FIXED POINTS OF UNIVALENT FUNCTIONS II

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(Communicated by Juha M. Heinonen)

To my friend Masakazu Shiba on the occasion of his 60th birthday

ABSTRACT. For a closed nowhere dense subset C of $\partial\mathbb{D}$ a bounded univalent holomorphic function f on $\mathbb{D}$ is found such that C equals the cluster set of its fixed points.

1. INTRODUCTION

Let f be a univalent (i.e. holomorphic and injective, but not necessarily normalized) function of the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$.

Such a function $f \neq id_{\mathbb{D}}$ may have infinitely many fixed points. In [2] it is proved that for each Blaschke sequence (z_n) there exists a univalent function $f \neq id_{\mathbb{D}}$ with $f(z_n) = z_n$ for all $n \in \mathbb{N}$, provided that the cluster set $F_c(f)$ of (z_n) is an arbitrary Carleson set, i.e. a closed set of Lebesgue measure 0 on $\partial\mathbb{D}$ with $\sum_n \varepsilon_n \log \varepsilon_n > -\infty$, where ε_n denotes the length of the open components (arcs) of its complement with respect to $\partial\mathbb{D}$. In [4] it is proved that F_c is a nowhere dense subset of the unit circle unless $f \equiv id_{\mathbb{D}}$. It remains open whether $F_c(f)$ can be a proper subset of $\partial\mathbb{D}$ of positive measure. Here we will prove:

Theorem 1. *Let C be a nowhere dense, closed subset of $\partial\mathbb{D}$. Then there exists a bounded univalent function $g : \mathbb{D} \rightarrow \mathbb{C}$ with the property that the cluster set $F_c(g) \subset \partial\mathbb{D}$ of its fixed points is C .*

Cantor sets on $\partial\mathbb{D}$ fulfill the assumption of Theorem 1. Their Lebesgue measure with respect to $\partial\mathbb{D}$ can be arbitrarily close to 2π .

If $f : \mathbb{D} \rightarrow \mathbb{C}$ is holomorphic and $\alpha \in [0, 2\pi[$, then the cluster set $C_\alpha(f)$ of the restriction $f_\alpha(t) := f(te^{i\alpha})$ for $t \rightarrow 1-$ is called the radial cluster set of f with respect to $e^{i\alpha}$. It is well known that this cluster set coincides with the cluster set of $f(z)$, where z tends to $e^{i\alpha}$ in an arbitrary triangle $\Delta_\alpha \subset \mathbb{D}$ with $e^{i\alpha} \in \partial\Delta_\alpha$. One can also describe $C_\alpha(f)$ as the set of principal points of the prime end of the image domain $f(\mathbb{D})$ which belongs to the boundary point $e^{i\alpha}$ of $\mathbb{D}$ (see [5] or [6]). An accumulation point of $f(z)$ for $z \rightarrow e^{i\alpha}$ which is not radial is called tangential. In terms of prime ends of $f(\mathbb{D})$ these are the points of the corresponding prime end impressions which are not principal points.

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By a result in [4] (Thm. 1), which combines a result of Beurling ([1], Thm. 3.5) and the theorem of F. and M. Riesz ([1], Thm. 2.5), the set $\{e^{i\alpha} : e^{i\alpha} \in C_\alpha(f)\}$ has measure zero if $f : \mathbb{D} \rightarrow \mathbb{C}$ is univalent. This shows that the fixed points of the function f in Theorem 1 have to approach their accumulation points on the boundary of $\mathbb{D}$ tangentially up to a zero set.

The crucial tool in the proof is Theorem 2, which states the possibility of fusion of univalent functions. This is a consequence of an improved version of Alice Roth's well-known fusion lemma, due to Masakazu Shiba and the author ([7], [8]).

2. FUSION OF UNIVALENT FUNCTIONS

In this section we will prove the following result.

Theorem 2. *Let G_1, G_2 be simply connected domains, bounded by Jordan curves γ_1 , resp. γ_2 , and $\overline{G_1} \cap \overline{G_2} = \{z_0\}$. Moreover let f_1 , resp. f_2 , be continuous on $\overline{G_1}$, resp. on $\overline{G_2}$, univalent on G_1 , resp. on G_2 .*

If $f_1(\overline{G_1}) \cap f_2(\overline{G_2})$ consists of the single point $w_0 = f_1(z_0) = f_2(z_0)$, then, for all $\varepsilon > 0$, there exists a neighborhood U of $\overline{G_1} \cup \overline{G_2}$ and an entire function f , which is univalent on U and fulfills $\|f - f_1\|_{G_1} < \varepsilon$ as well as $\|f - f_2\|_{G_2} < \varepsilon$.

