CONFORMAL MAPPING AND ELLIPSES

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ABSTRACT. We answer a question raised by M. Chuaqui, P. Duren, and B. Osgood by showing that a conformal mapping of a simply connected domain cannot take *two* circles onto two proper ellipses.

The goal of this note is to prove the following.

Proposition 1. *Let Ω be a simply connected domain on $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ containing circles (on $\overline{\mathbb{C}}$) C_1 and C_2 , $C_1 \neq C_2$. Let f be a conformal mapping from Ω into $\mathbb{C}$. If f maps each of the circles C_1 and C_2 onto an ellipse, then f is a Möbius transformation and therefore the ellipses are actually circles.*

Thus this proposition answers affirmatively a question raised by M. Chuaqui, P. Duren, and B. Osgood in [1] that a conformal mapping of a simply connected domain cannot take *two* circles onto two ellipses.

Precomposing f with a Möbius transformation and postcomposing it with a linear transformation, we may assume that $C_1 = \{z : |z| = 1\}$ and that f maps the unit disc $\mathbb{D} = \{z : |z| < 1\}$ onto the interior of an ellipse $\mathcal{L}_1$ with foci ± 1 such that $f(r) = 1$, $f(-r) = -1$ for some $0 < r < 1$. Then it is a standard exercise in complex analysis to verify that $w = f(z)$ is defined by

$$(1) \quad w = \frac{1}{2}(\zeta + 1/\zeta) \quad \text{with} \quad \zeta = i \exp \left\{ -\frac{\pi i}{2\mathbf{K}(r^2)} \int_0^{z/r} \frac{dt}{\sqrt{(1-t^2)(1-r^4t^2)}} \right\},$$

where $\mathbf{K}(\cdot)$ denotes the complete elliptic integral of the first kind. Since the image $f(\mathbb{D})$ is symmetric with respect to the coordinate axes, it follows that f is an odd function having real coefficients, i.e.,

$$f(z) = pz + c_3z^3 + c_5z^5 + \dots \quad \text{with} \quad p = \pi/(2r\mathbf{K}(r^2)) > 0.$$

We want to stress that our proof below *does not require* the explicit form (1) of the mapping function, which is included here for completeness.

Instead, we will use the fact that every ellipse $\mathcal{L}$ having foci w_1 and w_2 is a trajectory of the quadratic differential

$$(2) \quad Q_1(w) dw^2 = -\frac{dw^2}{(w-w_1)(w-w_2)};$$

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we refer to [2] for the definition and necessary properties of quadratic differentials. Let $w = f(z)$ map a disc D on $\overline{\mathbb{C}}$ conformally onto the interior of $\mathcal{L}$. Let z_1 and z_2 be the preimages in D of w_1 and w_2 and let z_3, z_4 be the reflections of z_1, z_2 in the circle $C = \partial D$. Pulling the quadratic differential (2) back to the disc D we obtain a quadratic differential $Q_2(z) dz^2 = Q_1(f(z)) f'^2(z) dz^2$ positive on C whose only singularities in D are simple poles at z_1 and z_2 . The latter implies that

$$(3) \quad Q_2(z) dz^2 = -k \frac{dz^2}{(z - z_1)(z - z_2)(z - z_3)(z - z_4)},$$

with some nonzero $k \in \mathbb{C}$. In (3), we assume that all singularities are finite. If one of them, say z_j , is infinite, the corresponding term $(z - z_j)$ in (3) must be replaced by 1.

