BASIS PROPERTIES OF EIGENFUNCTIONS OF THE p -LAPLACIAN

PAUL BINDING, LYONELL BOULTON, JAN ČEPIČKA, PAVEL DRÁBEK,
 AND PETR GIRG

(Communicated by Carmen C. Chicone)

ABSTRACT. For $p \geq \frac{12}{11}$, the eigenfunctions of the non-linear eigenvalue problem for the p -Laplacian on the interval $(0, 1)$ are shown to form a Riesz basis of $L_2(0, 1)$ and a Schauder basis of $L_q(0, 1)$ whenever $1 < q < \infty$.

1. INTRODUCTION

Generalized sine functions S_p , $1 < p < \infty$, were studied by Elbert [6] as Dirichlet eigenfunctions for the one-dimensional p -Laplacian equation

$$(1) \quad -(y'(t)|y'(t)|^{p-2})' = (p-1)\lambda y(t)|y(t)|^{p-2}$$

on $0 \leq t \leq \pi_p$. Here

$$(2) \quad \pi_p := \frac{2\pi}{p \sin(\pi/p)}$$

is the first zero of S_p , which is the eigenfunction (normalized by $S_p'(0) = 1$) corresponding to the minimal Dirichlet eigenvalue $\lambda = 1$ of (1). Elbert also deduced the relation

$$(3) \quad |S_p|^p + |S_p'|^p = 1$$

and considered the higher-order eigenfunctions, which are given by $S_p(nt)$ with corresponding eigenvalues $\lambda = n^p$, $n = 2, 3, \dots$. When $p = 2$, (1) corresponds to Fourier's equation, $S_2(t) = \sin(t)$, $\pi_2 = \pi$ and (3) is the Pythagorean relation.

In [15], Ôtani examined analogous functions $\sin_p$ with the factor $(p-1)$ on the right side of (1) replaced by 1. Lindqvist [12], [13] has studied the $\sin_p$ functions in some detail, noting that hyperbolic versions were introduced as early as 1879. We remark that the S_p and $\sin_p$ functions have become standard tools in the analysis of more complicated equations with various applications; cf. e.g. [1], [2], [3], [4], [5], [6], [7], [8], [14] and [16].

Received by the editors May 5, 2004 and, in revised form, October 19, 2004.

2000 *Mathematics Subject Classification*. Primary 34L30; Secondary 34L10, 42A65.

Key words and phrases. p -Laplacian eigenvalues, eigenfunction completeness.

The research of the first author was supported by I. W. Killam Foundation and NSERC of Canada.

The second author was supported by a PIMS Postdoctoral Fellowship at the University of Calgary.

The research of the third, fourth, and fifth authors was supported by GAČR, no. 201/03/0671.

Despite this activity, it seems that analogues for $1 < p \neq 2$ of the standard completeness and expansion theorems for sine functions have not been discussed previously. While the S_p and $\sin_p$ functions can easily be transformed into each other, it turns out that (3) must be modified when S_p is replaced by $\sin_p$. Since relation (3) simplifies our arguments below, we shall work with S_p normalized to the interval $[0, 1]$. To this end, we define

$$f_n(t) := S_p(n\pi_p t), \quad n \in \mathbb{N},$$

which comprise eigenfunctions of the nonlinear eigenvalue problem (1) on $(0, 1)$ with $y(0) = y(1) = 0$, corresponding to eigenvalues $\lambda_n = (n\pi_p)^p$, $n = 1, 2, \dots$. They admit characterizations, e.g., via a modification of Prüfer's transformation [6] and via Ljusternik-Schnirelmann variational formulae [2].

The functions f_n depend on p , and in the case $p = 2$ they become

$$e_n(t) := \sin(n\pi t), \quad n \in \mathbb{N},$$

which are proportional to a standard orthonormal basis of the Hilbert space $L_2(0, 1)$ and form a Schauder basis of the Banach space $L_q(0, 1)$ for all $q > 1$ (see the proof of Theorem 6 below).