The proof requires some preparation.

Definition 1. We call a pair K_1, K_2 of compact sets in the plane normal, if

- (1) $\mathbb{C} \setminus (K_1 \cup K_2)$ is connected.
- (2) The interiors K_1°, K_2° of both sets are connected.
- (3) $K_1 \cap K_2^\circ = K_2 \cap K_1^\circ = \emptyset$.
- (4) $K_1 = \overline{K_1^\circ}$ and $K_2 = \overline{K_2^\circ}$.

Definition 2. We say that a pair of compact plane sets K_1, K_2 has the rational fusion property if there exists some positive number $a = a(K_1, K_2)$ such that the following is true: For each pair r_1, r_2 of rational functions and for each compact set K there is some rational function r with

$$|r(z) - r_j(z)| \leq a \cdot \sup\{|r_1(w) - r_2(w)| : w \in K \cup (K_1 \cap K_2)\}$$

for all $z \in K_j \cup K$ simultaneously for $j = 1, 2$.

We say that the pair K_1, K_2 has the holomorphic fusion property with respect to some domain $G \subset \mathbb{C}$ if we may replace the rational functions r_1, r_2, r by holomorphic functions on G .

In [7] the following extension of Alice Roth's famous Fusion Lemma (cf. [3]) is proved, basing on a result of M. Shiba and the author [8].

Proposition 1. *A normal pair of compact sets K_1, K_2 has the rational fusion property if and only if $\partial K_1 \cup \partial K_2 = \partial(K_1 \cup K_2)$.*

Proof of Theorem 2. Let $\gamma_{1,\delta}$ be a Jordan curve in $\mathbb{C} \setminus \overline{G_1}$ with $\text{dist}(\gamma_1, \gamma_{1,\delta}) < \delta$ and let $G_{1,\delta}$ denote the bounded component of $\mathbb{C} \setminus \gamma_{1,\delta}$. Then $\overline{G_1} \subset G_{1,\delta}$, and we have a biholomorphic map $\phi_\delta : G_{1,\delta} \rightarrow G$. An equicontinuity argument shows that, for each given ε , the function $f_{1,\delta} = f_1 \circ \phi_\delta$ fulfills $\|f_1 - f_{1,\delta}\|_{G_1} < \varepsilon$ if $\delta \leq \delta_0(\varepsilon)$. We may also assume that $f_{1,\delta}(z_0) = f_1(z_0)$, otherwise we replace ε by $\varepsilon/2$ and consider

the function $f_{1,\delta}(z_0) + f_1(z_0) - f_{1,\delta}(z_0)$. This shows that we may provide that f_1 (as well as f_2) is univalent on a neighborhood of $K_1 := \overline{G_1}$ (resp. on $K_2 = \overline{G_2}$).

Without loss of generality we may assume that $z_0 = 0$, $f_1(0) = 0 = f_2(0)$.

In the first part of the proof we consider the case that $f_1'(0) = f_2'(0)$. Without loss of generality we may assume that f_1, f_2 are entire functions, because we may approximate our starting functions uniformly on suitable compact sets containing K_1 , resp. K_2 , in its interior. This will preserve the univalence on a neighborhood of K_j (as the argument principle shows) if the approximation bound is sufficiently small.

Obviously K_1, K_2 is a normal pair, and we also find a normal pair of compact sets C_j , also bounded by a Jordan curve respectively, containing $K_j \setminus \{0\}$ ($j = 1, 2$) in the interior, such that $C_1 \cap C_2 = \{0\}$, and f_1, f_2 are univalent in an open neighborhood of C_1, C_2 . We may also provide that C_1, C_2 is a rational fusion pair by Proposition 1 and, obviously, that it is a holomorphic fusion pair. As the “bridge” K joining C_1, C_2 we take a disc $D_\rho := \{|z| \leq \rho\}$; its radius will be fixed later. So we find some entire function f with the property that

$$|f'(z) - f_j'(z)| \leq a \cdot \max\{|f_1'(w) - f_2'(w)| : w \in D_\rho\} \quad (z \in C_j \cup D_\rho, j = 1, 2),$$

where a depends only on C_1, C_2 . This shows that (note that $f_1(0) = f_2(0)$ has been assumed)

$$|f(z) - f_j(z)| \leq b \cdot \max\{|f_1'(w) - f_2'(w)| : w \in D_\rho\} \quad (z \in C_j \cup D_\rho, j = 1, 2)$$

with some b depending only on geometrical properties of C_1, C_2 . Thus, by the argument principle, we see that f is univalent on a neighborhood of C_1 as well as on a neighborhood of C_2 if $\max\{|f_1'(w) - f_2'(w)| : w \in D_\rho\}$ is sufficiently small, and thus for small ρ , as follows from the assumption that $f_1'(0) = f_2'(0)$ and from the continuity of f_j' .

