kernels D_N . The existence of a sequence of polynomials D_N which satisfy both (1.9) and (1.10) but assume negative values on $[-1, 1]$ is well known. See, for example, the sequence of polynomials $K_{[N/2]}$ defined by (2.4) in Section 2.

Remark 1.3. If, however, we assume that all the kernels D_N are nonnegative, then we have the following.

Corollary 1.2. *Suppose that*

$$D_N(t) = \sum_{k=0}^N a_{N,k} G_k(t), \quad N = 1, 2, \dots,$$

is a sequence of nonnegative polynomials on $[-1, 1]$, and $GL_N \equiv GL_{N,D_N}$ ($N = 1, 2, \dots$) denote the generalized hyperinterpolation operators defined by (1.6). Then in order that

$$\|f - GL_N f\|_\infty \leq C \omega_2(f, N^{-1})$$

(with $C > 0$ depending only on d) holds for all $f \in C(\mathbb{S}^{d-1})$, it is necessary and sufficient that

$$a_{N,0} = 1$$

and

$$a_{N,1} \geq 1 - C_d N^{-2}.$$

Corollary 1.2 can easily be deduced from Theorem 1.1 and the following two identities:

$$\int_{\mathbb{S}^{d-1}} D_N(x \cdot y) d\sigma(y) = a_{N,0}, \quad x \in \mathbb{S}^{d-1},$$

and

$$\int_{\mathbb{S}^{d-1}} (z \cdot y) D_N(x \cdot y) d\sigma(y) = a_{N,1} z \cdot x, \quad z, x \in \mathbb{S}^{d-1}.$$

The paper is organized as follows. In Section 2, we give an equivalent characterization of the regularity condition (1.4), which is in terms of the Marcinkiewicz-Zygmund inequality. Based on this characterization, we prove the main result in Section 3.

2. THE REGULARITY CONDITION

Our main purpose in this section is to give an equivalent characterization of the regularity condition of Sloan and Womersley [SW]. Our main result can be stated as follows.

Theorem 2.1. *Suppose that Ω is a finite subset of $\mathbb{S}^{d-1}$, $\{\mu_\omega : \omega \in \Omega\}$ is a set of positive numbers, and n is a positive integer. If for some $0 < p_0 < \infty$ the inequality*

$$(2.1) \quad \sum_{\omega \in \Omega} \mu_\omega |f(\omega)|^{p_0} \leq C_1 \int_{\mathbb{S}^{d-1}} |f(x)|^{p_0} d\sigma(x)$$

with C_1 independent of f holds for all $f \in \Pi_n^d$, then the following regularity condition must be satisfied:

$$(2.2) \quad \sup_{x \in \mathbb{S}^{d-1}} \sum_{\omega \in \Omega \cap B(x, \frac{1}{n})} \mu_\omega \leq C_2 n^{-(d-1)},$$

where $C_2 = CC_1$ and $C > 0$ depends only on d and p_0 .

Conversely, if the regularity condition (2.2) is satisfied for some constant $C_2 > 0$, then for any $0 < p < \infty$ and any $f \in \Pi_m^d$ with $m \geq n$,

$$(2.3) \quad \sum_{\omega \in \Omega} \mu_\omega |f(\omega)|^p \leq CC_2 \left(\frac{m}{n}\right)^{d-1} \int_{\mathbb{S}^{d-1}} |f(y)|^p d\sigma(y),$$

where $C > 0$ depends only on d and p .

We point out that while the second assertion of Theorem 2.1 with $p = 1$ will be enough for the proof of our main result (Theorem 1.1) in this paper, we shall give the proof of the whole theorem in this section for the sake of completeness.

After this paper was completed, we learned that a result similar to Theorem 2.1 for $1 \leq p < \infty$ was obtained independently by H. N. Mhaskar in a recent paper [M2, Theorem 3.3].