Recall that $\{g_n\}$ is a Schauder basis [17, p. 152] of $L_q(0, 1)$, $1 < q < \infty$, if for any $g \in L_q(0, 1)$, there exist unique coefficients c_n , depending continuously on g , so that

$$\left\| \sum_{n=1}^N c_n g_n - g \right\|_q \rightarrow 0$$

as $N \rightarrow \infty$. For $q = 2$, such a basis is a Riesz basis [10, p. 74] if the ℓ_2 norm of the above coefficients c_n generates a norm equivalent to that in $L_2(0, 1)$.

The aim of this note is to establish the following.

Theorem 1. *For $\frac{12}{11} \leq p < \infty$, the family $\{f_n\}_{n=1}^\infty$ forms a Riesz basis of $L_2(0, 1)$ and a Schauder basis of $L_q(0, 1)$ whenever $1 < q < \infty$.*

Our main device is a linear mapping T of the space $L_q(0, 1)$, satisfying $Te_n = f_n$, $n \in \mathbb{N}$, and decomposing into a linear combination of certain isometries. In Section 3 we show that T is a bounded operator for $1 < p < \infty$. In Section 4 we prove boundedness of the inverse for the range $\frac{12}{11} \leq p < \infty$.

We should point out that the bound $\frac{12}{11}$ for p is certainly not optimal, although it is convenient for explicit calculations. We shall make some remarks at the end about this bound, but we note here that our methods will not reduce it to 1.

2. PRELIMINARIES

In this section we describe some elementary properties of π_p and S_p , which will be employed in the sequel. We start with

Lemma 2. *The number π_p in (2) is strictly decreasing in $p > 1$.*

Indeed, a simple computation gives $\frac{d\pi_p}{dp} < 0$ whenever $p > 1$. Notice that $\pi_p \nearrow \infty$ as $p \searrow 1$ and $\pi_p \searrow 2$ as $p \nearrow \infty$. We shall also need the specific value

$$(4) \quad \pi_{12/11} = \frac{11\pi\sqrt{2}}{3(\sqrt{3}-1)}.$$

FIGURE 1.

Let x be an arbitrary point in the interval $(0, \pi_p/2)$. By virtue of (3), $0 \leq S_p(x) \leq 1$ and

$$(5) \quad S'_p(x) = (1 - S_p(x)^p)^{1/p} > 0.$$

Furthermore, by construction, $S_p(0) = 0$ and $S_p(\pi_p/2) = 1$.

Lemma 3. *Let $p > 1$. Then S_p is strictly concave on $(0, \pi_p/2)$, and, when $p \leq 2$,*

$$(6) \quad |S''_p(x)| \leq 1.$$

Proof. By differentiating (5), one readily sees that

$$(7) \quad S''_p(x) = -h(S_p(x))$$

where $h(y) := y^{p-1}(1 - y^p)^{\frac{2-p}{p}} > 0$ for $y \in (0, 1)$. When $p \leq 2$ it is clear that $|h(y)| \leq 1$. $\square$

Remark 4. The above results apply to $f_1 : t \mapsto S_p(\pi_p t)$ on $[0, \frac{1}{2}]$. We extend f_1 symmetrically across $\frac{1}{2}$ by $f_1(1 - t) = f_1(t)$. When $p \leq 2$, it is easily seen that Lemma 3 extends to this situation, and so f''_1 is continuous on $(0, 1)$. When $p > 2$, $f''_1(\frac{1}{2})$ does not exist, but

$$(8) \quad \int_0^{1/2} |f''_1(t)| dt = -\pi_p^2 \int_0^{1/2} S''_p(\pi_p t) dt = -\pi_p \left[S'_p(\pi_p t) \right]_{t=0}^{1/2} = \pi_p,$$

and so $f''_1 \in L_1(0, 1)$.

For any function f on $[0, 1]$, we define its successive antiperiodic extension f^* over $\mathbb{R}_+$ by $f^* = f$ on $[0, 1]$, and

$$(9) \quad f^*(t) = -f^*(2k - t) \text{ if } k \leq t \leq k + 1, \quad k \in \mathbb{N}.$$

Then we extend f_1 over $\mathbb{R}_+$ via f_1^* , in agreement with [6].

