AN ITÔ FORMULA FOR A FRACTIONAL BROWNIAN SHEET WITH ARBITRARY HURST PARAMETERS

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ABSTRACT. By using the white noise theory for a fractional Brownian sheet, we derive an Itô formula for the fractional Brownian sheet with arbitrary Hurst parameters $H_1, H_2 \in (0, 1)$.

1. INTRODUCTION AND PRELIMINARIES

For $H_1, H_2 \in (0, 1)$, a real-valued fractional Brownian sheet B^H with Hurst parameter $H = (H_1, H_2)$ is a centered Gaussian random field with covariance

$$\mathbb{E}(B_{t_1, t_2}^{H_1, H_2} B_{s_1, s_2}^{H_1, H_2}) = \prod_{i=1}^2 \frac{1}{2} (|t_i|^{2H_i} + |s_i|^{2H_i} - |t_i - s_i|^{2H_i}).$$

The fractional Brownian sheet with Hurst parameter H has the following integral representation:

$$\begin{aligned} B_{t_1, t_2}^{H_1, H_2} &= \frac{C(H)}{\Gamma(H_1 + (1/2))\Gamma(H_2 + (1/2))} \\ (1) \quad &\times \int_{\mathbb{R}} \int_{\mathbb{R}} \prod_{i=1}^2 [((t_i - s_i)_+)^{H_i - (1/2)} - ((-s_i)_+)^{H_i - (1/2)}] dW_{s_1, s_2}, \end{aligned}$$

where W is a standard Brownian sheet, $s_+ = \max\{s, 0\}$ and $C(H)$ is the normalizing constant.

The purpose of this paper is to obtain an Itô formula for the fractional Brownian sheet with arbitrary Hurst parameters $H_1, H_2 \in (0, 1)$. The main tool is the white noise theory for the fractional Brownian sheet with $H_1, H_2 \in (0, 1)$, given by Hu *et al.* [4]. Our result holds for Hurst parameters $H_1, H_2 \in (0, 1)$, whereas that of Tudor and Viens [5] is valid only for Hurst parameters $H_1, H_2 \in (1/2, 1)$.

Now we briefly mention the white noise theory of fractional Brownian sheet, given in [4], to be necessary for our main result. Let $\mathcal{S}(\mathbb{R}^2)$ be the Schwartz space

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of rapidly decreasing smooth functions on $\mathbb{R}^2$. We consider the white noise space $(\Omega, \mathbf{F}, \mathbb{P})$ as the underlying probability space, i.e., $\Omega := \mathcal{S}'(\mathbb{R}^2)$ is the space of tempered distributions and $\mathbb{P}$ is a unique probability measure such that for all $f \in \mathcal{S}(\mathbb{R}^2)$,

$$\int_{\mathcal{S}'(\mathbb{R}^2)} e^{i\langle \omega, f \rangle} d\mathbb{P}(\omega) = e^{-(1/2)\|f\|_{L^2(\mathbb{R}^2)}^2}.$$

We can extend $\langle \cdot, f \rangle$ to $f \in L^2(\mathbb{R}^2)$ and $\mathbb{E}[\langle \cdot, f \rangle \langle \cdot, g \rangle] = (f, g)_{L^2(\mathbb{R}^2)}$. For $s = (s_1, s_2), t = (t_1, t_2)$ and $x = (x_1, x_2) \in \mathbb{R}^2$, we define $\mathbf{1}_{(s,t)}(x) = \prod_{i=1}^2 \mathbf{1}_{(s_i, t_i)}(x_i)$, where the indicator function $\mathbf{1}_{(s_i, t_i)}(x_i)$ is given by

$$\mathbf{1}_{(s_i, t_i)}(x_i) = \begin{cases} 1 & \text{for } s_i \leq x_i < t_i, \\ -1 & \text{for } t_i \leq x_i < s_i, \\ 0 & \text{otherwise.} \end{cases}$$