On D_ρ we have for small ρ that $|f'(z) - f_1'(0)| < |f_1'(0)|$. Since $f_1'(0) \neq 0$ this implies that $\Re f' > 0$ on D_ρ , and a well-known argument shows that therefore f is univalent on D_ρ . Finally we may assume that $f(C_1 \setminus D_\rho) \cap f(C_2 \setminus D_\rho) = \emptyset$, as follows from the assumption that $f_1(K_1) \cap f_2(K_2) = \{0\}$, provided that the approximation was done with a bound sufficiently close to zero.

So f has the desired properties if ρ is small enough.

Now we consider the case that $f_1'(0) \neq f_2'(0)$.

Let $\sigma(z) = \sigma(z, \eta) = 1 - (1 - z)^\eta$ with some (small) $\eta > 0$. This function is holomorphic on each simply connected domain which does not contain 1. We obtain $\sigma'(1 - \eta^{1/(1-\eta)}, \eta) = 1$. Without loss of generality we may assume that $1 \notin K_1$.

Now we consider the functions $\varphi(z, \eta) = \sigma(z + 1 - \eta^{1/(1-\eta)}, \eta)$ and $g_1(z) = f_1(z) + (f_2'(0) - f_1'(0))\varphi(z, \eta)$ on K_1 for small η . Note that this expression is defined on a neighborhood of K_1 . Then, by the mentioned properties of σ , we have that

$$\|\varphi(\cdot, \eta)\|_{K_1} = |\varphi(0)| = 1 - \eta^{\eta/(1-\eta)} \rightarrow 0 \text{ if } \eta \rightarrow 0,$$

provided that $\eta = \eta(K, \varepsilon)$ is small. Then g_1 is univalent on a neighborhood of K_1 for all small η , and if some ε is given, we have that $\|g_1 - f_1\|_{K_1} < \varepsilon$ for those η . On the other hand we see that $g_1'(0) = f_2'(0)$. The remaining argumentation is as in the first case, if we replace f_1 by g_1 . $\square$

3. A PREPARATORY LEMMA ON CONFORMAL MAPPINGS

We will need the following statement on conformal mappings.

Lemma 1. *Let $D = D_\kappa = \{|z - 1| < \kappa\}$ ($\kappa > 0$) and let H be a (simply connected) Jordan domain with $H \subset \mathbb{D} \cup D$ and $\mathbb{D} \setminus D \subset H$. The set $H \cap \partial\mathbb{D} \subset D$ is assumed to be nonempty and connected.*

Let Θ denote the Riemann map of $\mathbb{D}$ onto H with $\Theta(0) = 0$ and let some $\varepsilon > 0$ be given. With respect to the continuous extension of Θ on $\overline{\mathbb{D}}$ we define the set

$$E = \{z : |z| = 1 \text{ and } |\Theta(z)| > 1\}.$$

Then E is an interval on $\partial\mathbb{D}$, and there exists some $\kappa_0 > 0$ such that the length of E is less than ε if $\kappa < \kappa_0$.

Proof. That E is an interval on $\partial\mathbb{D}$ follows from the assumption on $H \cap \partial\mathbb{D} \subset D$.