Finally we need a monotonicity property of this function.

Lemma 5. *For each $t \in (0, 1)$, $f_1(t) = S_p(\pi_p t)$ is strictly decreasing in $p > 1$.*

Proof. From (3), $f_1^{-1}(y) = (\pi_p)^{-1} \int_0^y (1 - \tau^p)^{-1/p} d\tau$, for $0 \leq y \leq 1$. Then the proof follows from the fact that $(\pi_p)^{-1}$ and $(1 - \tau^p)^{-1/p}$ are both positive and strictly increasing in $p > 1$ for fixed $\tau \in (0, 1)$. $\square$

For the reader's convenience, we depict typical profiles of $S_p(\pi_p t)$, together with the limiting cases $p = 1$ and $p = \infty$, in Figure 1.

3. THE OPERATOR T

We start by justifying the route we shall take in proving Theorem 1.

Theorem 6. *If there is a linear homeomorphism T of $L_q(0, 1)$ satisfying $Te_n = f_n, n \in \mathbb{N}$, for $\frac{12}{11} \leq p < \infty$ and $1 < q < \infty$, then Theorem 1 holds.*

Proof. By [17, Example 11.1, pp. 342-345], any function in $L_q(-1, 1)$ has a unique sine-cosine series representation. For any $f \in L_q(0, 1)$, we can thus represent its odd extension to $L_q(-1, 1)$ uniquely in a sine series, so the e_n form a basis of $L_q(0, 1)$. According to [10, p. 75] the same is true for the f_n , and so by [17, Theorem 3.1, p. 20] they form a Schauder basis of $L_q(0, 1)$. The argument for a Riesz basis when $q = 2$ is similar and follows, e.g., [9, Section VI.2]. $\square$

In the remainder of this section we define T as a linear combination of certain isometries of $L_q(0, 1)$. Then we show that T is a bounded operator satisfying $Te_n = f_n, n \in \mathbb{N}$, for all $p, q > 1$.

As in the above proof, the f_n (which belong to $L_q(0, 1)$) have Fourier sine series expansions

$$f_n(t) = \sum_{k=1}^{\infty} \widehat{f_n}(k) \sin(k\pi t), \text{ where } \widehat{f_n}(k) := 2 \int_0^1 f_n(t) \sin(k\pi t) dt.$$

An argument involving symmetry with respect to the midpoint $x = \frac{1}{2}$ easily shows that $\widehat{f_1}(k) = 0$ whenever k is even. Furthermore

$$\begin{aligned} \widehat{f_n}(k) &= 2 \int_0^1 f_1(nt) \sin(k\pi t) dt \\ (10) \quad &= 2 \sum_{m \text{ odd}} \widehat{f_1}(k) \int_0^1 \sin(m\pi nt) \sin(k\pi t) dt \\ &= \begin{cases} \widehat{f_1}(m) & \text{if } mn = k \text{ for some } m \text{ odd,} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

In what follows we will often denote $\widehat{f_1}(j)$ by τ_j . We first find a bound on $|\tau_j|$ which will be crucial in the definition of T below. Integration by parts ensures that

$$\begin{aligned} (11) \quad \tau_j &= 2 \int_0^1 f_1(t) \sin(j\pi t) dt = 4 \int_0^{1/2} f_1(t) \sin(j\pi t) dt \\ &= -\frac{4}{j\pi} \int_0^{1/2} f_1'(t) \cos(j\pi t) dt = -\frac{4}{j^2\pi^2} \int_0^{1/2} f_1''(t) \sin(j\pi t) dt, \end{aligned}$$

where the integrals exist because $f_1'' \in L_1(0, 1)$ by Remark 4. In fact (8) shows that

$$\int_0^{1/2} |f_1''(t) \sin(j\pi t)| dt < \int_0^{1/2} |f_1''(t)| dt = \pi_p,$$

so

$$(12) \quad |\tau_j| < \frac{4\pi_p}{j^2\pi^2}, \quad \text{for all } p > 1, j \in \mathbb{N}.$$

In order to construct the linear operator T , we next define isometries M_m of the Banach space $L_q(0, 1)$ by $M_m g(t) := g^*(mt), m \in \mathbb{N}$, in the notation of (9). Notice that $M_m e_n = e_{mn}$.

