ON AFFINE FRAMES WITH TRANSCENDENTAL DILATIONS

R. S. LAUGESEN

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ABSTRACT. We answer a question of O. Christensen about affine systems in $L^2(\mathbb{R})$. Specifically, we show that if the dilation factor $a > 1$ is transcendental, then cancellations cannot occur between different scales, in the conditions for the affine system to form a frame. Such cancellations are known to occur when a is an integer.

1. INTRODUCTION

Consider a *generating function* $\psi \in L^2 = L^2(\mathbb{R})$ and a *dilation factor* $a > 1$. Define

$$\psi_{j,y}(x) = a^{j/2}\psi(a^jx - y), \quad j \in \mathbb{Z}, \quad y \in \mathbb{R}, \quad x \in \mathbb{R}.$$

Call $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ the *affine system* generated by ψ and a .

We are interested in when the affine system forms a *frame* for L^2 , in other words, when ψ is a *frame wavelet*. Recall that a countable collection of functions $\{g_m\}$ in L^2 forms a *frame* if constants $0 < A \leq B < \infty$ exist such that

$$A\|f\|^2 \leq \sum_m |\langle f, g_m \rangle|^2 \leq B\|f\|^2 \quad \text{for all } f \in L^2,$$

where $\|f\|$ denotes the L^2 -norm of f . The numbers A and B are *frame bounds*. The frame is *tight* if $A = B$. For example an orthonormal basis forms a tight frame with $A = B = 1$.

The affine system $\{\psi_{j,k}\}$ forms a frame if and only if $0 < A_\psi \leq B_\psi < \infty$, where

$$(1.1) \quad A_\psi := \inf_{\|f\|=1} \sum_{j,k \in \mathbb{Z}} |\langle f, \psi_{j,k} \rangle|^2, \quad B_\psi := \sup_{\|f\|=1} \sum_{j,k \in \mathbb{Z}} |\langle f, \psi_{j,k} \rangle|^2.$$

In his survey on frame theory, O. Christensen [8] inquired whether formulas for A_ψ and B_ψ could be found that involve “cancellation” between terms at different dilation scales (different values of j). Such cancellation is known to occur when a is an integer. This paper shows it cannot occur when the dilation a is transcendental.

Quick tutorial on frames. Frame theory began with Duffin and Schaeffer [12], who were motivated by nonharmonic Fourier series. Frames have become particularly important in the last 20 years in connection with Gabor systems (time-frequency

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analysis) and affine systems (wavelet analysis). See for example the books [13, 16], the survey articles [5, 8], and the open problems in Section 5 below.

Note that if the $\{g_m\}$ form a frame, then every $f \in L^2$ has a reconstruction $f = \sum_m c_m g_m$ as a linear combination of the g_m . A frame need not be a basis, though: these coefficients c_m need not be unique. One choice of coefficients is $c_m = \langle f, h_m \rangle$, where the $\{h_m\}$ form a “dual frame”.

Frames are at first sight less elegant than orthonormal bases, due to their redundancy or overcompleteness, but this redundancy gives them considerable power in applications such as noise reduction [2].

Frame theory in finite-dimensional spaces has flowered recently too. See [3, 6], and the results on signal reconstruction without phase information in [1].

2. RESULTS

Recall that a real number is *algebraic* if it is a root of some nontrivial polynomial equation with integer coefficients. Otherwise the number is *transcendental*. The algebraic numbers are countable, and thus most real numbers are transcendental.

Define a dense subclass of the unit sphere in L^2 by

$$\mathcal{C} = \{f \in L^2 : \|f\| = 1, \text{ and } \hat{f} \text{ is bounded and compactly supported in } \mathbb{R} \setminus \{0\}\},$$

and for $f \in L^2$ define a family of nonnegative 1-periodic functions by

$$(T_j f)(z) = \sum_{k \in \mathbb{Z}} |\langle f, \psi_{j, k-z} \rangle|^2, \quad j \in \mathbb{Z}, \quad z \in \mathbb{R}.$$

$T_j f$ is continuous if $f \in \mathcal{C}$, by [17, Lemma A.4].

Theorem 2.1. *Assume $\psi \in L^2$ and $a > 1$. Then*

$$B_\psi \leq \sup_{f \in \mathcal{C}} \sum_{j \in \mathbb{Z}} \max T_j f$$

with equality if a is transcendental. Also, if $B_\psi < \infty$, then

$$A_\psi \geq \inf_{f \in \mathcal{C}} \sum_{j \in \mathbb{Z}} \min T_j f$$

with equality if a is transcendental.

Theorem 2.1 implies a sufficient frame condition involving only ψ , and not f .