For the proof of Theorem 2.1, we need three previously known lemmas. To state our first lemma, we let $\eta \in C^\infty[0, \infty)$ be such that $\eta(t) = 1$ for $0 \leq t \leq 1$ and $\eta(t) = 0$ for $t \geq 2$, and we define

$$(2.4) \quad K_N(t) \equiv K_{N,\eta}(t) = \sum_{k=0}^{2N} \eta\left(\frac{k}{N}\right) G_k(t), \quad t \in [-1, 1], \quad N = 1, 2, \dots$$

By (1.1) and (1.2), it follows that for very $f \in \Pi_N^d$

$$(2.5) \quad f(x) = \int_{\mathbb{S}^{d-1}} f(y) K_N(x \cdot y) d\sigma(y), \quad x \in \mathbb{S}^{d-1}.$$

We will keep the function η and the notation K_N for the rest of this section. Our first lemma, which was proved in [BD, Lemma 3.3], can be stated as follows.

Lemma 2.2. For $\theta \in [0, \pi]$ and any positive integer ℓ , we have

$$|K_N^{(i)}(\cos \theta)| \leq C_{\ell,i} N^{d-1+2i} \min\{1, (N\theta)^{-\ell}\}, \quad i = 0, 1, \dots, \quad N = 1, 2, \dots,$$

where $K_N^{(0)}(t) = K_N(t)$, $K_N^{(i)}(t) = \left(\frac{d}{dt}\right)^i \{K_N(t)\}$ for $i \geq 1$, and the constant $C_{\ell,i}$ depends only on ℓ , i and η .

The point of Lemma 2.2 is that the positive integer ℓ can be chosen arbitrarily big. An estimate analogous to Lemma 2.2 was obtained independently by H. N. Mhaskar in a recent paper [M1, Lemma 4.10].

Our second lemma, Lemma 2.3 below, was proved by Q. T. Le Gia and I. H. Sloan in [LS, Lemma 5.1].

Lemma 2.3. Suppose that Ω is a finite subset of $\mathbb{S}^{d-1}$, $\{\mu_\omega : \omega \in \Omega\}$ is a set of positive numbers satisfying the regularity condition (2.2) for some $C_2 > 0$. Then we have

(a) For every spherical cap $B(x, \alpha)$ with radius $\alpha \geq \frac{1}{n}$,

$$\sum_{\omega \in \Omega \cap B(x, \alpha)} \mu_\omega \leq CC_2 |B(x, \alpha)|,$$

where $C > 0$ depends only on d .

(b) For every spherical collar $B(x, \alpha, \alpha + \beta)$ with spherical height $\beta \geq \frac{1}{n}$,

$$\sum_{\omega \in B(x, \alpha, \alpha + \beta) \cap \Omega} \mu_\omega \leq CC_2 |B(x, \alpha, \alpha + \beta)|,$$

where $C > 0$ depends only on d .

Our final lemma, Lemma 2.4 below, is well known (see, for instance, [BDS], [SW] or [MNW]).

Lemma 2.4. For each positive integer N , there exists a positive cubature formula of degree $4N$ on $\mathbb{S}^{d-1}$, that is,

$$(2.6) \quad \int_{\mathbb{S}^{d-1}} f(y) d\sigma(y) = \sum_{j=1}^{M_N} w_{j,N} f(t_{j,N}), \quad \text{for all } f \in \Pi_{4N}^d,$$

where $t_{j,N} \in \mathbb{S}^{d-1}$, $0 < w_{j,N} \leq C_d N^{-(d-1)}$, and $M_N \sim N^{d-1}$.

Proof of Theorem 2.1. First, we assume (2.1) is true for some $0 < p_0 < \infty$, all $f \in \Pi_n^d$ and for some finite set $\Omega \subset \mathbb{S}^{d-1}$, and we will prove the regularity condition (2.2). Let

$$P_{[n/2]}(\cos \theta) = \frac{P_{[n/2]}^{(\frac{d}{p_0}-\frac{1}{2}, -\frac{1}{2})}(\cos \theta)}{P_{[n/2]}^{(\frac{d}{p_0}-\frac{1}{2}, -\frac{1}{2})}(1)}, \quad \theta \in [0, \pi],$$

where $P_k^{(\alpha, \beta)}$ denotes the Jacobi polynomial of degree k and with indices α, β , as defined in [Sz, p. 58]. Then by (4.1.1) and (8.21.18) of [Sz], we have

$$(2.7) \quad |P_{[n/2]}(\cos \theta)| \leq \min\{1, C_{d,p_0} (n\theta)^{-d/p_0}\}, \quad \theta \in [0, \pi].$$