Comparing (2) and (3) we find that the mapping $w = f(z)$ from D onto the interior of $\mathcal{L}$ satisfies the differential equation

$$(4) \quad w'^2 = k \frac{(w - w_1)(w - w_2)}{(z - z_1)(z - z_2)(z - z_3)(z - z_4)}.$$

For a particular choice when $D = \mathbb{D}$, $w_1 = 1$, $w_2 = -1$, $z_1 = r$, and $z_2 = -r$ with $0 < r < 1$, (4) becomes

$$(5) \quad w'^2 = -p^2 \frac{(w^2 - 1)}{(z^2 - r^2)(z^2 - 1/r^2)}, \quad \text{where } p = w'(0) > 0.$$

Now we assume that Ω is a simply connected domain on $\overline{\mathbb{C}}$ containing the discs $\mathbb{D}$ and $D \neq \mathbb{D}$. If there is a function $w = f(z)$ univalent in Ω , which maps the circles $C_1 = \partial \mathbb{D}$ and $C_2 = \partial D$ onto ellipses, then, by the principle of analytic continuation, $w = f(z)$ will satisfy near the origin both equations (4) and (5). Therefore the Taylor coefficients of f at $z = 0$ can be found from either one of these equations.

To simplify our computation, we first take logarithmic derivatives in (4) and (5), then expand them into power series at $z = 0$. For our purposes we need the first eight coefficients, so all our computations can be easily verified with *Maple* or *Mathematica*. Equating the first eight Taylor coefficients of the results and assuming that $z_1, z_2, z_3, z_4, w_1, w_2$ all are nonzero and finite we obtain in this generic case the following system of eight polynomial equations, which we call *system* (*):

$$\begin{aligned} a + b + c + d &= p(A + B), \\ a^2 + b^2 + c^2 + d^2 &= p^2(A^2 + B^2) + 2((r^2 + 1/r^2) - p^2), \\ a^3 + b^3 + c^3 + d^3 &= p^3(A^3 + B^3) + \frac{1}{2}p(A + B)((r^2 + 1/r^2) - p^2), \\ a^4 + b^4 + c^4 + d^4 &= p^4(A^4 + B^4) + \frac{2}{3}p^2(A^2 + B^2 - 2)((r^2 + 1/r^2) - p^2) \\ &\quad + 2((r^4 + 1/r^4) - p^4), \\ a^5 + b^5 + c^5 + d^5 &= p^5(A^5 + B^5) + \frac{1}{4}p(A + B) + \frac{3}{8}p(A + B)((r^4 + 1/r^4) - p^4) \\ &\quad + \frac{5}{12}p^3[2(A^3 + B^3) - (A + B)]((r^2 + 1/r^2) - p^2), \\ a^6 + b^6 + c^6 + d^6 &= p^6(A^6 + B^6) + 2((r^6 + 1/r^6) - p^6) + \frac{7}{15}p^2(A^2 + B^2 - 2) \\ &\quad + \frac{8}{15}p^2(A^2 + B^2 - 2)((r^4 + 1/r^4) - p^4) \\ &\quad + p^4[(A^4 + B^4) - \frac{2}{3}(A^2 + B^2 + 1)]((r^2 + 1/r^2) - p^2), \end{aligned}$$

$$\begin{aligned}
a^7 + b^7 + c^7 + d^7 &= p^7(A^7 + B^7) + \frac{5}{16}p(A+B)((r^6 + 1/r^6) - p^6) \\
&\quad + \frac{259}{720}p^3[2(A^3 + B^3) - (A+B)]((r^4 + 1/r^4) - p^4) \\
&\quad + \left\{ \frac{7}{144}p^5[24(A^5 + B^5) - 20(A^3 + B^3) + (A+B)] \right. \\
&\quad \left. + \frac{3}{16}p(A+B) \right\}((r^2 + 1/r^2) - p^2) \\
&\quad + \frac{133}{180}p^3[(A^3 + B^3) - (A+B)] + \frac{401}{720}p^3(A+B), \\
a^8 + b^8 + c^8 + d^8 &= p^8(A^8 + B^8) + 2((r^8 + 1/r^8) - p^8) \\
&\quad + \frac{16}{35}p^2(A^2 + B^2 - 2)((r^6 + 1/r^6) - p^6) \\
&\quad + \frac{14}{45}p^4[3(A^4 + B^4) - 2(A^2 + B^2) - 2]((r^4 + 1/r^4) - p^4) \\
&\quad + p^6\left[\frac{4}{3}((A^6 + B^6) - (A^4 + B^4)) \right. \\
&\quad \left. + \frac{8}{45}(A^2 + B^2) - \frac{16}{45}\right]((r^2 + 1/r^2) - p^2) \\
&\quad + \frac{8}{21}p^2(A^2 + B^2 - 2)((r^2 + 1/r^2) - p^2),
\end{aligned}$$

