ON A SEMILINEAR PARABOLIC EQUATION

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ABSTRACT. We introduce a general class of potentials $V = V(x, t)$ so that the semilinear parabolic equation $a\Delta u - \frac{\partial}{\partial t}u + Vu^p = 0$ in $\mathbb{R}^n \times]0, \infty[$, $n \geq 3$, $p > 1$, $a > 0$, has global positive continuous solutions. These results extend the recent ones proved by Zhang to a more general class of potentials.

1. INTRODUCTION

In this paper we will be interested in the following semilinear parabolic problem:

$$(\mathcal{P}_V) \quad \begin{cases} a\Delta u - \frac{\partial}{\partial t}u + V(x, t)u^p = 0 & \text{in } \mathbb{R}^n \times]0, \infty[, \\ u > 0 & \text{in } \mathbb{R}^n \times]0, \infty[, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where $n \geq 3$, $p > 1$, Δ is the Laplacian operator and $V \in L^1_{\text{loc}}(\mathbb{R}^n \times \mathbb{R})$. We first recall that Zhang [5] studied the global existence for this problem when the potential V is in the parabolic Kato class at infinity P^∞ , and in the time-independent case when $V = V(x)$ is in the subclass of Green tight functions, the asymptotic behavior of solutions for this problem is studied by Zhang and Zhao [6]. The elliptic counterpart problem $\Delta u(x) + V(x)u^p(x) = 0$, $x \in \mathbb{R}^n$, $n \geq 3$, was studied by some authors during the last two decades (see Kenig and Ni [2], Lin [3], Ni [4], Zhao [7] and the references therein).

Our aim in this paper is to introduce a new class of potentials V denoted by P^∞_{loc} so that the problem $(\mathcal{P}_V)$ has a global continuous solution comparable to that of $(\mathcal{P}_0)$. We also prove that when $V \geq 0$, the condition $V \in P^\infty_{\text{loc}}$ is optimal for this result to hold. In particular, we extend the results proved by Zhang [5] to a new functional class more general than the parabolic Kato class P^∞ . We also give interesting examples of potentials in $P^\infty_{\text{loc}} \setminus P^\infty$.

Let us introduce the class P^∞_{loc} .

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Definition 1.1. A Borel measurable function $V = V(x, t)$ on $\mathbb{R}^n \times \mathbb{R}$ is said to be in the class P_{loc}^∞ if it satisfies for all $c > 0$,

$$\begin{aligned} N_c(V) &\equiv \sup_{(x,t) \in \mathbb{R}^n \times \mathbb{R}} \int_{-\infty}^t \int_{\mathbb{R}^n} \Gamma_c(x, t; y, s) |V(y, s)| dy ds \\ &\quad + \sup_{(y,s) \in \mathbb{R}^n \times \mathbb{R}} \int_s^{+\infty} \int_{\mathbb{R}^n} \Gamma_c(x, t; y, s) |V(x, t)| dx dt < \infty, \end{aligned}$$

and for any compact subset $A \subset \mathbb{R}^n \times \mathbb{R}$,

$$\begin{aligned} \lim_{r \rightarrow 0} &\left\{ \sup_{(x,t) \in A} \int_{t-r}^t \int_{|x-y| < \sqrt{r}} \Gamma_c(x, t; y, s) |V(y, s)| dy ds \right. \\ &\quad \left. + \sup_{(y,s) \in A} \int_s^{s+r} \int_{|x-y| < \sqrt{r}} \Gamma_c(x, t; y, s) |V(x, t)| dx dt \right\} = 0, \end{aligned}$$

where $\Gamma_c(x, t; y, s) = \frac{1}{(t-s)^{n/2}} \exp(-c \frac{|x-y|^2}{t-s})$, for $t > s$.

Let P^∞ be the class introduced by Zhang in [5]. Obviously $P^\infty \subset P_{\text{loc}}^\infty \subset L_{\text{loc}}^1(\mathbb{R}^n \times \mathbb{R})$ and we have the following.