Lemma 7. *The maps M_m are isometric linear transformations of $L_q(0, 1)$ for all $m \in \mathbb{N}$ and $q > 1$.*

Proof.

$$\begin{aligned} \int_0^1 |M_m g(t)|^q dt &= \int_0^1 |g^*(mt)|^q dt \\ &= \frac{1}{m} \int_0^m |g^*(u)|^q du = \frac{1}{m} \sum_{k=1}^m \int_{k-1}^k |g^*(u)|^q du \\ &= \frac{1}{m} \sum_{k=1}^m \int_0^1 |g(u)|^q du = \int_0^1 |g(t)|^q dt. \end{aligned}$$

□

We now define $T : L_q(0, 1) \longrightarrow L_q(0, 1)$ by

$$(13) \quad Tg(t) = \sum_{m=1}^{\infty} \tau_m M_m g(t).$$

The above lemma, the triangle inequality and (12) ensure that T is a bounded everywhere-defined operator with $\|T\|_{(L_q \rightarrow L_q)} \leq \frac{\pi_p}{2}$.

We conclude this section by showing that $Te_n = f_n$. Indeed, by virtue of (10),

$$Te_n = \sum_{m=1}^{\infty} \tau_m e_{mn} = \sum_{m-\text{odd}} \widehat{f}_1(m) e_{mn} = \sum_{k=1}^{\infty} \widehat{f}_n(k) e_k = f_n.$$

4. BOUNDED INVERTIBILITY OF T

In this section we complete the proof of our main result by showing that T has a bounded inverse for all $p \geq \frac{12}{11}$ and $1 < q < \infty$.

Observe that (13), Lemma 7 and the triangle inequality give

$$\|T - \tau_1 M_1\|_{(L_q \rightarrow L_q)} \leq \sum_{j=3}^{\infty} |\tau_j|.$$

Since $M_1 = I$, [11, p. 196] shows that it suffices to prove

$$(14) \quad \sum_{j=3}^{\infty} |\tau_j| < |\tau_1|.$$

We will carry out the verification of (14) separately for $p \geq 2$ and $p < 2$.

Case 1 ($p \geq 2$). By virtue of (12) and Lemma 2,

$$\sum_{j=3}^{\infty} |\tau_j| = \sum_{3 \leq j \text{ odd}} |\tau_j| < \frac{4\pi_2}{\pi^2} \left(\frac{\pi^2}{8} - 1 \right) < \frac{8}{\pi^2}.$$

On the other hand, Lemma 3 ensures that

$$\begin{aligned} \tau_1 &= 2 \int_0^1 S_p(\pi_p t) \sin \pi t dt = 4 \int_0^{1/2} S_p(\pi_p t) \sin \pi t dt \\ &> 4 \int_0^{1/2} 2t \sin \pi t dt = \frac{8}{\pi^2}. \end{aligned}$$

This guarantees (14).

FIGURE 2.

Case 2 ($\frac{12}{11} \leq p < 2$). By virtue of Lemma 5,

$$\tau_1 = 2 \int_0^1 S_p(\pi_p t) \sin \pi t \, dt > 2 \int_0^1 (\sin \pi t)^2 \, dt = 1.$$

On the other hand, (11), Lemma 2 and the Cauchy-Schwartz inequality yield

$$\begin{aligned} |\tau_j| &= \left| \frac{4\pi_p}{j^2\pi^2} \int_0^{1/2} S_p''(\pi_p t) \sin j\pi t \, dt \right| \\ &\leq \frac{4\pi_{12/11}}{j^2\pi^2} \left(\int_0^{1/2} (S_p''(\pi_p t))^2 \, dt \right)^{1/2} \left(\int_0^{1/2} (\sin j\pi t)^2 \, dt \right)^{1/2}. \end{aligned}$$

Then (4) and (6) give

$$\sum_{j=3}^{\infty} |\tau_j| \leq \frac{22}{3(\sqrt{3}-1)\pi} \left(\frac{\pi^2}{8} - 1 \right) < 1.$$

This completes the proof of Theorem 1.