Thus, using Bernstein's inequality for trigonometric polynomials, we have, for all $\theta \in [0, \pi]$,

$$|P_{[n/2]}(\cos \theta) - 1| = |P_{[n/2]}(\cos \theta) - P_{[n/2]}(1)| \leq \frac{n\theta}{2} \|P_{[n/2]}\|_\infty = \frac{n\theta}{2},$$

which implies that for $0 \leq \theta \leq \frac{1}{n}$,

$$\frac{1}{2} \leq P_{[n/2]}(\cos \theta) \leq 1.$$

Now, for any fixed $x \in \mathbb{S}^{d-1}$, using (2.1) with $f(y) = P_{[n/2]}(y \cdot x)$, we deduce

$$\sum_{\omega \in \Omega \cap B(x, \frac{1}{n})} \mu_\omega \leq 2^{p_0} \sum_{\omega \in \Omega} \mu_\omega |P_{[n/2]}(x \cdot \omega)|^{p_0} \leq 2^{p_0} C_1 \int_{\mathbb{S}^{d-1}} |P_{[n/2]}(x \cdot y)|^{p_0} d\sigma(y),$$

which, by (2.7) and a straightforward calculation, is controlled by $C_1 C_{d,p_0} n^{-(d-1)}$. This proves (2.2) with $C_2 = C_{d,p_0} C_1$.

Next, we prove the second assertion of Theorem 2.1. Suppose that the regularity condition (2.2) is satisfied. We will prove that (2.3) holds for all $f \in \Pi_m^d$ with $m \geq n$. Set

$$(2.8) \quad A_m(\theta) = m^{d-1} (1 + m\theta)^{-d}, \quad \theta \in [0, \pi].$$

The following estimate will play a crucial role in our proof:

$$(2.9) \quad \max_{x \in \mathbb{S}^{d-1}} \sum_{\omega \in \Omega} \mu_\omega A_m(d(\omega, x)) \leq C C_2 \left(\frac{m}{n}\right)^{d-1},$$

where $C > 0$ depends only on d and p when p is small.

For the moment, we take (2.9) for granted and proceed with the proof. We start with the equation

$$(2.10) \quad f(x) = \int_{\mathbb{S}^{d-1}} f(y) K_m(x \cdot y) d\sigma(y), \quad x \in \mathbb{S}^{d-1}, \quad f \in \Pi_m^d,$$

and consider the following two cases:

Case 1. $1 \leq p < \infty$.

In this case, using (2.5) and Hölder's inequality, we have, for $f \in \Pi_m^d$ and all $x \in \mathbb{S}^{d-1}$,

$$|f(x)|^p \leq \left[\int_{\mathbb{S}^{d-1}} |f(y)|^p |K_m(x \cdot y)| d\sigma(y) \right] \left[\int_{\mathbb{S}^{d-1}} |K_m(x \cdot y)| d\sigma(y) \right]^{p-1}.$$

Therefore, by Lemma 2.2 applied to $\ell = d$ and $i = 0$, and a straightforward calculation, it follows that for $f \in \Pi_m^d$ and all $x \in \mathbb{S}^{d-1}$

$$|f(x)|^p \leq C \int_{\mathbb{S}^{d-1}} |f(y)|^p A_m(d(x, y)) d\sigma(y),$$

where A_m is defined by (2.8). Now combining this last inequality with (2.9), we obtain

$$\sum_{\omega \in \Omega} \mu_\omega |f(\omega)|^p \leq C \|f\|_p^p \max_{y \in \mathbb{S}^{d-1}} \left[\sum_{\omega \in \Omega} \mu_\omega A_m(d(\omega, y)) \right] \leq C C_2 \left(\frac{m}{n}\right)^{d-1} \|f\|_p^p.$$

Case 2. $0 < p < 1$.