Remark 1.1. Let V be a Borel measurable function on $\mathbb{R}^n \times \mathbb{R}$ satisfying $N_c(V) < \infty$, for all $c > 0$. Then $V \in P_{\text{loc}}^\infty$ if and only if $V\varphi \in P^\infty$ for all $\varphi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R})$.

In the time-independent case we have the following.

Remark 1.2. Let $\mathcal{M}_b$ be the set of functions $V = V(x)$ on $\mathbb{R}^n$ which define bounded Newtonian potentials, i.e. $\sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|V(y)|}{|x-y|^{n-2}} dy < \infty$. Then $P^\infty = K_n \cap \mathcal{M}_b$ and $P_{\text{loc}}^\infty = K_n^{\text{loc}} \cap \mathcal{M}_b$, where K_n and K_n^{loc} are respectively the Kato and the local Kato classes introduced by Aizenman and Simon [1].

The main results about the problem $(\mathcal{P}_V)$ are the following.

Theorem I. *Let $V \in P_{\text{loc}}^\infty$. For any $M > 1$, there is a constant $b_0 > 0$ such that for each nonnegative $u_0 \in C^2(\mathbb{R}^n)$ satisfying $\|u_0\|_\infty \leq b_0$, there exists a positive continuous solution u of $(\mathcal{P}_V)$ such that*

$$M^{-1} \int_{\mathbb{R}^n} G_a(x, t; y, 0) u_0(y) dy \leq u(x, t) \leq M \int_{\mathbb{R}^n} G_a(x, t; y, 0) u_0(y) dy,$$

for all $(x, t) \in \mathbb{R}^n \times]0, \infty[$. Here G_a is the fundamental solution of $a\Delta_x - \partial/\partial t$ on $\mathbb{R}^n \times \mathbb{R}$.

Theorem II. *Assume that V is defined on $\mathbb{R}^n \times]0, \infty[$, $V \geq 0$, and the result of Theorem I holds for all $a > 0$. Then, we have for all $c > 0$,*

$$\sup_{(x,t) \in \mathbb{R}^n \times \mathbb{R}} \int_{-\infty}^t \int_{\mathbb{R}^n} \Gamma_c(x, t; y, s) |V(y, s)| dy ds < \infty$$

and

$$\lim_{r \rightarrow 0} \sup_{(x,t) \in A} \int_{t-r}^t \int_{|x-y| < \sqrt{r}} \Gamma_c(x, t; y, s) |V(y, s)| dy ds = 0,$$

for any compact subset $A \subset \mathbb{R}^n \times \mathbb{R}$.

By using the same arguments as in [6], we also have the following result about the large-time behavior.

Theorem III. *Let $V = V(x) \in P_{\text{loc}}^\infty$, $u_0 \in C^2(\mathbb{R}^n)$ with $0 < \beta \leq u_0 \leq \alpha$ and $\Delta u_0 \in P_{\text{loc}}^\infty$. Then the problem (P_V) has a global positive solution which converges pointwise to a positive solution of the elliptic problem $a\Delta u + V(x)u^p = 0$.*

In section 2, we prove the continuity of the parabolic potentials $\int \int G_a |V| dy ds$ when $V \in P_{\text{loc}}^\infty$. In section 3, we present an alternative and shorter proof of Theorem I based on a fixed point argument more simple than the Schauder theorem used by Zhang in [5]. In section 4, we prove Theorem II. In section 5, we characterize the classes P^∞ and P_{loc}^∞ in special cases which allow us to give interesting examples of potentials in the class P_{loc}^∞ which do not belong to the class P^∞ .

Throughout the paper the letter C denotes a positive constant which may vary in value from line to line but not on dependence.

2. CONTINUITY OF PARABOLIC POTENTIALS

In this section we prove the continuity of the potentials $\int \int G_a |V| dy ds$ when $V \in P_{\text{loc}}^\infty$. The continuity is required to establish the global existence result of Theorem I.