If the assertion were not true, we could find some positive ε and a sequence $\delta_k \rightarrow 0, \kappa_k \rightarrow 0$ and simply connected domains H_k with the described properties, such that the lengths of the intervals E_k (defined with respect to the associated conformal maps Θ_k with $\Theta_k(0) = 0$) all are at least ε . Let V be a disc centered at 1. Then it follows by standard arguments that the sequence of the conformal maps Θ_k converges uniformly on $\overline{\mathbb{D}} \setminus V$ to the identity map. Thus E_k must be contained in V for all sufficiently large k . This leads to a contradiction if we take V so small that the length of $\partial\mathbb{D} \cap V$ is less than ε . $\square$

4. A SEQUENCE OF FUNCTIONS

Let $C \subset \partial\mathbb{D}$ be a nowhere dense closed set. Then the complement with respect to $\partial\mathbb{D}$ is open, and thus is the countable union of pairwise disjoint open intervals I_j on $\partial\mathbb{D}$. Let $e^{i\alpha_j}$ be the center point of I_j . For each $j \in \mathbb{N}$ we fix some open disk V_j with $e^{i\alpha_j} \in V_j$ and $V_j \cap \partial\mathbb{D} \subset I_j$. Moreover we may assume that these V_j are pairwise disjoint.

Remark 1. If a sequence $w_j \in \overline{V_j} \cap \mathbb{D}$ ($j \in \mathbb{N}$) is given, then the cluster set of (w_j) consists of all nonisolated points of C , as follows from the properties of C as a closed and nowhere dense set. In order to construct a function g which also fulfills the assertion of Theorem 1 with respect to possible isolated points of C , we can enlarge this set by adding a discrete subset of $\partial\mathbb{D}$, which contains for each isolated point $c_1 \in C$ a sequence $c_1' \in \partial\mathbb{D} \setminus C$ tending to c_1 . Then the fixed point cluster $F_c(g)$ of the function g we will construct below contains c_1 , but not the additional points c_1' . We may assume in the following that C has been prepared in this way.

Now we describe a sequence of positive numbers ε_j ($j \geq 2$) with $\varepsilon_2 > \varepsilon_3 > \dots$ and $\sum_{j=1}^{\infty} \varepsilon_j < \frac{1}{3}$.

The aim of this section is to construct a sequence of functions $g_n : \mathbb{D} \rightarrow \mathbb{C}$ which fulfills:

- (i) g_n is holomorphic and univalent on $\overline{\mathbb{D}}$.
- (ii) $|g_n(z) - g_{n-1}(z)| < \varepsilon_n$ for all $z \in \mathbb{D} \setminus V_n$ if $n > 1$.
- (iii) $\mathbb{C} \setminus g_n(\mathbb{D})$ is connected.
- (iv) $e^{i\alpha_j} \notin g_n(\mathbb{D})$ for $j > n$.
- (v) For each $n \in \mathbb{N}$ the function g_n has fixed points $z_{n,k} \in V_k$ ($k = 1, \dots, n$), and it holds that $|z_{n,k} - z_{k,k}| < \frac{1}{2} \text{dist}(z_{k,k}, \partial V_k)$ for all $k = 1, \dots, n-1$.

- (vi) g_n has in $\overline{\mathbb{D}} \setminus V_n$ the same number of fixed points as g_{n-1} for all $n \geq 2$.
- (vii) g_n has no fixed point on $\partial\mathbb{D}$ ($n \in \mathbb{N}$).

We try $g_1(z) := \frac{z}{4} + e^{i\alpha_1} \cdot \sigma\left(\frac{ze^{-i\alpha_1}}{1+\eta}, \eta\right)$ with $0 < \eta < 1$ (the functions $\sigma(\cdot, \eta)$ have been defined close to the end of section 2). The derivative of this function has, for each choice of the parameters, positive real part in $\mathbb{D}$, and therefore it is univalent. The number η can be taken in such a way that g_1 has a fixed point $z_{1,1} = \rho_1 e^{i\alpha_1}$ in V_1 . Concerning (v) we have nothing more to check in this first step. Note that $\sigma(\cdot, \eta)$ tends uniformly to 0 on each compact subset of $\mathbb{C} \setminus [1, \infty[$. The conditions (i), (iii), (iv) and (vii) are obviously fulfilled for all sufficiently small positive η , while (ii) and (vi) are empty in this case. If some number b bigger than $\frac{1}{4}$ is given we can arrange that $|g_1| < b$ on $\mathbb{D} \setminus V_1$.

Let, for some $n \in \mathbb{N}$, a function g_n with the mentioned properties be found, and we will construct the function g_{n+1} .