For $f \in \mathcal{S}(\mathbb{R}^2)$, we define $I_{\pm}^{H_i} f : \mathbb{R}^2 \rightarrow \mathbb{R}$, $i = 1, 2$, by

$$(2) \quad I_{\pm}^{H_i} f(x) = \begin{cases} \frac{1}{\Gamma(H_i - (1/2))} \int_0^\infty \frac{f(x \mp u\epsilon_i)}{u^{(3/2) - H_i}} du & \text{for } 1/2 < H_i < 1, \\ f & \text{for } H_i = \frac{1}{2}, \\ \frac{(1/2) - H_i}{\Gamma(H_i + (1/2))} \int_0^\infty \frac{f(x) - f(x \mp u\epsilon_i)}{u^{(3/2) - H_i}} du & \text{for } 0 < H_i < 1/2, \end{cases}$$

where $\epsilon_1 = (1, 0)$ and $\epsilon_2 = (0, 1)$. Let $M_{\pm\pm}^{H_1, H_2} f(x) = I_{\pm}^{H_1}(I_{\pm}^{H_2})f(x)$. Then a continuous version of $\langle \cdot, M_{\pm\pm}^{H_1, H_2} \mathbf{1}_{(0,t)} \rangle$ is a fractional Brownian sheet with arbitrary Hurst parameters $H_1, H_2 \in (0, 1)$ on $(\Omega, \mathbf{F}, \mathbb{P})$.

Let $\mathbf{H}_n(x)$ be the n -th Hermite polynomial and h_n the n -th Hermite function, $n = 0, 1, \dots$. For $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ (with $\mathbb{N} = \{1, 2, \dots\}$), let us set $\mathbf{e}_\alpha(x_1, x_2) = \prod_{i=1}^2 h_{\alpha_i}(x_i)$. Denote by $\mathbf{A}$ the set of all finite sequences $\mathbf{a} = (a_1, a_2, \dots, a_m)$ with $a_i \in \{0\} \cup \mathbb{N}$, $m = 1, 2, \dots$. For $\mathbf{a} \in \mathbf{A}$, we set $\mathbf{a}! = \prod_{i=1}^\infty a_i!$ and $|\mathbf{a}| = \sum_{i=1}^\infty a_i$. Let $\alpha^{(i)}$, $i = 1, 2, \dots$, be a fixed ordering of $\mathbb{N}^2$ such that for $i < j$, $|\alpha^{(i)}| \leq |\alpha^{(j)}|$. With these notations, we define

$$\mathbf{H}_\mathbf{a}(\omega) = \prod_{i=1}^\infty \mathbf{H}_{a_i}(\langle \omega, \mathbf{e}_{\alpha^{(i)}} \rangle).$$

We recall the following chaos expansion theorem.

Theorem 1. *Let $F \in \mathbb{L}^2 := L^2(\Omega, \mathbf{F}, \mathbb{P})$. Then there exist constants $c_\mathbf{a} \in \mathbb{R}$ for $\mathbf{a} \in \mathbf{A}$ such that*

$$(3) \quad F(\omega) = \sum_{\mathbf{a} \in \mathbf{A}} c_\mathbf{a} \mathbf{H}_\mathbf{a}(\omega) \text{ limit in } \mathbb{L}^2.$$

Furthermore, we have the isometry $\|F\|_{\mathbb{L}^2}^2 = \sum_{\mathbf{a} \in \mathbf{A}} \mathbf{a}! c_\mathbf{a}^2$.

Let $\mathbb{Z}$ be the set of all integers. For $p \in \mathbb{Z}$ and F given in (3), we define the norm $\|F\|_p^2 = \sum_{\mathbf{a} \in \mathbf{A}} c_\mathbf{a}^2 \mathbf{a}! (2\mathbb{N})^{p\mathbf{a}}$, where $(2\mathbb{N})^\mathbf{a} = \prod_{l=1}^m (2l)^{a_l}$ with $\mathbf{a} = (a_1, \dots, a_m)$. If $p \in \mathbb{N}$, we define the space $(\mathbb{S})_p = \{F \in \mathbb{L}^2 : \|F\|_p^2 < \infty\}$ and endow $(\mathbb{S})_p$ with the norm $\|\cdot\|_p$. Also, define $(\mathbb{S})_{-p} = \{F \in \mathbb{L}^2 : \|F\|_{-p}^2 < \infty\}$.

Definition 1. (1) The projective limit of the spaces $(\mathbb{S})_p$, $p \in \mathbb{N}$, is called the space of the stochastic test functions and denoted by $(\mathbb{S})$.

(2) The inductive limit of the spaces $(\mathbb{S})_{-p}$, $p \in \mathbb{N}$, is called the space of stochastic distributions and denoted by $(\mathbb{S})^*$.