For $a > 0$, $x, y \in \mathbb{R}^n$ and $t > s$, let

$$\Gamma_a(x, t; y, s) = \frac{1}{(t-s)^{n/2}} \exp(-a \frac{|x-y|^2}{t-s}).$$

The fundamental solution of $a\Delta_x - \partial/\partial t$ on $\mathbb{R}^n \times \mathbb{R}$ is given by

$$G_a(x, t; y, s) = \begin{cases} (4a\pi)^{-n/2} \Gamma_{\frac{1}{4a}}(x, t; y, s) & \text{if } t > s, \\ 0 & \text{if } t < s. \end{cases}$$

For $\alpha, \beta \geq 0$ we use the notation $\alpha \vee \beta$ and $\alpha \wedge \beta$ to mean $\max(\alpha, \beta)$ and $\min(\alpha, \beta)$, respectively. We also need to use the inequality

$$(2.1) \quad e^{-\theta} \leq \frac{1 \vee (m/e)^m}{(1 \vee \theta)^m}, \quad \text{for all } \theta \geq 0 \text{ and } m > 0.$$

For $(x, t), (y, s) \in \mathbb{R}^n \times \mathbb{R}$, let $d((x, t), (y, s)) \equiv |x-y| \vee |t-s|^{1/2}$ be the parabolic distance on $\mathbb{R}^n \times \mathbb{R}$. In particular it satisfies the triangle inequality. For $r > 0$, let $B((x, t), r)$ be the ball with center (x, t) and radius r with respect to the parabolic distance and $B^c((x, t), r)$ its complement. We first prove the following.

Lemma 2.1. *Let $a > 0$. Then there exist constants $C = C(a) > 0$ and $c = c(a) > 0$ such that for all $r \in]0, 1[$, $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$, $(x, t) \in B((x_0, t_0), \sqrt{r}/8)$, $t \geq t_0$ and $(y, s) \in B^c((x_0, t_0), \sqrt{r}/2)$, we have*

$$|G_a(x, t; y, s) - G_a(x_0, t_0; y, s)| \leq C \left(\frac{|x-x_0| \vee |t-t_0|^{1/2}}{r^{1/2}} \right) G_c(x, t; y, s).$$

Proof. Case 1. $s \geq t$. The left-hand side term is equal to zero and so the inequality is trivial.

Case 2. $t_0 \leq s < t$. From the assumptions we have $0 < t-s \leq t-t_0$ and $|x-y| \vee |t-s|^{1/2} \geq 3\sqrt{r}/8$. By noting that $G_a(x_0, t_0; y, s) = 0$ and applying (2.1)

with $m = 1/2$, we have

$$\begin{aligned} |G_a(x, t; y, s) - G_a(x_0, t_0; y, s)| &= \frac{(4a\pi)^{-n/2}}{(t-s)^{n/2}} \exp\left(-\frac{|x-y|^2}{4a(t-s)}\right) \\ &\leq C \frac{(t-s)^{1/2}}{|x-y| \vee (t-s)^{1/2}} \frac{\exp\left(-\frac{|x-y|^2}{8a(t-s)}\right)}{(t-s)^{n/2}} \leq C \frac{(t-t_0)^{1/2}}{r^{1/2}} G_{2a}(x, t; y, s). \end{aligned}$$

Case 3. $s < t_0$. We have

$$(2.2) \quad \begin{aligned} |G_a(x, t; y, s) - G_a(x_0, t_0; y, s)| &\leq |G_a(x, t; y, s) - G_a(x_0, t; y, s)| \\ &\quad + |G_a(x_0, t; y, s) - G_a(x_0, t_0; y, s)|. \end{aligned}$$

Moreover, we have

$$(2.3) \quad \begin{aligned} |G_a(x, t; y, s) - G_a(x_0, t; y, s)| &\leq |x - x_0| \sup_{z \in [x, x_0]} |\nabla_z G_a(z, t; y, s)| \\ &\leq C|x - x_0| \sup_{z \in [x, x_0]} |z - y| \frac{\exp\left(-\frac{|z-y|^2}{4a(t-s)}\right)}{(t-s)^{n/2+1}}. \end{aligned}$$

On the other hand, by applying (2.1) with $m = 2$, we obtain

$$(2.4) \quad |z - y| \frac{\exp\left(-\frac{|z-y|^2}{4a(t-s)}\right)}{(t-s)^{n/2+1}} \leq \frac{C}{|z - y| \vee (t-s)^{1/2}} \frac{\exp\left(-\frac{|z-y|^2}{8a(t-s)}\right)}{(t-s)^{n/2}}.$$