Corollary 2.2 (Casazza and Christensen [7, Theorem 2.5], [8, Theorem 5.1]). *Assume $\psi \in L^2$ and $a > 1$. Then*

$$B_\psi \leq B := \text{ess. sup}_{1 \leq |\xi| \leq a} \sum_{j \in \mathbb{Z}} \left(|\hat{\psi}(\xi a^j)|^2 + \sum_{n \neq 0} |\hat{\psi}(\xi a^j) \hat{\psi}(\xi a^j - n)| \right).$$

Also if $B_\psi < \infty$, then

$$A_\psi \geq A := \text{ess. inf}_{1 \leq |\xi| \leq a} \sum_{j \in \mathbb{Z}} \left(|\hat{\psi}(\xi a^j)|^2 - \sum_{n \neq 0} |\hat{\psi}(\xi a^j) \hat{\psi}(\xi a^j - n)| \right),$$

provided the sum over j in this quantity A is well defined.

Corollary 2.2 implies that if $0 < A \leq B < \infty$, then ψ generates an affine frame.

When $a = 2$, the bounds in Corollary 2.2 can be replaced by ones in which summations over certain j and n values are carried out *inside* the absolute value signs. This is due to Ph. Tchamitchian, as presented by I. Daubechies in [11, Theorem 2.9]. Getting these summations inside the absolute value signs can lead to helpful cancellations [11, p. 984].

Christensen [8, p. 288] wrote that it “would be interesting if [Corollary 2.2] could be generalized that way” for general $a > 1$. Theorem 2.1 makes such a generalization impossible when a is transcendental, because the max and min are inside the sum over j and so oscillations in different $T_j f$ functions cannot possibly cancel each other. (This statement is best understood in terms of formula (3.1) in the proof below.)

Theorem 2.1 leaves open the possibility of modifying Corollary 2.2 to allow cancellations within the sum over n . But it is unclear how to do that.

Notes.

1. It is an open question whether equality must hold in Theorem 2.1 whenever $a^j \notin \mathbb{Q}$ for all $j > 0$. This condition is strictly weaker than transcendentality of a . For example $a = 1 + \sqrt{2}$ is algebraic rather than transcendental, being a root of $a^2 = 2a + 1$, but $a^j \notin \mathbb{Q}$ for all $j > 0$. Incidentally, the condition $a^j \notin \mathbb{Q}$ for all $j > 0$ already arises in tight frame characterizations [9, §3].

2. If $a^j \in \mathbb{Q}$ for some $j > 0$, then cancellations can occur in the style of Tchamitchian’s result, basically because the Fourier series of $T_j f(a^j z)$ has terms such as $\exp(2\pi i n a^j z)$, and it is possible to have $n_1 a^{j_1} = n_2 a^{j_2}$ (when $a^{j_1 - j_2} = n_2/n_1 \in \mathbb{Q}$).

3. PROOF OF THEOREM 2.1

Note that $B_\psi = \sup_{f \in \mathcal{C}} \sum_{j,k \in \mathbb{Z}} |\langle f, \psi_{j,k} \rangle|^2$, by definition (1.1) of B_ψ and the density of $\mathcal{C}$ in the unit sphere of L^2 , using [17, Lemma B.1]. Hence

$$B_\psi = \sup_{f \in \mathcal{C}} \sum_{j \in \mathbb{Z}} T_j f(0) \leq \sup_{f \in \mathcal{C}} \sum_{j \in \mathbb{Z}} \max T_j f.$$

For the equality statement in the theorem, we start with two trivial but important observations: if $f \in \mathcal{C}$, then the translates $f(\cdot - z)$ of f also belong to $\mathcal{C}$ whenever $z \in \mathbb{R}$, and $\langle f(\cdot - z), \psi_{j,k} \rangle = \langle f, \psi_{j,k-a^j z} \rangle$. Therefore

$$\begin{aligned} B_\psi &= \sup_{f \in \mathcal{C}} \sup_{z \in \mathbb{R}} \sum_{j \in \mathbb{Z}} T_j (f(\cdot - z))(0) \\ (3.1) \quad &= \sup_{f \in \mathcal{C}} \sup_{z \in \mathbb{R}} \sum_{j \in \mathbb{Z}} T_j f(a^j z). \end{aligned}$$

The equality statement for B_ψ in Theorem 2.1 now follows from the next lemma.

Lemma 3.1. *If $a \in \mathbb{R}$ is transcendental and the functions $h_j : \mathbb{R} \rightarrow \mathbb{R}$ are continuous, nonnegative and 1-periodic, then*

$$\sup_{z \in \mathbb{R}} \sum_{j \in \mathbb{Z}} h_j(a^j z) = \sum_{j \in \mathbb{Z}} \max h_j.$$

Proof of Lemma 3.1. The “ $\leq$ ” direction is immediate. For the “ $\geq$ ” direction, the task is to show that for each $J \in \mathbb{N}$

$$(3.2) \quad \sup_{z \in \mathbb{R}} \sum_{|j| \leq J} h_j(a^j z) = \sum_{|j| \leq J} \max h_j.$$

To accomplish this, write $m_j \in [0, 1)$ for a point at which h_j attains its maximum, and then fix J and define two vectors

$$\alpha = (a^J, a^{J-1}, \dots, a^{-J+1}, a^{-J}), \quad \mu = (m_J, m_{J-1}, \dots, m_{-J+1}, m_{-J}).$$