where $a = 1/z_1$, $b = 1/z_2$, $c = 1/z_3$, $d = 1/z_4$, $A = 1/w_1$, and $B = 1/w_2$.

From this point, our argument is purely algebraic. Our goal is to show that the only solution to system (*) is $a = r$, $b = -r$, $c = 1/r$, $d = -1/r$ (up to permutation), and $A = 1$, $B = -1$ (again up to permutation).

Since the left-hand sides of the equations in system (*) are Newton power sums in a, b, c and d , they are, in particular, symmetric polynomials, and can therefore be written in terms of the elementary symmetric symbols $s_1 = a + b + c + d$, $\dots$, $s_4 = abcd$. For necessary properties of the Newton power sums and symmetric polynomials, one may consult [3, Section 5.7]. Similarly, A and B only occur on the right-hand sides of system (*) in the form of Newton power sums, which can be expressed in terms of the elementary symmetric symbols $S_1 = A + B$ and $S_2 = AB$. The goal is then to show that the only solution to the rewritten system is $s_1 = 0$, $s_2 = -(r^2 + 1/r^2)$, $s_3 = 0$, $s_4 = 1$, $S_1 = 0$, and $S_2 = -1$.

The first equation of system (*) now takes the form $s_1 = pS_1$, which determines s_1 . The second equation becomes

$$s_1^2 - 2s_2 = p^2(S_1^2 - 2S_2) + 2((r^2 + 1/r^2) - p^2),$$

which we can solve to find

$$s_2 = p^2S_2 - ((r^2 + 1/r^2) - p^2).$$

The third equation is

$$s_1^3 - 3s_1s_2 + 3s_3 = p^3(S_1^3 - 3S_1S_2) + 3pS_1,$$

and we solve this to find

$$s_3 = -\frac{5pS_1}{6}((r^2 + 1/r^2) - p^2).$$

The fourth equation becomes

$$\begin{aligned}
&s_1^4 + 4s_1s_3 - 4s_1^2s_2 + 2s_2^2 - 4s_4 \\
&= 4pq(S_1^2 - 2S_2) + p^4(S_1^4 - 4S_1^2S_2 + 2S_2^2) - 8pq + 2((r^4 + 1/r^4) - p^4),
\end{aligned}$$

which we solve to find

$$s_4 = 1 - \frac{2}{3}p^2(S_2 + 1)((r^2 + 1/r^2) - p^2).$$

This determines s_1, s_2, s_3, s_4 as functions of S_1 and S_2 . Also, for $S_1 = 0$ and $S_2 = -1$, this gives the required values for s_1, s_2, s_3, s_4 . Thus, the only thing remaining is to show that $S_1 = 0$ and $S_2 = -1$ is the only solution to the last four equations of system (*).

The fifth equation becomes

$$pS_1(61p^4 + 100p^2 - 50p^2R^2 - 11R^4 + 44R^2 + 64) = 0,$$

where $R = r + 1/r$. Therefore since $p \neq 0$ by (5), we must either have $S_1 = 0$ or

$$(6) \quad 61p^4 + 100p^2 - 50p^2R^2 - 11R^4 + 44R^2 + 64 = 0.$$

Now, assume $S_1 \neq 0$. Then (6) is satisfied.