Moreover, by the triangle inequality of d , for $z \in [x, x_0]$, we have

$$\begin{aligned} |z - y| \vee (t-s)^{1/2} &\geq |x - y| \vee (t-s)^{1/2} - |x - z| \\ &\geq |x - y| \vee (t-s)^{1/2} - \sqrt{r}/8 \geq \frac{2}{3}|x - y| \vee (t-s)^{1/2} \geq \sqrt{r}/4, \end{aligned}$$

which yields $|z - y|^2 + t - s \geq \frac{2}{9}(|x - y|^2 + t - s)$ and so

$$(2.5) \quad \frac{|z - y|^2}{t - s} \geq \frac{2}{9} \frac{|x - y|^2}{t - s} - \frac{7}{9}.$$

Combining (2.3), (2.4) and (2.5), we obtain

$$(2.6) \quad |G_a(x, t; y, s) - G_a(x_0, t; y, s)| \leq C \frac{|x - x_0|}{r^{1/2}} G_{9a}(x, t; y, s).$$

In the same way, there is a $\theta \in]t_0, t[$ such that

$$(2.7) \quad |G_a(x_0, t; y, s) - G_a(x_0, t_0; y, s)| = |t - t_0| \left| \frac{\partial}{\partial \theta} G_a(x_0, \theta; y, s) \right|.$$

Moreover by applying (2.1) with $m = n/2 + 2$ and using the inequality $0 < \theta - s \leq t - s$, we have

$$\begin{aligned} \left| \frac{\partial}{\partial \theta} G_a(x_0, \theta; y, s) \right| &\leq \left(\frac{n}{2} + \frac{|x_0 - y|^2}{4a(\theta - s)} \right) \frac{\exp\left(-\frac{|x_0 - y|^2}{4a(\theta - s)}\right)}{(\theta - s)^{n/2+1}} \\ &\leq \frac{C}{(|x_0 - y| \vee (\theta - s)^{1/2})^{n+2}} \exp\left(-\frac{|x_0 - y|^2}{8a(t-s)}\right). \end{aligned}$$

By noting that $|x_0 - y| \vee (\theta - s)^{1/2} \geq \frac{2}{3}|x - y| \vee (t-s)^{1/2} \geq \sqrt{r}/4$ and $\frac{|x_0 - y|^2}{t-s} \geq \frac{2}{9} \frac{|x - y|^2}{t-s} - \frac{7}{9}$, it follows that

$$(2.8) \quad \left| \frac{\partial}{\partial \theta} G_a(x_0, \theta; y, s) \right| \leq \frac{C}{r} G_{9a}(x, t; y, s).$$

Combining (2.7) and (2.8), we obtain

$$(2.9) \quad |G_a(x_0, t; y, s) - G_a(x_0, t_0; y, s)| \leq C \left(\frac{|t - t_0|}{r} \right)^{1/2} G_{9a}(x, t; y, s).$$

Combining (2.2), (2.6) and (2.9), we obtain the inequality stated in the lemma. $\square$

Remark 2.2. If we replace $t \geq t_0$ by $t \leq t_0$, we obtain the same inequality provided that $G_c(x, t; y, s)$ is replaced by $G_c(x_0, t_0; y, s)$.

For $(x, t), (x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$, we put

$$K_a(x, t; x_0, t_0) = \int_{-\infty}^{t \vee t_0} \int_{\mathbb{R}^n} |G_a(x, t; y, s) - G_a(x_0, t_0; y, s)| |V(y, s)| dy ds.$$

We have the following.