We first note that $(f, M_{--}^{H_1, H_2} \mathbf{1}_{(0,t)})_{L^2(\mathbb{R}^2)} = (M_{++}^{H_1, H_2} f, \mathbf{1}_{(0,t)})_{L^2(\mathbb{R}^2)}$. Hence with the above notations, the chaos expansion of fBm is given by

$$B_t^{H_1, H_2}(\omega) = \sum_{l=1}^{\infty} (M_{++}^{H_1, H_2} \mathbf{e}_{\alpha^{(l)}}, \mathbf{1}_{(0,t)})_{L^2(\mathbb{R}^2)} \mathbf{H}_{\epsilon_l}(\omega),$$

where $\epsilon_l = (0, \dots, 0, 1, 0, \dots, 0)$ is the l -th unit vector. The two-parameter fractional white noise $W_{t_1, t_2}^{H_1, H_2}(\omega)$ is defined by the derivative in $(\mathbb{S})^*$ of $B_{t_1, t_2}^{H_1, H_2}(\omega)$:

$$(4) \quad W_{t_1, t_2}^{H_1, H_2} = \sum_{l=1}^{\infty} (M_{++}^{H_1, H_2} \mathbf{e}_{\alpha^{(l)}})(t_1, t_2) \mathbf{H}_{\epsilon_l}(\omega).$$

For $\alpha^{(l)} = (\alpha_1^{(l)}, \alpha_2^{(l)})$, we define two partial fractional white noises:

$$(5) \quad \partial_{t_1} B_{t_1, t_2}^{H_1, H_2}(\omega) = \sum_{l=1}^{\infty} (I_+^{H_1} h_{\alpha_1^{(l)}})(t_1) (I_+^{H_2} h_{\alpha_2^{(l)}} \mathbf{1}_{(0, t_2)})_{L^2(\mathbb{R}^2)} \mathbf{H}_{\epsilon_l}(\omega),$$

$$(6) \quad \partial_{t_2} B_{t_1, t_2}^{H_1, H_2}(\omega) = \sum_{l=1}^{\infty} (I_+^{H_1} h_{\alpha_1^{(l)}} \mathbf{1}_{(0, t_1)})_{L^2(\mathbb{R}^2)} (I_+^{H_2} h_{\alpha_2^{(l)}})(t_2) \mathbf{H}_{\epsilon_l}(\omega).$$

By the direct extension of the method used by Elliott and Van der Hoek [3] in the one-parameter case, we easily prove that these noises, defined by (4), (5) and (6), satisfy that for each $(t_1, t_2) \in \mathbb{R}^2$,

$$\partial_{t_1} B_{t_1, t_2}^{H_1, H_2}, \partial_{t_2} B_{t_1, t_2}^{H_1, H_2} \text{ and } W_{t_1, t_2}^{H_1, H_2} \in (\mathbb{S})^*.$$

We give the definition on $(\mathbb{S})^*$ -valued integrals for the two-parameter case.

Definition 2. Let $Y : \mathbb{R}^2 \rightarrow (\mathbb{S})^*$ be a given function satisfying

$$(7) \quad \int_{\mathbb{R}^2} \langle \langle Y(t_1, t_2), \varphi \rangle \rangle_{(\mathbb{S})^*, (\mathbb{S})} dt_1 dt_2 < \infty \text{ for all } \varphi \in (\mathbb{S}),$$

where $\langle \langle \cdot, \cdot \rangle \rangle_{(\mathbb{S})^*, (\mathbb{S})}$ is the bi-pairing $(\mathbb{S})^*$ and $(\mathbb{S})$. Then we define $(\mathbb{S})^*$ -valued integral $\int_{\mathbb{R}^2} Y(t_1, t_2) dt_1 dt_2$ to be the unique element of $(\mathbb{S})^*$ such that for all $\varphi \in (\mathbb{S})$,

$$\langle \langle \int_{\mathbb{R}^2} Y(t_1, t_2) dt_1 dt_2, \varphi \rangle \rangle_{(\mathbb{S})^*, (\mathbb{S})} = \int_{\mathbb{R}^2} \langle \langle Y(t_1, t_2), \varphi \rangle \rangle_{(\mathbb{S})^*, (\mathbb{S})} dt_1 dt_2.$$