The sixth equation of system (*) written in terms of symmetric symbols S_1 and S_2 becomes

$$(7) \quad 60p^2[3(61p^4 + 100p^2 - 50p^2R^2 - 11R^4 + 44R^2 + 64)S_1^2 + 8(-8 - 28R^2 + 7R^4 - 20p^2 + 10p^2R^2 - 17p^4)(S_2 + 1)] = 0.$$

So assuming (6), we must in fact have

$$(-8 - 28R^2 + 7R^4 - 20p^2 + 10p^2R^2 - 17p^4)(S_2 + 1) = 0.$$

Hence, $S_2 = -1$ or

$$-8 - 28R^2 + 7R^4 - 20p^2 + 10p^2R^2 - 17p^4 = 0.$$

In the second case, we obtain

$$\begin{aligned} & (61p^4 + 100p^2 - 50p^2R^2 - 11R^4 + 44R^2 + 64) \\ & + 5(-8 - 28R^2 + 7R^4 - 20p^2 + 10p^2R^2 - 17p^4) \\ & = 24(R^4 - 4R^2 + (1 - p^4)) = 0, \end{aligned}$$

meaning that

$$R = \sqrt{2 \pm \sqrt{3 + p^4}}.$$

Plugging this into (6), we get

$$25(3 + 2p^4 \pm 2p^2\sqrt{3 + p^4}) = 0,$$

which is easily seen to be impossible. Therefore, we must have $S_2 = -1$.

The seventh equation of system (*) can be written as

$$\begin{aligned} (8) \quad & pS_1[42(61p^4 + 100p^2 - 50p^2R^2 - 11R^4 + 44R^2 + 64)p^2S_1^2 \\ & + (1246R^4 - 4984R^2 + 3220p^2R^2 - 3584 - 6440p^2 - 4466p^4)p^2S_2 \\ & + (5355p^4R^2 + 3690R^4 - 2040R^2 - 615R^6 - 3081p^6 - 12264p^2 + 6636p^2R^2 \\ & - 10710p^4 - 1659p^2R^4 - 5760)] = 0, \end{aligned}$$

which simplifies to

$$-2(49p^4 + 44)R^2 + 98p^6 + 196p^4 + 235p^2 + 176 = 0,$$

when we use (6) and $S_2 = -1$. Consequently,

$$R = \sqrt{\frac{98p^6 + 196p^4 + 235p^2 + 176}{2(49p^4 + 44)}},$$

and (6) becomes

$$\frac{1089(768 + 637p^4)}{4(49p^4 + 44)^2} = 0,$$

which is impossible as well, since p is real.

All in all, we must therefore have that $S_1 = 0$. Again, (7) reduces to

$$(-8 - 28R^2 + 7R^4 - 20p^2 + 10p^2R^2 - 17p^4)(S_2 + 1) = 0,$$

so if S_2 does *not* equal -1 , we have

$$(9) \quad -8 - 28R^2 + 7R^4 - 20p^2 + 10p^2R^2 - 17p^4 = 0.$$

Now, the eighth equation of system (*) is

$$\begin{aligned} (1 + S_2) & [-7(-8 - 28R^2 + 7R^4 - 20p^2 + 10p^2R^2 - 17p^4)p^2S_2 \\ & + (132 + 378p^2 - 252p^2R^2 + 486R^2 + 63p^2R^4 \\ & - 189p^4R^2 - 414R^4 + 378p^4 + 57p^6 + 69R^6)] = 0, \end{aligned}$$

and so assuming (9),

$$132 + 378p^2 - 252p^2R^2 + 486R^2 + 63p^2R^4 - 189p^4R^2 - 414R^4 + 378p^4 + 57p^6 + 69R^6 = 0.$$

Taking this times 49, adding $-483R^2 + 996 + 249p^2$ times (9), and dividing the result by 90, this reduces to

$$(16p^4 + 7)R^2 - (16p^6 + 32p^4 + 31p^2 + 14) = 0,$$

i.e.,

$$R = \sqrt{\frac{16p^6 + 32p^4 + 31p^2 + 14}{16p^2 + 7}}.$$

Substituting this into (9), we get

$$-\frac{1764}{(16p^4 + 7)^2} = 0.$$

This is of course absurd, so we conclude that $S_2 = -1$.