Proposition 2.3. *Let $V \in P_{\text{loc}}^\infty$. Then for all $a > 0$,*

$$\lim_{|x - x_0| \vee |t - t_0|^{1/2} \rightarrow 0} K_a(x, t; x_0, t_0) = 0.$$

Proof. Let $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$ be fixed. Put $A = \overline{B}((x_0, t_0), 1)$. Since $V \in P_{\text{loc}}^\infty$, for $\varepsilon > 0$, there exists $r > 0$ sufficiently small such that

$$(2.10) \quad \sup_{(x, t) \in A} \int_{t-r}^t \int_{|x-y| < \sqrt{r}} G_a(x, t; y, s) |V(y, s)| dy ds < \varepsilon.$$

For $|x - x_0| \vee |t - t_0|^{1/2} \leq \sqrt{r}/8$, we have

$$\begin{aligned} K_a(x, t; x_0, t_0) &= \int_{-\infty}^{t \vee t_0} \int_{\mathbb{R}^n} |G_a(x, t; y, s) - G_a(x_0, t_0; y, s)| |V(y, s)| dy ds \\ &= \int \int_{B((x_0, t_0), \sqrt{r}/2)} \dots dy ds + \int \int_{B^c((x_0, t_0), \sqrt{r}/2)} \dots dy ds \\ &\equiv I_1(x, t; x_0, t_0) + I_2(x, t; x_0, t_0). \end{aligned}$$

By (2.10), for $|x - x_0| \vee |t - t_0|^{1/2} \leq \sqrt{r}/8$, we have

$$\begin{aligned} I_1(x, t; x_0, t_0) &\leq \int \int_{B((x, t), \sqrt{r})} G_a(x, t; y, s) |V(y, s)| dy ds \\ &\quad + \int \int_{B((x_0, t_0), \sqrt{r})} G_a(x_0, t_0; y, s) |V(y, s)| dy ds \\ &\leq \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

By Lemma 2.1, for $|x - x_0| \vee |t - t_0|^{1/2} \leq \sqrt{r}/8$, we have

$$I_2(x, t; x_0, t_0) \leq C \left(\frac{|x - x_0| \vee |t - t_0|^{1/2}}{r^{1/2}} \right) N_c(V).$$

This ends the proof. $\square$

Remark 2.4. Similarly, for $(y, s), (y_0, s_0) \in \mathbb{R}^n \times \mathbb{R}$ and $a > 0$, let

$$K_a^*(y, s; y_0, s_0) = \int_{s \wedge s_0}^{+\infty} \int_{\mathbb{R}^n} |G_a^*(y, s; x, t) - G_a^*(y_0, s_0; x, t)| |V(x, t)| dx dt.$$

If $V \in P_{\text{loc}}^\infty$, then $\lim_{|y - y_0| \vee |s - s_0|^{1/2} \rightarrow 0} K_a^*(y, s; y_0, s_0) = 0$.

3. PROOF OF THEOREM I

For simplicity, we put

$$h(x, t) = \int_{\mathbb{R}^n} G_a(x, t; y, 0) u_0(y) dy.$$

The function h is continuous on $\mathbb{R}^n \times [0, \infty[$ and $0 \leq h \leq \|u_0\|_\infty$. Let $C_b(\mathbb{R}^n \times [0, \infty[)$ be the set of all bounded continuous solutions on $\mathbb{R}^n \times [0, \infty[$. It is well known that $(C_b(\mathbb{R}^n \times [0, \infty[), \|\cdot\|_\infty)$ is a Banach space. For $M > 1$, let us define the set

$$S = \{u \in C_b(\mathbb{R}^n \times [0, \infty[) : M^{-1}h \leq u \leq Mh\}.$$

Clearly S is a nonempty closed subset of $C_b(\mathbb{R}^n \times [0, \infty[)$. We define the integral operator T on $C_b(\mathbb{R}^n \times [0, \infty[)$ by

$$Tu(x, t) = h(x, t) + \int_0^t \int_{\mathbb{R}^n} G_a(x, t; y, s) V(y, s) u^p(y, s) dy ds.$$

Since u is bounded, by Proposition 2.3, $Tu \in C_b(\mathbb{R}^n \times [0, \infty[)$. Moreover, as in [5] (see p. 1550), there are constants $C = C(a, p) > 0$ and $c = c(a, p) > 0$ such that for all $u \in S$, $|Tu - h| \leq M(Mb_0)^{p-1} CN_c(V)h$. It follows that for b_0 sufficiently small, $M^{-1}h \leq Tu \leq Mh$, which means that $Tu \in S$ and so T maps S into itself. Moreover for all $u, v \in S$, we clearly have $\|Tu - Tv\|_\infty \leq p(Mb_0)^{p-1} CN_c(V) \|u - v\|_\infty$. In particular for b_0 small enough, we obtain $\|Tu - Tv\|_\infty \leq \frac{1}{2} \|u - v\|_\infty$. Thus T is a $\frac{1}{2}$ -Lipschitz mapping from S into itself, and so by the well-known fixed point theorem there exists $u \in S$ such that $Tu = u$.