If (7) holds, we then say that Y is *integrable in $(\mathbb{S})^*$* . If $F, G \in (\mathbb{S})^*$, then their Wick product $F \diamond G \in (\mathbb{S})^*$. Since $\partial_{t_1} B_{t_1, t_2}^{H_1, H_2} \in (\mathbb{S})^*$ and $\partial_{t_2} B_{t_1, t_2}^{H_1, H_2} \in (\mathbb{S})^*$, we define a mixed fractional white noise by

$$\partial_{t_1} B_{t_1, t_2}^{H_1, H_2} \diamond \partial_{t_2} B_{t_1, t_2}^{H_1, H_2} \in (\mathbb{S})^*.$$

So the stochastic integrals of various types can be defined.

Definition 3. Suppose that $Y : \mathbb{R}^2 \rightarrow (\mathbb{S})^*$ is a given function such that

$$\begin{aligned} & \int_{\mathbb{R}^2} Y(t_1, t_2) \diamond \left(\partial_{t_1} B_{t_1, t_2}^{H_1, H_2} \diamond \partial_{t_2} B_{t_1, t_2}^{H_1, H_2} \right) dt_1 dt_2, \\ & \int_{\mathbb{R}^2} Y(t_1, t_2) \diamond W_{t_1, t_2}^{H_1, H_2} dt_1 dt_2, \\ & \int_{\mathbb{R}^2} Y(t_1, t_2) t_1^{2H_1-1} t_2^{2H_2} \diamond \partial_{t_2} B_{t_1, t_2}^{H_1, H_2} dt_1 dt_2, \end{aligned}$$

and

$$\int_{\mathbb{R}^2} Y(t_1, t_2) t_1^{2H_1} t_2^{2H_2-1} \diamond \partial_{t_1} B_{t_1, t_2}^{H_1, H_2} dt_1 dt_2$$

are integrable in $(\mathbb{S})^*$. Then we define the following $(\mathbb{S})^*$ -valued integrals:

$$\begin{aligned} & \int_{\mathbb{R}^2} Y(t_1, t_2) d\tilde{M}(t_1, t_2) \\ &= \int_{\mathbb{R}^2} Y(t_1, t_2) \diamond \left(\partial_{t_1} B_{t_1, t_2}^{H_1, H_2} \diamond \partial_{t_2} B_{t_1, t_2}^{H_1, H_2} \right) dt_1 dt_2, \\ & \int_{\mathbb{R}^2} Y(t_1, t_2) dB_{t_1, t_2}^{H_1, H_2} \\ &= \int_{\mathbb{R}^2} Y(t_1, t_2) \diamond W_{t_1, t_2}^{H_1, H_2} dt_1 dt_2, \\ & \int_{\mathbb{R}^2} Y(t_1, t_2) t_1^{2H_1-1} t_2^{2H_2} dt_1 dt_2 B_{t_1, t_2}^{H_1, H_2} \\ &= \int_{\mathbb{R}^2} Y(t_1, t_2) t_1^{2H_1-1} t_2^{2H_2} \diamond \partial_{t_2} B_{t_1, t_2}^{H_1, H_2} dt_1 dt_2, \\ & \int_{\mathbb{R}^2} Y(t_1, t_2) t_1^{2H_1} t_2^{2H_2-1} dt_1 B_{t_1, t_2}^{H_1, H_2} dt_2 \\ &= \int_{\mathbb{R}^2} Y(t_1, t_2) t_1^{2H_1} t_2^{2H_2-1} \diamond \partial_{t_1} B_{t_1, t_2}^{H_1, H_2} dt_1 dt_2. \end{aligned}$$

Remark 1. We adopt the same notation $\tilde{M}(t_1, t_2)$ as that used by Tudor and Viens [5] in which $d\tilde{M}(t_1, t_2)$ is interpreted as $d_{t_1} B_{t_1, t_2}^{H_1, H_2} \cdot d_{t_2} B_{t_1, t_2}^{H_1, H_2}$. Here the notation $d_{t_1} B_{t_1, t_2}^{H_1, H_2}$ is used to denote the differential of the fractional Brownian motion given by $t_1 \rightarrow B_{t_1, t_2}^{H_1, H_2}$ for fixed t_2 , and similarly for $d_{t_2} B_{t_1, t_2}^{H_1, H_2}$. In our case, by (4), (5) and (6), we have