Remark 8. A more precise but less explicit statement of Theorem 1 can be given as follows:

There exists $p_0 \in (1, \frac{12}{11})$ such that for $p_0 \leq p < \infty$, the family $\{f_n\}_{n=1}^{\infty}$ forms a Riesz basis of $L_2(0, 1)$ and a Schauder basis of $L_q(0, 1)$ for all $1 < q < \infty$.

As already mentioned at the end of Section 1 the value $p = \frac{12}{11}$ is by no means the optimal one. For example, our proof already shows that inequality (14) holds also for $p \in (\frac{12}{11} - \varepsilon, \frac{12}{11})$ with $\varepsilon > 0$ small enough.

Numerical results indicate that (14) holds when $p = 1.05$, but not when $p = 1.04$, so the method here will not suffice for all $p > 1$.

Remark 9. It is obvious that $\{f_n\}_{n=1}^{\infty}$ is not an orthogonal basis in $L^2(0, 1)$ for $p \neq 2$. Indeed, the sign of

$$I_{i,j} = \int_0^1 f_i(t) f_j(t) \, dt, \quad i, j = 1, 2, \dots$$

depends on p and can change as p varies from 1 to ∞ . Figure 2 illustrates the dependence of $I_{1,3}$, $I_{1,5}$ and $I_{3,7}$ on p .

More values of $I_{i,j}$, and further numerical experiments with the bases $\{f_n\}$, can be found at <http://drift.cam.zcu.cz>.

FIGURE 3.

5. THE LIMIT AS $p \rightarrow \infty$

Although (1) does not represent a differential equation when $p = \infty$, we can still discuss the pointwise limits f_n^∞ of the f_n as $p \rightarrow \infty$, constructed as follows. On $[0, \frac{2}{n})$ we set

$$f_n^\infty(t) := \begin{cases} 2nt, & t \in [0, \frac{1}{2n}], \\ 2 - 2nt, & t \in (\frac{1}{2n}, \frac{3}{2n}), \\ 2nt - 4, & t \in [\frac{3}{2n}, \frac{2}{n}), \end{cases}$$

and extend it periodically to the intervals $[\frac{2k}{n}, \frac{2(k+1)}{n})$, $k \in \mathbb{Z}$. The restriction to $[0, 1]$ is a piecewise linear function whose graph is depicted in Figure 3.

Similarly to Section 3 we can define $\tau_j^\infty := \widehat{f_1^\infty}(j)$. As before, $\tau_j^\infty = 0$ when j is even. For odd j , integration by parts yields

$$\begin{aligned} \tau_j^\infty &= 2 \int_0^1 f_1^\infty(t) \sin(j\pi t) dt = 4 \int_0^{1/2} f_1^\infty(t) \sin(j\pi t) dt \\ &= \frac{4}{j\pi} \int_0^{1/2} 2 \cos(j\pi t) dt = (-1)^j \frac{8}{j^2 \pi^2}; \end{aligned}$$

so

$$|\tau_j^\infty| = \frac{8}{j^2 \pi^2}.$$

Hence, as in Section 4, we have

$$\sum_{j=3}^{\infty} |\tau_j^\infty| < \frac{8}{\pi^2}$$

and so

$$\sum_{j=3}^{\infty} |\tau_j^\infty| < |\tau_1^\infty|.$$

The analogue of Theorem 1 for $p = \infty$ follows as for $p < \infty$, the analogue $T^\infty : L_q(0, 1) \rightarrow L_q(0, 1)$ of T being defined via

$$T^\infty g(t) = \sum_{m=1}^{\infty} \tau_m^\infty M_m g(t).$$

Remark 10. The results of this paper appear to be useful in numerical approaches to boundary value problems for quasilinear equations involving the p -Laplacian, e.g., of the type considered in [3] and [4]. We hope to report on this elsewhere.