4. PROOF OF THEOREM II

Assume that V is defined on $\mathbb{R}^n \times]0, \infty[$, $V \geq 0$ and for all $a > 0$ the result of Theorem I holds. By the assumptions, for all $M > 0$ there is a constant b_0 such that for each nonnegative $u_0 \in C^2(\mathbb{R}^n)$ with $\|u_0\|_\infty \leq b_0$, there exists a solution u of the integral equation

$$u(x, t) = \int_{\mathbb{R}^n} G_a(x, t; y, 0) u_0(y) dy + \int_0^t \int_{\mathbb{R}^n} G_a(x, t; y, s) V(y, s) u^p(y, s) dy ds$$

satisfying

$$M^{-1} \int_{\mathbb{R}^n} G_a(x, t; y, 0) u_0(y) dy \leq u(x, t) \leq M \int_{\mathbb{R}^n} G_a(x, t; y, 0) u_0(y) dy.$$

For $u_0 \equiv b_0$, we have $u(x, t) = b_0 + \int_0^t \int_{\mathbb{R}^n} G_a(x, t; y, s) V(y, s) u^p(y, s) dy ds$ and $M^{-1}b_0 \leq u(x, t) \leq Mb_0$, which implies

$$(4.1) \quad \sup_{(x, t) \in \mathbb{R}^n \times \mathbb{R}} \int_{-\infty}^t \int_{\mathbb{R}^n} G_a(x, t; y, s) |V(y, s)| dy ds \leq (M - 1) M^p b_0^{1-p} < \infty.$$

Moreover,

$$p(x, t) \equiv \int_{-\infty}^t \int_{\mathbb{R}^n} G_a(x, t; y, s) V(y, s) u^p(y, s) dy ds = u(x, t) - b_0$$

is a continuous potential. For $r \in]0, 1[$, let

$$p_r(x, t) = \int \int_{B((x, t), \sqrt{r})} G_a(x, t; y, s) V(y, s) u^p(y, s) dy ds$$

and

$$q_r(x, t) = \int \int_{B^c((x, t), \sqrt{r})} G_a(x, t; y, s) V(y, s) u^p(y, s) dy ds.$$

We will prove that q_r is continuous. Let $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$. For $|x - x_0| \vee |t - t_0|^{1/2} \leq \sqrt{r}/8$, we have

$$|q_r(x, t) - q_r(x_0, t_0)| \leq (Mb_0)^p (J_1(x, t; x_0, t_0) + J_2(x, t; x_0, t_0)),$$

where

$$J_1 = \int \int_{B^c((x, t), \sqrt{r})} |G_a(x, t; y, s) - G_a(x_0, t_0; y, s)| V(y, s) dy ds$$

and

$$J_2 = \int \int |\mathbf{1}_{B^c((x, t), \sqrt{r})} - \mathbf{1}_{B^c((x_0, t_0), \sqrt{r})}| G_a(x_0, t_0; y, s) V(y, s) dy ds.$$

If $|x - y| \vee |t - s|^{1/2} \geq \sqrt{r}$, then $|x_0 - y| \vee |t_0 - s|^{1/2} \geq \sqrt{r}/2$ and so by Lemma 2.1 and (4.1), we obtain

$$J_1(x, t; x_0, t_0) \leq C \frac{|x - x_0| \vee |t - t_0|^{1/2}}{r^{1/2}}.$$

On the other hand, by the dominated convergence theorem, we also have

$$J_2(x, t; x_0, t_0) \rightarrow 0 \quad \text{as} \quad |x - x_0| \vee |t - t_0|^{1/2} \rightarrow 0.$$

Hence q_r is continuous. We then have $p_r = p - q_r$ is continuous and $\lim_{r \rightarrow 0} p_r(x, t) = 0$. So, by the Dini theorem, $\lim_{r \rightarrow 0} \sup_{(x, t) \in A} p_r(x, t) = 0$, for any compact subset $A \subset \mathbb{R}^n \times \mathbb{R}$.