In this case, we use (2.10) and the cubature formulas in Lemma 2.4 to obtain

$$f(x) = \sum_{j=1}^{M_m} w_{j,m} f(\rho t_{j,m}) K_m(x \cdot \rho t_{j,m}), \quad x \in \mathbb{S}^{d-1}, \quad \rho \in SO(d),$$

where $f \in \Pi_m^d$, $SO(d)$ denotes the group of rotations on $\mathbb{S}^{d-1}$. Since $0 < w_{j,m} \leq C_d m^{-(d-1)}$, it follows by Lemma 2.2 with $\ell = \frac{d}{p}$ that for $x \in \mathbb{S}^{d-1}$ and $\rho \in SO(d)$,

$$|f(x)|^p \leq C \sum_{j=1}^{M_m} m^{-(d-1)p} |f(\rho t_{j,m})|^p m^{(d-1)(p-1)} A_m(d(x, \rho t_{j,m})).$$

This implies that

$$\begin{aligned} \sum_{\omega \in \Omega} \mu_\omega |f(\omega)|^p &\leq C m^{-(d-1)} \left[\inf_{\rho \in SO(d)} \sum_{j=1}^{M_m} |f(\rho t_{j,m})|^p \right] \left[\max_{y \in \mathbb{S}^{d-1}} \sum_{\omega \in \Omega} \mu_\omega A_m(d(\omega, y)) \right] \\ &\leq C C_2 \left(\frac{m}{n} \right)^{d-2} m^{-(d-1)} \sum_{j=1}^{M_m} \int_{SO(d)} |f(\rho t_{j,m})|^p d\rho \quad (\text{by (2.9)}) \\ &\leq C C_2 \left(\frac{m}{n} \right)^{d-2} \|f\|_p^p, \quad (\text{by the fact that } M_m \sim m^{d-1}), \end{aligned}$$

where $d\rho$ denotes the usual Haar measure on the group $SO(d)$.

Therefore, in either case, we have proven the desired inequality (2.3), taking (2.9) for granted.

It remains to prove (2.9). The proof is based on Lemma 2.3. In fact, using Lemma 2.3 and the definition of A_m , we have, for any $x \in \mathbb{S}^{d-1}$,

$$\begin{aligned} &\sum_{\omega \in \Omega} \mu_\omega A_m(d(x, \omega)) \\ &\leq m^{d-1} \sum_{\omega \in \Omega \cap B(x, \frac{1}{n})} \mu_\omega + \sum_{k=1}^{n-1} \sum_{\omega \in B(x, k/n, (k+1)/n) \cap \Omega} \mu_\omega m^{d-1} \left(\frac{mk}{n} \right)^{-d} \\ &\quad + m^{-1} \sum_{\omega \in B(-x, \pi-1) \cap \Omega} \mu_\omega \\ &\leq C C_2 m^{d-1} |B(x, \frac{1}{n})| + C C_2 m^{-1} \sum_{k=1}^{n-1} \left(\frac{n}{k} \right)^d |B(x, k/n, (k+1)/n)| \\ &\quad + C C_2 m^{-1} |B(-x, \pi-1)| \\ &\leq C C_2 \left(\frac{m}{n} \right)^{d-1}, \end{aligned}$$

since $|B(x, \alpha)| \sim \alpha^{d-1}$ and $|B(x, k/n, (k+1)/n)| \sim k^{d-2}/n^{d-1}$ for $1 \leq k \leq n-1$. This proves (2.9) and hence completes the proof of Theorem 2.1. $\square$

3. PROOF OF THEOREM 1.1

For the proof of Theorem 1.1, we need three lemmas. The first two lemmas are previously known.

Lemma 3.1 ([N]). *Suppose that T is a continuously differentiable, 2π -periodic function on $\mathbb{R}$ with $\int_0^{2\pi} T(x) dx = 0$. Then*

$$\|T\|_{C(\mathbb{T})} \leq C \|T'\|_{C(\mathbb{T})},$$

where $C > 0$ is an absolute constant, and throughout this section, $\|\cdot\|_{C(\mathbb{T})}$ is defined as $\|T\|_{C(\mathbb{T})} = \max_{x \in [0, 2\pi]} |T(x)|$.