We start with this part of (v), which claims that there are appropriate fixed points $z_{n+1,1}, \dots, z_{n+1,n}$ of our candidate function F . The “new” fixed point $z_{n+1,n+1}$ we will discuss later. From (v) we know that g_n has a fixed point $z_{n,k}$ in each of the discs

$$\Delta_k := \left\{ |z - z_{k,k}| < \frac{1}{2} \operatorname{dist}(z_{k,k}, \partial V_k) \right\} \quad (k = 1, \dots, n).$$

Since a fixed point of a function f is a solution of the equation $f(z) - z = 0$, we know that 0 is an inner point of every domain $W_k := (g_n - id)(\Delta_k)$ ($k = 1, \dots, n$). So, if $\varepsilon < \operatorname{dist}(z_{n,k}, \partial W_k)$ for all $k = 1, \dots, n$, we indeed find fixed points $z_{n+1,k} \in \Delta_k \subset V_k$ ($k = 1, \dots, n$) of F , provided that $|F - g_n| < \varepsilon$ on $\Delta_1 \cup \dots \cup \Delta_n$. From (vii) we conclude that the fixed point set of g_n in $\mathbb{D}$ has positive distance d to $\partial\mathbb{D}$. Therefore we will approximate g_n with a bound

$$\varepsilon < \min \left\{ \varepsilon_{n+1}, \min_{k=1, \dots, n} \operatorname{dist}(z_{n,k}, \partial W_k), d \right\}.$$

Now we fix some ε according to this condition. Note that each holomorphic $F : \overline{\mathbb{D}} \rightarrow \mathbb{C}$ with $\|F - g_n\|_{\mathbb{D} \setminus V_{n+1}} < \varepsilon$ will have fixed points $z_{n+1,1}, \dots, z_{n+1,n}$ with the properties mentioned in (v). Moreover, the number of fixed points in $\overline{\mathbb{D}} \setminus V_{n+1}$ of such a function F is the same as of g_n in this set. Thus (vi) is automatically fulfilled if we take F as described.

In particular, F cannot have any fixed point in $\partial\mathbb{D} \setminus V_{n+1}$, as follows from (vii).

Now we consider the point $z_0 := e^{i\alpha_{n+1}}$. Then $z_0 \in \mathbb{C} \setminus \overline{g_n(\mathbb{D})}$ by (v). Since $z_0 \notin V_1 \cup \dots \cup V_n$ it follows from (ii) that $|g_n(z_0)| \leq |g_1(z_0)| + \sum_{j=2}^n \varepsilon_j \leq \frac{1}{3} + \frac{1}{3}$.

By (iii) we find some smooth Jordan arc

$$\Gamma : [0, 1] \rightarrow \mathbb{C} \setminus \bigcup_{j \in \mathbb{N} \setminus \{n+1\}} \overline{V_j}$$

with $\Gamma(0) = g_n(z_0)$, $\Gamma(1) = z_0$ and $\Gamma(t) \in \mathbb{C} \setminus \overline{g_n(\mathbb{D})}$ for all $t \in]0, 1[$.

Moreover there exists some simply connected domain

$$(1) \quad V \subset \mathbb{C} \setminus \left(\overline{g_n(\mathbb{D})} \cup \bigcup_{j \in \mathbb{N} \setminus \{n+1\}} \overline{V_j} \right)$$

(think of V as a small tube containing $\Gamma([0, 1])$) with $\partial V \cap \partial g_n(\mathbb{D}) = \{g_n(z_0)\}$.

Remark 2. The domain V depends only on g_n and on z_0 , and it contains some closed disc $D^* = \{|z - z_0| \leq r\}$ of positive radius.

We find, by Riemann's mapping theorem, some surjective univalent function

$$f_1 : G_1 := \{z : |z - (1 + \frac{\kappa}{4})z_0| < \frac{\kappa}{4}\} \rightarrow V$$

with $f_1(z_0) = g_n(z_0)$, where the parameter $\kappa > 0$ is arbitrary in the moment, but it will be fixed below. With the notations $K_1 = \overline{G_1}$, $G_2 = \mathbb{D}$, $K_2 = \overline{G_2}$ and $f_2 = g_n$ we obtain:

- (a) $\Gamma([0, 1]) \subset f_1(K_1)$,
- (b) G_1 and G_2 are Jordan domains,
- (c) $K_1 \cap K_2 = \{z_0\}$,
- (d) f_1 , resp. f_2 , is continuous on K_1 , resp. on K_2 , and univalent on G_1 , resp. on G_2 ,
- (e) $f_1(K_1) \cap f_2(K_2) = \{g_n(z_0)\}$ and $g_n(z_0) = f_1(z_0) = f_2(z_0)$.

