A SHORT PROOF OF AN INEQUALITY OF LITTLEWOOD AND PALEY

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ABSTRACT. A very short proof is given of the inequality

$$\int_{|z|<1} |\nabla u(z)|^p (1 - |z|)^{p-1} dx dy \leq C_p \left(\frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|^p dt - |u(0)|^p \right),$$

where $p > 2$, and u is the Poisson integral of $f \in L^p(\partial\mathbb{D})$, $\mathbb{D} = \{z : |z| < 1\}$.

Let $\mathbb{D}$ denote the open unit disk of the complex plane, and $\mathbb{T} = \partial\mathbb{D}$. The following theorem was proved by Littlewood and Paley in [2].

Theorem LP. *If f is a real valued function of class $L^p(\mathbb{T})$, $p > 2$, and if u is the Poisson integral of f , then*

$$(1) \quad \int_{\mathbb{D}} |\nabla u(z)|^p (1 - |z|^2)^{p-1} dA(z) \leq C_p \|f\|_p^p,$$

where C_p is a constant depending only on p and

$$\|f\|_p^p = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|^p dt.$$

Here dA stands for the Lebesgue measure normalized so that $A(\mathbb{D}) = 1$.

This theorem can easily be proved by using the Riesz-Thorin interpolation theorem. In [3] Luecking gave an elementary but rather long proof based on the formula

$$(2) \quad \|f\|_p^p - |u(0)|^p = \frac{(p^2 - p)}{2} \int_{\mathbb{D}} |\nabla u|^2 |u|^{p-2} \log \frac{1}{|z|} dA(z).$$

This formula, a consequence of the Green formula, was used by P. Stein [5] to prove the Riesz theorem on conjugate functions (see [1], p. 55). We also start from (2), and reduce to the case of positive harmonic functions, which satisfy the following inequality:

If u is a positive harmonic function on $\mathbb{D}$, then

$$(3) \quad |\nabla u(z)| \leq 2(1 - |z|^2)^{-1} u(z) \quad (z \in \mathbb{D}).$$

This inequality is obtained by applying the special case $z = 0$ to the function

$$w \mapsto u \left(\frac{z - w}{1 - \bar{z}w} \right).$$

In fact, we shall prove a slightly improved version of Theorem LP.

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Theorem 1. *If f is a real valued function of class $L^p(\mathbb{T})$, $p > 2$, and if u is the Poisson integral of f , then*

$$(4) \quad \int_{\mathbb{D}} |\nabla u(z)|^p (1 - |z|^2)^{p-1} dA(z) \leq C_p (\|f\|_p^p - |u(0)|^p),$$

where C_p is a constant depending only on p .

Proof. Let $f \in L^p(\mathbb{T})$, $p > 2$. Let u_i ($i = 1, 2$) denote the Poisson integral of f_i , where $f_1 = \max(f, 0)$ and $f_2 = \max(-f, 0)$. Then $u_i \geq 0$, $u = u_1 - u_2$ and

$$(5) \quad \|f\|_p^p = \|f_1\|_p^p + \|f_2\|_p^p.$$

Also, since

$$(6) \quad |\nabla u|^p \leq 2^{p-1} (|\nabla u_1|^p + |\nabla u_2|^p),$$

the proof reduces to the case where $u > 0$. Then it follows from (2), (3) and the inequality

$$\log \frac{1}{|z|} \geq \frac{1 - |z|^2}{2}$$

that

$$\begin{aligned} \|f\|_p^p - |u(0)|^p &\geq \frac{p^2 - p}{4} \int_{\mathbb{D}} |\nabla u|^2 u^{p-2} (1 - |z|^2) dA(z) \\ &\geq \frac{p^2 - p}{4} \int_{\mathbb{D}} |\nabla u|^2 2^{2-p} |\nabla u|^{p-2} (1 - |z|^2)^{p-1} dA(z). \end{aligned}$$

This proves (1) for $f > 0$ with $C_p = 2^p/(p^2 - p)$. If f is arbitrary, then we use (5), (6) and the inequality

$$|a - b|^p \leq a^p + b^p \quad (a \geq 0, b \geq 0)$$

to get

$$\begin{aligned} \|f\|_p^p - |u(0)|^p &\geq \|f_1\|_p^p - |u_1(0)|^p + \|f_2\|_p^p - |u_2(0)|^p \\ &\geq (p^2 - p)/2^p \int_{\mathbb{D}} (|\nabla u_1|^p + |\nabla u_2|^p) (1 - |z|)^{p-1} dA(z) \\ &\geq 2^{1-p} (p^2 - p)/2^p \int_{\mathbb{D}} |\nabla u|^p (1 - |z|)^{p-1} dA(z). \end{aligned}$$

Hence

$$\int_{\mathbb{D}} |\nabla u|^p (1 - |z|^2)^{p-1} dA(z) \leq C_p (\|f\|_p^p - |u(0)|^p)$$

with

$$C_p = 2^{2p-1}/(p^2 - p).$$

This completes the proof. $\square$

Remark 1. Inequality (4) can be written as

$$\|f\|_p^p - |f(0)|^p \geq c_p \int_{\mathbb{D}} |\nabla u|^p (1 - |z|^2)^{p-1} dA(z),$$

which can be viewed as a refinement of the inequality $\|f\|_p^p - |f(0)|^p \geq 0$, a consequence of the subharmonicity of the function $|u(z)|^p$.

Remark 2. Inequality (4) holds for functions with values in a Hilbert space, but the proof is more delicate. See [4].

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