REFERENCES

- [1] C. BENNEWITZ, Y. SAITŌ. “An embedding norm and the Lindqvist trigonometric functions”, *Electronic Journal of Differential Equations* 86 (2002), 1–6. MR1938382 (2003j:46028)
- [2] P. BINDING, P. DRÁBEK. “Sturm-Liouville theory for the p -Laplacian”, *Studia Scientiarum Mathematicarum Hungarica* 40 (2003), 375 – 396. MR2037324 (2004j:34068)
- [3] M. DEL PINO, P. DRÁBEK AND R. F. MANÁSEVICH, *The Fredholm alternative at the first eigenvalue for the one-dimensional p -Laplacian*, *J. Differential Equations*, 151 (1999), 386–419. MR1669705 (99m:34042)
- [4] P. DRÁBEK, P. GIRG AND R. F. MANÁSEVICH, *Generic Fredholm alternative for the one dimensional p -Laplacian*, *Nonlin. Diff. Equations and Applications* 8 (2001), 285–298. MR1841260 (2002f:34027)
- [5] W. EBERHART, Á. ELBERT, “On the eigenvalues of a half-linear boundary value problem”, *Math. Nachr.* 213 (2000), 57 – 76. MR1755246 (2001b:34035)
- [6] Á. ELBERT. “A half-linear second order differential equation”. *Coll. Math. Soc. J. Bolyai* 30 (1979) 153 – 179. MR0680591 (84g:34008)
- [7] Á. ELBERT, K. TAKÁSI, T. TINAGAWA, “An Oscillatory half-linear differential equation”, *Arch. Math.* 33 (1997), 355 – 361. MR1601353 (98m:34071)
- [8] C. FABRY, D. FAYYAD, “Periodic solutions of second order differential equations with a p -Laplacian and asymmetric nonlinearities”, *Rend. Ist. Mat. Univ. Trieste* 24 (1992), 207 – 227. MR1310080 (96b:34027)
- [9] I. GOHBERG, M. KREĬN, *Introduction to the Theory of Linear Nonselfadjoint Operators*. Translations of Mathematical Monographs, Vol. 18 American Mathematical Society, 1969. MR0246142 (39:7447)
- [10] J.R. HIGGINS, *Completeness and the Basis Properties of Sets of Special Functions*, Cambridge Tracts in Mathematics, Vol. 72. Cambridge University Press, 1977. MR0499341 (58:17240)
- [11] T. KATO, *Perturbation Theory for Linear Operators*, Springer-Verlag, 1976. MR0407617 (53:11389)
- [12] P. LINDQVIST, “Note on a nonlinear eigenvalue problem”, *Rocky Mount. J. Math.* 23 (1993), 281 – 288. MR1212743 (94d:34031)
- [13] P. LINDQVIST, “Some remarkable sine and cosine functions”, *Ricerche di Matematica* 44 (1995), 269 – 290. MR1469702 (99g:33001)
- [14] Y. NAITO, “Uniqueness of positive solutions of quasilinear differential equations”, *Diff. Int. Equations* 8 (1995), 1813 – 1822. MR1347982 (96g:34039)
- [15] M. ÔTANI, “A Remark on certain nonlinear elliptic equations”, *Proc. Fac. Sci. Tokai Univ.* 19 (1984), 23 – 28. MR0753635 (86c:35009)
- [16] W. REICHEL, W. WALTER, “Sturm-Liouville type problems for the p -Laplacian under asymptotic nonresonance conditions”, *J. Differential Equations* 156 (1999), 50 – 70. MR1701814 (2000e:34036)
- [17] I. SINGER, *Bases in Banach Spaces I*, Springer-Verlag, 1970. MR0298399 (45:7451)

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF CALGARY, CALGARY, ALBERTA, CANADA T2N 1N4

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF CALGARY, CALGARY, ALBERTA, CANADA T2N 1N4

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WEST BOHEMIA, PILSEN, CZECH REPUBLIC

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WEST BOHEMIA, PILSEN, CZECH REPUBLIC

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WEST BOHEMIA, PILSEN, CZECH REPUBLIC