$$\begin{aligned} \tilde{M}(t_1, t_2) &= \int_0^{t_1} \int_0^{t_2} \int_0^{s_2} W_{s_1, u_2}^{H_1, H_2} du_2 \diamond \int_0^{s_1} W_{u_1, s_2}^{H_1, H_2} du_1 ds_1 ds_2 \\ &= \int_{s_1=0}^{t_1} \int_{s_2=0}^{t_2} d_{s_1} B_{s_1, s_2}^{H_1, H_2} \diamond d_{s_2} B_{s_1, s_2}^{H_1, H_2}. \end{aligned}$$

Hence we can also interpret $d\tilde{M}(t_1, t_2)$ as $d_{s_1} B_{s_1, s_2}^{H_1, H_2} \diamond d_{s_2} B_{s_1, s_2}^{H_1, H_2}$, which is the reason for using the notation $\tilde{M}$. $\square$

2. ITÔ FORMULA

In this section we give the Itô formula for the fractional Brownian sheet. The following theorem is the extension of the one-parameter case given by Bender [1] and Biagini *et al.* [2].

Theorem 2. *Let $f \in C^4(\mathbb{R})$ and let $(B_{t_1, t_2}^{H_1, H_2})$, $(t_1, t_2) \in \mathbb{R}^2$, be a fractional Brownian sheet with $H_1, H_2 \in (0, 1)$. Also, we assume that for each $(t_1, t_2) \in \mathbb{R}^2$, $f(B_{t_1, t_2}^{H_1, H_2}) \in \mathbb{L}^2$. Then we have that in $\mathbb{L}^2$,*

$$\begin{aligned}
 f(B_{t_1, t_2}^{H_1, H_2}) &= f(0) + \int_0^{t_1} \int_0^{t_2} f^{(1)}(B_{s_1, s_2}^{H_1, H_2}) dB_{s_1, s_2}^{H_1, H_2} \\
 &\quad + 2H_1 H_2 \int_0^{t_1} \int_0^{t_2} f^{(2)}(B_{s_1, s_2}^{H_1, H_2}) s_1^{2H_1-1} s_2^{2H_2-1} ds_1 ds_2 \\
 &\quad + \int_0^{t_1} \int_0^{t_2} f^{(2)}(B_{s_1, s_2}^{H_1, H_2}) d\tilde{M}(s_1, s_2) \\
 &\quad + H_1 \int_0^{t_1} \int_0^{t_2} f^{(3)}(B_{s_1, s_2}^{H_1, H_2}) s_1^{2H_1-1} s_2^{2H_2} ds_1 ds_2 B_{s_1, s_2}^{H_1, H_2} \\
 &\quad + H_2 \int_0^{t_1} \int_0^{t_2} f^{(3)}(B_{s_1, s_2}^{H_1, H_2}) s_1^{2H_1} s_2^{2H_2-1} ds_1 B_{s_1, s_2}^{H_1, H_2} ds_2 \\
 &\quad + H_1 H_2 \int_0^{t_1} \int_0^{t_2} f^{(4)}(B_{s_1, s_2}^{H_1, H_2}) s_1^{4H_1-1} s_2^{4H_2-1} ds_1 ds_2,
 \end{aligned} \tag{8}$$

where $f^{(k)}(x) = (d^k/dx^k)f(x)$ for $k = 1, \dots, 4$.

Proof. For any constant $c \in \mathbb{R}$, we define $g(x) = e^{cx}$ and put $Y(t_1, t_2) = g(B_{t_1, t_2}^{H_1, H_2})$. By definition of Wick exponential, we have

$$Y(t_1, t_2) = \exp^\diamond \left(c B_{t_1, t_2}^{H_1, H_2} + \frac{1}{2} c^2 t_1^{2H_1} t_2^{2H_2} \right).$$