Lemma 3.2 ([St]). *If T_n is a trigonometric polynomial of degree at most n and $r \geq 1$ is a positive integer, then for $0 < h < \frac{2\pi}{n}$,*

$$\|T_n^{(r)}\|_{C(\mathbb{T})} \leq \left(\frac{n}{2\sin(nh/2)}\right)^r \|\Delta_h^r T_n\|_{C(\mathbb{T})},$$

where

$$\Delta_h^r T_n(x) = \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} T_n(x + jh).$$

Lemma 3.3. *Suppose that D_N satisfies the assumptions in Theorem 1.1, and $GL_N \equiv GL_{N,D_N}$ is defined by (1.6). Then for $f \in C(\mathbb{S}^{d-1})$, we have*

$$\|GL_N f\|_\infty \leq CK \|f\|_\infty,$$

where $C > 0$ depends only on d .

Proof. By the definition (1.6), we have, for $f \in C(\mathbb{S}^{d-1})$ and $x \in \mathbb{S}^{d-1}$,

$$\begin{aligned} |GL_N f(x)| &\leq \left(\sum_{\omega \in \Lambda_N} \lambda_{N,\omega} |D_N(x \cdot \omega)| \right) \|f\|_\infty \\ &\leq C \left(\int_{\mathbb{S}^{d-1}} |D_N(x \cdot y)| d\sigma(y) \right) \|f\|_\infty \quad (\text{by (1.4) and Theorem 2.1}) \\ &\leq CK \|f\|_\infty \quad (\text{by (1.10)}). \end{aligned}$$

This completes the proof. $\square$

Proof of Theorem 1.1. Let $g \in \Pi_N^d$ be such that

$$\|f - g\|_\infty = E_N(f)_\infty := \inf_{p \in \Pi_N^d} \|f - p\|_\infty.$$

Then by Lemma 3.3, we have

$$\begin{aligned} \|f - GL_N(f)\|_\infty &\leq \|GL_N(f - g)\|_\infty + \|GL_N(g) - g\|_\infty + E_N(f)_\infty \\ (3.1) \quad &\leq CK E_N(f)_\infty + \|GL_N(g) - g\|_\infty. \end{aligned}$$

It will be shown that

$$(3.2) \quad \|g - GL_N(g)\|_\infty \leq CK \omega_2(g, \frac{1}{N}).$$

Once (3.2) is proved, then by (3.1) and the Jackson-type estimate (see, for example, [Re2, Theorem 7]), we obtain

$$\|f - GL_N(f)\|_\infty \leq CK E_N(f)_\infty + CK \omega_2(f - g, \frac{1}{N}) + CK \omega_2(g, \frac{1}{N}) \leq CK \omega_2(f, \frac{1}{N}),$$

which gives the desired inequality (1.11). Thus, the proof of Theorem 1.1 is reduced to the proof of (3.2).

To show (3.2), we claim that for $x = (x_1, \dots, x_d) \in \mathbb{S}^{d-1}$ and $y = (y_1, \dots, y_d) \in \mathbb{S}^{d-1}$,

$$(3.3) \quad \left| g(y) - g(x) - \sum_{j=1}^d g'_j(x) (y_j - x_j(x \cdot y)) \right| \leq CN^2 (1 - x \cdot y) \omega_2(g, \frac{1}{N}),$$

where $g'_j(x) = \frac{\partial G(x)}{\partial x_j}$ and $G(z) = g(\frac{z}{|z|})$ for $z \in \mathbb{R}^d \setminus \{0\}$.