Thus, $S_1 = 0$ and $S_2 = -1$ is the only solution, and the proposition is proved in the generic case.

Turning to the remaining special cases, we note that if $z_k = \infty$, then the corresponding term a , b , c , or d in the left-hand side of system (*) is absent, which does not effect further computation.

Next, since $f(z)$ is analytic at $z = 0$, it follows from (4) that $z_k = 0$ for some k if and only if $w_1 = 0$ or $w_2 = 0$. Assume $w_1 = 0$. Then also $z_1 = 0$ since z_1 denotes the preimage of w_1 . Now we can of course find equations similar to those

of system (*), where the left-hand side of the equations are Newton power sums in $b = 1/z_2$, $c = 1/z_3$, and $d = 1/z_4$, and the right-hand side are expressions in r , p and $B = 1/w_2$. In this case, having six variables, we need the first six equations. Again, the Newton power sums are rewritten in terms of the elementary symmetric symbols $s_1 = b + c + d$, $s_2 = bc + bd + cd$, and $s_3 = bcd$, and the first three equations are used to solve for s_1 , s_2 , and s_3 in terms of r , p , and B . As a further simplification, we replace $r + 1/r$ by R , to get the remaining three equations

$$\begin{aligned} L_4 &:= 11R^4 + (50p^2 - 44)R^2 - (61p^4 + 100p^2 + 64) = 0, \\ L_5 &:= (24R^4 - (70p^2 + 92)R^2 + (47p^4 + 140p^2 + 128))B = 0, \\ L_6 &:= -7p^2(24R^4 - (70p^2 + 92)R^2 + (47p^4 + 140p^2 + 128))B^2 \\ &\quad + 269R^6 + (1085p^2 - 1614)R^4 + (936 - 4340p^2 - 1729p^4)R^2 \\ &\quad + (375p^6 + 3458p^4 + 4760p^2 + 2432) = 0. \end{aligned}$$

Here, we note an obvious simplification: $L_6 + 7p^2BL_5$ does not depend on B , but is simply a polynomial in R and p . Furthermore,

$$\begin{aligned} &\frac{1}{480}(121(L_6 + 7p^2BL_5) + (5918 + 1515p^2 - 2959R^2)L_4) \\ &= 2(49p^2 + 44)R^2 - (98p^6 + 196p^4 + 235p^2 + 176) = 0, \end{aligned}$$

so we must have

$$R = \sqrt{\frac{98p^6 + 196p^4 + 235p^2 + 176}{2(49p^2 + 44)}}.$$

But then

$$L_4 = -\frac{1089(637p^4 + 768)}{4(49p^2 + 44)^2},$$

which cannot be zero, since p is real.

Therefore, there are no solutions in this case. This finishes the proof of Proposition 1.

Remarks. (1) Actually our proof gives a stronger result: Let f admit analytic continuation along any path in a domain (not necessarily simply connected) $\Omega \subset \overline{\mathbb{C}}$. Suppose there are two branches f_1 and f_2 of f and two discs D_1 and D_2 , $D_1 \neq D_2$, such that f_k maps D_k one-to-one onto an interior of an ellipse. Then f is a Möbius transformation.

(2) Equating the right-hand sides of (4) and (5) we can express w as an algebraic function of z . Then one may finish the proof of Proposition 1 by invoking the fact that the elliptic integral in (1) cannot be reduced to an algebraic form for any value of r under consideration. But we could not find a reference for such a result in the accessible literature.

(3) It is worth mentioning that the existence of a common solution $w = f(z)$ to two equations $w'^2 = R_1(z, w)$ and $w'^2 = R_2(z, w)$ with rational R_1 and R_2 does not guarantee in general that $R_1 \equiv R_2$. Indeed, the Koebe function $w = z/(1 - z)^2$, being extremal in numerous problems of geometric function theory, satisfies a variety of such equations; see [2, Ch. VI].

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