We apply Theorem 2 and find an entire function f , which is univalent on a neighborhood U of $K_1 \cup K_2$ and fulfills

$$(2) \quad \|f_1 - f\|_{K_1} < \frac{\varepsilon}{2} \text{ and } \|f_2 - f\|_{K_2} < \frac{\varepsilon}{2}.$$

Let $H \subset \mathbb{D} \cup \{|z - z_0| < \kappa\}$ be a simply connected Jordan domain such that $\mathbb{C} \setminus \overline{H}$ is connected and that f is univalent on H .

Now we apply Lemma 1 and consider the conformal map $\Theta : \mathbb{D} \rightarrow H$. We will have that

$$(3) \quad \|\Theta - id\|_{\mathbb{D}} < \frac{\varepsilon}{2} \text{ and } G_1 \subset \Theta(V_{n+1} \cap \mathbb{D}),$$

if κ is small enough, and for small κ we also see that $F^{-1}(D^*) \subset D^*$, where $F := f \circ \Theta$ (which is our candidate for the desired function g_{n+1}). We fix such a number κ .

F is continuous on $\overline{\mathbb{D}}$ and univalent on $\mathbb{D}$ by construction. This gives (i).

On $\mathbb{D} \setminus V_{n+1}$ we see that $|F(z) - g_n(z)| < \varepsilon \leq \varepsilon_{n+1}$ by (2) and (3). So (ii) is fulfilled (with F as the candidate for g_{n+1}).

$\mathbb{C} \setminus \overline{F(\mathbb{D})}$ is the complement of the closure of the domain H , and thus this set is connected by construction. This shows that (iii) holds.

Since $e^{i\alpha_j} \in V_j$ ($j \in \mathbb{N}$) we verify (iv) by (1).

Now we prove the part of (v) concerning the desired fixed point $z_{n+1, n+1}$ of F in V_{n+1} .

It has been remarked that the closed disc D^* is mapped under F^{-1} into itself. By Brouwer's fixed point theorem, F^{-1} has a fixed point z_1 in D^* . But then $z_1 = F^{-1}(z_1) \in V_{n+1}$ by (3) and by $D^* \subset V$.

Finally we discuss property (vii) with respect to F . We discuss the $h(z) := F(z) - z$. For each $\delta > 0$ there is of course some complex number w_0 with $w_0 \notin h(\partial\mathbb{D})$. If we consider, for suitably small δ , the function $F(z) - w_0$ instead of F , then (vii) is fulfilled and (i), ..., (vi) remain true.

Now we are ready to define $g_{n+1} := F$ and $z_{n+1, n+1} := z_1$.

The construction of the functions g_n can be done in such a way that, for example, $|g_n(z)| < 2$ for all $n \in \mathbb{N}$ and for all $z \in \mathbb{D}$. The bound 2 could be replaced by each number bigger than 1.

5. PROOF OF THEOREM 1

From (ii) and $\sum_{j=1}^{\infty} \varepsilon_j < \frac{1}{3}$ we see that $g = \lim_{n \rightarrow \infty} g_n$ exists, and g can be neither the identity nor a constant function. By standard arguments we conclude that g is univalent on $\mathbb{D}$, and it follows from the construction that $|g|$ is bounded. The inequality

$$|z_{n,k} - z_{k,k}| < \frac{1}{2} \operatorname{dist}(z_{k,k}, \partial V_k)$$

for all $1 \leq k < n$ in (v) guarantees that none of the sequences $s_k = (z_{n,k})_{n \geq k}$ can tend to $\partial \mathbb{D}$. Thus g has in each V_k a fixed point z_k , which obviously fulfills $|z_k - z_{k,k}| \leq \frac{1}{2} \operatorname{dist}(z_{k,k}, \partial V_k)$.

The fixed point cluster $F_c(g)$ of the function g contains the cluster set of the sets V_k , which is C (or, to be more precise, the set of all nonisolated points of C , compare Remark 1).

The validity of the opposite inclusion can be obtained as follows. Let some $\zeta \in \partial \mathbb{D}$ be given, which is not a nonisolated point of C (compare Remark 1). Then there is some open disk $U(\zeta)$, centered at ζ , and some $n_0 \in \mathbb{N}$ with $U(\zeta) \cap V_n = \emptyset$ for all $n \geq n_0$. Therefore, by (vi), we see that the number of fixed points of g_n in $U(\zeta) \cap \mathbb{D}$ has a (with respect to n) uniform finite bound. Thus ζ does not belong to $F_c(g)$. $\square$

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