For the proof of the claim (3.3), we assume that $y = x \cos \theta + u \sin \theta$, where $\theta = d(x, y)$ and $u = (u_1, \dots, u_d) \in \mathbb{S}^{d-1}$ with $u \perp x$. We set

$$g_{x,u}(\varphi) = g(x \cos \varphi + u \sin \varphi), \quad \varphi \in \mathbb{R}.$$

Then $g_{x,u}$ is a trigonometric polynomial of degree at most N , and moreover,

$$g'_{x,u}(\varphi) = \sum_{j=1}^d g'_j(x \cos \varphi + u \sin \varphi) (-x_j \sin \varphi + u_j \cos \varphi).$$

It follows that

$$\begin{aligned} I &: \equiv \left| g(y) - g(x) - \sum_{j=1}^d g'_j(x)(y_j - x_j(x \cdot y)) \right| \\ &= \left| g_{x,u}(\theta) - g_{x,u}(0) - \left(\sum_{j=1}^d g'_j(x) u_j \right) \sin \theta \right| \\ &\leq \left| g_{x,u}(\theta) - g_{x,u}(0) - g'_{x,u}(0) \theta \right| + C \|g'_{x,u}\|_{C(\mathbb{T})} \theta^3 \\ (3.4) \quad &= \frac{\theta^2}{2} |g''_{x,u}(\xi)| + C \theta^3 \|g'_{x,u}\|_{C(\mathbb{T})}, \quad \text{for some } \xi \in [0, \theta]. \end{aligned}$$

However, noting that

$$\int_0^{2\pi} g'_{x,u}(t) dt = g_{x,u}(2\pi) - g_{x,u}(0) = 0,$$

using Lemma 3.1, we deduce

$$\theta^3 \|g'_{x,u}\|_{C(\mathbb{T})} \leq C \theta^2 \|g''_{x,u}\|_{C(\mathbb{T})}, \quad \theta \in [0, \pi].$$

Therefore, combining this last inequality with (3.4), we obtain

$$\begin{aligned} I &\leq C \theta^2 \|g''_{x,u}\|_{C(\mathbb{T})} \\ &\leq C N^2 (1 - x \cdot y) \|\Delta_{N-1}^2(g_{x,u})\|_{C(\mathbb{T})} \quad (\text{by Lemma 3.2}) \\ &\leq C N^2 (1 - x \cdot y) \omega_2(g, \frac{1}{N}) \quad (\text{by the definition (1.8)}), \end{aligned}$$

which proves the claim (3.3)).

Now we are ready to prove (3.2). By (1.9) and the definition (1.6), we have

$$(3.5) \quad GL_N(g)(x) - g(x) = \sum_{\omega \in \Lambda_N} \lambda_{N,\omega} (g(\omega) - g(x)) D_N(x \cdot \omega).$$

We note that for each $1 \leq j \leq d$,

$$\begin{aligned} &\sum_{\omega=(\omega_1, \dots, \omega_d) \in \Lambda_N} (\omega_j - x_j(x \cdot \omega)) \lambda_{N,\omega} D_N(x \cdot \omega) \\ &= \int_{\mathbb{S}^{d-1}} D_N(x \cdot y) (y_j - x_j(x \cdot y)) d\sigma(y) \\ &= a_{N,1} (x_j - x_j(x \cdot x)) = 0, \end{aligned}$$

where the first equality follows by the assumption that the cubature formula Q_N is of degree $N+1$, while the second follows from the fact that $h_j(y) = y_j - x_j(x \cdot y)$

is a spherical harmonic of degree 1. Therefore, using the claim (3.3), we conclude that for $x \in \mathbb{S}^{d-1}$,

$$\begin{aligned} |GL_N(g)(x) - g(x)| &\leq CN^2\omega_2(g, N^{-1}) \sum_{\omega \in \Lambda_N} \lambda_{N,\omega}(1 - x \cdot \omega) |D_N(x \cdot \omega)| \\ &\leq CN^2\omega_2(g, N^{-1}) \int_{\mathbb{S}^{d-1}} (1 - x \cdot y) |D_N(x \cdot y)| d\sigma(y), \\ &\quad \text{(by (1.4) and Theorem 2.1)} \\ &\leq CK\omega_2(g, N^{-1}) \quad \text{(by (1.10))} \end{aligned}$$

which proves (3.2) and hence completes the proof of Theorem 1.1. $\square$

ACKNOWLEDGEMENT

The author expresses his deep gratitude to an anonymous referee for reading this paper carefully and giving many helpful comments on it.

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