Now by recalling that $u^p \geq (Mb_0^{-1})^p$, we obtain

$$\lim_{r \rightarrow 0} \sup_{(x, t) \in A} \int \int_{B((x, t), \sqrt{r})} \Gamma_a(x, t; y, s) |V(y, s)| dy ds = 0,$$

for any compact subset $A \subset \mathbb{R}^n \times \mathbb{R}$, which completes the proof.

5. THE CLASSES P^∞ AND P_{loc}^∞

In this section we characterize the classes P^∞ and P_{loc}^∞ in interesting special cases so that the reader may see the kind of potentials admitted for the class P_{loc}^∞ and not for the class P^∞ . Let $C_b(\mathbb{R}^n \times \mathbb{R})$ be the set of all bounded continuous functions on $\mathbb{R}^n \times \mathbb{R}$ and let $C_0(\mathbb{R}^n \times \mathbb{R})$ be the subset of functions in $C_b(\mathbb{R}^n \times \mathbb{R})$ vanishing at infinity. For a Borel measurable function V on $\mathbb{R}^n \times \mathbb{R}$, we put

$$p_a^V(x, t) = \int_{-\infty}^t \int_{\mathbb{R}^n} G_a(x, t; y, s) |V(y, s)| dy ds$$

and

$$p_a^{*V}(y, s) = \int_s^\infty \int_{\mathbb{R}^n} G_a(x, t; y, s) |V(x, t)| dx dt.$$

We have the following result.

Proposition 5.1. *Let $V \in L^1(\mathbb{R}^n \times \mathbb{R})$. Then*

- (i) $V \in P_{\text{loc}}^\infty$ if and only if for all $a > 0$, the potentials p_a^V and $p_a^{*V} \in C_b(\mathbb{R}^n \times \mathbb{R})$.
- (ii) $V \in P^\infty$ if and only if for all $a > 0$, the potentials p_a^V and $p_a^{*V} \in C_0(\mathbb{R}^n \times \mathbb{R})$.

Proof. For simplicity we write, for $r > 0$,

$$p_{a,r}(x, t) = \int \int_{B((x,t), \sqrt{r})} G_a(x, t; y, s) |V(y, s)| dy ds$$

and

$$q_{a,r}(x, t) = \int \int_{B^c((x,t), \sqrt{r})} G_a(x, t; y, s) |V(y, s)| dy ds.$$

The analogous definitions for the adjoint potentials $p_{a,r}^*$ and $q_{a,r}^*$ are obvious. We will prove (i). Let $V \in P_{\text{loc}}^\infty$. Then $N_a(V) < \infty$ for all $a > 0$ and by Proposition 2.3 and Remark 2.4, p_a^V and p_a^{*V} are continuous. Hence p_a^V and $p_a^{*V} \in C_b(\mathbb{R}^n \times \mathbb{R})$.

Conversely, assume that for all $a > 0$, p_a^V and $p_a^{*V} \in C_b(\mathbb{R}^n \times \mathbb{R})$. Then $N_a(V) < \infty$ for all $a > 0$. Moreover, by using the same arguments as in the proof of Theorem II, we have, for all $r > 0$, $q_{a,r}$ is continuous, and so $p_{a,r} = p_a^V - q_{a,r}$ is continuous. Hence, by the Dini theorem, $\lim_{r \rightarrow 0} \sup_{(x,t) \in A} p_{a,r}(x, t) = 0$, for any compact subset $A \subset \mathbb{R}^n \times \mathbb{R}$. The same conclusion for $p_{a,r}^*$ holds. Thus $V \in P_{\text{loc}}^\infty$.