Note that (3.2) will hold if we find a sequence $\{z_\ell\}$ such that $\alpha z_\ell \rightarrow \mu \pmod{1}$ as $\ell \rightarrow \infty$ (with the convention that “ $\pmod{1}$ ” for vectors means $\pmod{1}$ in each component). Kronecker’s Theorem [14, Theorem 444] from number theory guarantees this by showing the trajectory $\{\alpha z \pmod{1} : z \in \mathbb{R}\}$ is dense in the unit cube of $\mathbb{R}^{2J+1}$. The transcendentality of a is used here to satisfy the hypothesis of Kronecker’s Theorem, namely that the entries of α should be linearly independent over the integers. $\square$

To prove the claims about A_ψ in Theorem 2.1, suppose $B_\psi < \infty$. Then the functional $f \mapsto \sum_{j,k \in \mathbb{Z}} |\langle f, \psi_{j,k} \rangle|^2$ is continuous on L^2 (see [17, Lemma B.1]) so that in the definition (1.1) of A_ψ we may replace $\{f : \|f\| = 1\}$ with its dense subclass $\mathcal{C}$:

$$A_\psi = \inf_{f \in \mathcal{C}} \sum_{j,k \in \mathbb{Z}} |\langle f, \psi_{j,k} \rangle|^2 = \inf_{f \in \mathcal{C}} \inf_{z \in \mathbb{R}} \sum_{j \in \mathbb{Z}} T_j f(a^j z),$$

by arguing as before. Hence $A_\psi \geq \inf_{f \in \mathcal{C}} \sum_{j \in \mathbb{Z}} \min T_j f$. For the equality statement on A_ψ , apply Lemma 3.1 to $h_j = (\max T_j f) - T_j f$, noting that $\sum_{j \in \mathbb{Z}} \max T_j f \leq B_\psi < \infty$ when a is transcendental, by the equality statement for B_ψ .

4. PROOF OF COROLLARY 2.2

We need some further properties of the $T_j f$.

Lemma 4.1 ([17, Section 8]). *Let $f \in \mathcal{C}$. Then $T_j f \in C[0, 1]$ and $T_j f$ equals its Fourier series at every point, for each $j \in \mathbb{Z}$. Obviously $\widehat{T_j f}(0) \geq 0$ because $T_j f \geq 0$. The Fourier coefficients are given by*

$$\widehat{T_j f}(n) = \int_{\mathbb{R}} \widehat{f}(\xi) \widehat{f}(\xi - na^j) \widehat{\psi}(\xi a^{-j}) \overline{\widehat{\psi}(\xi a^{-j} - n)} d\xi,$$

and their double sum satisfies

$$(4.1) \quad \sum_{j \in \mathbb{Z}} \sum_{n \neq 0} |\widehat{T_j f}(n)| < \infty.$$

Since $T_j f$ equals its Fourier series at every point, $T_j f(z) = \sum_{n \in \mathbb{Z}} \widehat{T_j f}(n) e^{2\pi i n z}$. Thus

$$\max T_j f \leq \widehat{T_j f}(0) + \sum_{n \neq 0} |\widehat{T_j f}(n)|, \quad \min T_j f \geq \widehat{T_j f}(0) - \sum_{n \neq 0} |\widehat{T_j f}(n)|.$$

We can substitute these bounds into Theorem 2.1. Note the resulting quantities are well defined because the double sum $\sum_{j \in \mathbb{Z}} \sum_{n \neq 0} |\widehat{T_j f}(n)|$ converges to a finite real number by (4.1) while $\sum_{j \in \mathbb{Z}} \widehat{T_j f}(0)$ converges in $[0, \infty]$ (each term being nonnegative).

To prove the corollary we now invoke the formula for $\widehat{T_j f}(n)$ from Lemma 4.1 and follow the original proof of Casazza and Christensen [8, Theorem 5.1], which judiciously employs Cauchy–Schwarz on both the integral and the sum over n .

Remark. The Fourier expansion of the function $T_j f$ was developed in [17] to study completeness of orthonormal affine systems with arbitrary real dilations. For higher dimensions see [18]. The idea was then applied for quite general Gabor–affine systems in [15].

5. OPEN PROBLEMS ON FRAME CHARACTERIZATION

Affine frames with band-limited ψ were characterized by Chui and Shi [10, p. 40]. For integer a and general ψ , Ron and Shen [19, Theorem 1.2] found a beautiful frame characterization in terms of ℓ^2 -norms of the dual Gramian and its inverse. Useful sufficient conditions for frames were known earlier, [11, Theorems 2.8 and 2.9] and [16, §8.3], and later [8, Theorem 5.1]. Tight affine frames ($A_\psi = B_\psi$) have been very successfully characterized, with the latest developments in [9, 15]. Bownik [4, §3] has extended the dual Gramian approach to rational dilations for tight frames. The nontight case seems to be open whenever a is noninteger: no necessary condition is known that comes close to the sufficient ones mentioned above.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS 61801
E-mail address: Laugesen@uiuc.edu