By using Wick calculus in $(\mathbb{S})^*$, we obtain

$$\begin{aligned}
 \frac{d^2}{dt_2 dt_1} Y(t_1, t_2) &= c Y(t_1, t_2) \diamond \frac{d^2}{dt_2 dt_1} B_{t_1, t_2}^{H_1, H_2} \\
 &\quad + c^2 Y(t_1, t_2) 2H_1 H_2 t_1^{2H_1-1} t_2^{2H_2-1} \\
 &\quad + c^2 Y(t_1, t_2) \diamond \left(\partial_{t_1} B_{t_1, t_2}^{H_1, H_2} \diamond \partial_{t_2} B_{t_1, t_2}^{H_1, H_2} \right) \\
 &\quad + c^2 Y(t_1, t_2) H_2 t_1^{2H_1} t_2^{2H_2-1} \diamond \frac{d}{dt_1} B_{t_1, t_2}^{H_1, H_2} \\
 &\quad + c^3 Y(t_1, t_2) H_1 t_1^{2H_1-1} t_2^{2H_2} \diamond \partial_{t_2} B_{t_1, t_2}^{H_1, H_2} \\
 &\quad + c^4 Y(t_1, t_2) 2H_1 H_2 t_1^{4H_1-1} t_2^{4H_2-1}.
 \end{aligned} \tag{9}$$

We write (9) as follows (with $g(x) = e^{cx}$): in $(\mathbb{S})^*$,

$$\begin{aligned}
 g(B_{t_1, t_2}^{H_1, H_2}) &= g(0) + \int_0^{t_1} \int_0^{t_2} g^{(1)}(B_{s_1, s_2}^{H_1, H_2}) \diamond \frac{d^2}{ds_2 ds_1} B_{s_1, s_2}^{H_1, H_2} ds_1 ds_2 \\
 &\quad + 2H_1 H_2 \int_0^{t_1} \int_0^{t_2} g^{(2)}(B_{s_1, s_2}^{H_1, H_2}) s_1^{2H_1-1} s_2^{2H_2-1} ds_1 ds_2 \\
 &\quad + \int_0^{t_1} \int_0^{t_2} g^{(2)}(B_{s_1, s_2}^{H_1, H_2}) \diamond \left(\partial_{s_1} B_{s_1, s_2}^{H_1, H_2} \diamond \partial_{s_2} B_{s_1, s_2}^{H_1, H_2} \right) ds_1 ds_2 \\
 (10) \quad &\quad + H_2 \int_0^{t_1} \int_0^{t_2} g^{(3)}(B_{s_1, s_2}^{H_1, H_2}) s_1^{2H_1} s_2^{2H_2-1} \diamond \partial_{s_1} B_{s_1, s_2}^{H_1, H_2} ds_1 ds_2 \\
 &\quad + H_1 \int_0^{t_1} \int_0^{t_2} g^{(3)}(B_{s_1, s_2}^{H_1, H_2}) s_1^{2H_1-1} s_2^{2H_2} \diamond \partial_{s_2} B_{s_1, s_2}^{H_1, H_2} ds_1 ds_2 \\
 &\quad + 2H_1 H_2 \int_0^{t_1} \int_0^{t_2} g^{(4)}(B_{s_1, s_2}^{H_1, H_2}) s_1^{4H_1-1} s_2^{4H_2-1} ds_1 ds_2.
 \end{aligned}$$

Let $\mathcal{E}$ be the family of linear combination of the functions e^{cx} , i.e.,

$$\mathcal{E} = \left\{ \sum_{k=1}^n a_k e^{c_k x} \mid n \in \mathbb{N}, a_k, c_k \in \mathbb{R} \right\}.$$

For given $f \in C^4(\mathbb{R})$, we can find a sequence $f_n \in \mathcal{E}$ such that as $n \rightarrow \infty$,

$$(11) \quad f_n(x) \rightarrow f(x) \text{ and } f_n^{(k)}(x) \rightarrow f^{(k)}(x), \quad k = 1, \dots, 4.$$

By linearity (10) is true for $f_n \in \mathcal{E}$. Note that the following functions are continuous in $(S)^*$:

$$(s_1, s_2) \rightarrow f^{(k)}(B_{s_1, s_2}^{H_1, H_2}), k = 1, \dots, 4, W_{s_1, s_2}^{H_1, H_2}, \partial_{s_1} B_{s_1, s_2}^{H_1, H_2}, \text{ and } \partial_{s_2} B_{s_1, s_2}^{H_1, H_2}.$$

Hence the integrals of the right-hand side in (8) exist. By (11) and the dominated convergence theorem, the asserted equation (8) holds in $(\mathbb{S})^*$. But the left-hand side is in $\mathbb{L}^2$ and the theorem is established. $\square$

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