Now we will prove (ii). Assume that $V \in P^\infty$. Then by (i), p_a^V and $p_a^{*V} \in C_b(\mathbb{R}^n \times \mathbb{R})$. It remains to prove that p_a^V and p_a^{*V} tend to 0 at infinity. Let $\varepsilon > 0$. Then, there exists $r > 0$ such that

$$(5.1) \quad \sup_{(x,t) \in \mathbb{R}^n \times \mathbb{R}} p_{a,r}(x, t) < \varepsilon.$$

On the other hand, by (2.1), for $|x - y| \vee |t - s|^{1/2} \geq \sqrt{r}$, we have

$$G_a(x, t; y, s) |V(y, s)| \leq C \frac{|V(y, s)|}{(|x - y| \vee |t - s|^{1/2})^n} \leq \frac{C}{r^{n/2}} |V(y, s)|$$

with $\int \int_{\mathbb{R}^n \times \mathbb{R}} |V(y, s)| dy ds < \infty$. Then by the dominated convergence theorem it follows that $\lim_{|(x,t)| \rightarrow \infty} q_{a,r}(x, t) = 0$, and so there exists a compact subset $A \subset \mathbb{R}^n \times \mathbb{R}$ such that for $(x, t) \in A^c$,

$$(5.2) \quad q_{a,r}(x, t) < \varepsilon.$$

Combining (5.1) and (5.2) it follows that for $(x, t) \in A^c$, $p_a^V(x, t) < \varepsilon$. Hence $p_a^V \in C_0(\mathbb{R}^n \times \mathbb{R})$. In the same way we obtain $p_a^{*V} \in C_0(\mathbb{R}^n \times \mathbb{R})$.

Conversely assume that for all $a > 0$, p_a^V and $p_a^{*V} \in C_0(\mathbb{R}^n \times \mathbb{R})$. Then, for any $\varepsilon > 0$, there exists a compact subset $A \subset \mathbb{R}^n \times \mathbb{R}$ such that for $(x, t) \in A^c$,

$$(5.3) \quad p_a^V(x, t) < \varepsilon.$$

Moreover by (i), $V \in P_{\text{loc}}^\infty$ and so there is $r > 0$ small such that

$$(5.4) \quad \sup_{(x,t) \in A} p_{a,r}(x, t) < \varepsilon.$$

Combining (5.3) and (5.4), we obtain that $\sup_{(x,t) \in \mathbb{R}^n \times \mathbb{R}} p_{a,r}(x, t) < 2\varepsilon$. This means that $\lim_{r \rightarrow 0} \sup_{(x,t) \in \mathbb{R}^n \times \mathbb{R}} p_{a,r}(x, t) = 0$. In the same way, $\lim_{r \rightarrow 0} \sup_{(y,s) \in \mathbb{R}^n \times \mathbb{R}} p_{a,r}^*(y, s) = 0$. Thus $V \in P^\infty$. $\square$

From Proposition 5.1, we deduce the following characterization of the classes P_{loc}^∞ and P^∞ in the time-independent case.

Corollary 5.2. *Let $V = V(x) \in L^1(\mathbb{R}^n)$. Then*

- (i) $V \in P_{\text{loc}}^\infty$ if and only if the potential $p^V(x) \equiv \int_{\mathbb{R}^n} \frac{|V(y)|}{|x-y|^{n-2}} dy \in C_b(\mathbb{R}^n)$.
- (ii) $V \in P^\infty$ if and only if the potential $p^V(x) \equiv \int_{\mathbb{R}^n} \frac{|V(y)|}{|x-y|^{n-2}} dy \in C_0(\mathbb{R}^n)$.

Example 5.3. We will provide examples of functions in $P_{\text{loc}}^\infty \setminus P^\infty$ in the time-independent and time-dependent cases.

1) Let $g(x, y) = c_n |x - y|^{2-n}$, where $c_n = \frac{\Gamma(n/2-1)}{4\pi^{n/2}}$, be the Green function of the Laplacian on $\mathbb{R}^n$, $n \geq 3$. For any integer $m \geq 1$, let $x_m = (2m, 0, \dots, 0) \in \mathbb{R}^n$ and choose $\alpha_m > 0$ such that $\alpha_m g(\cdot, x_m) < 2^{-m}$ on $B^c(x_m, 1)$. Let $q_m = 1 \wedge \alpha_m g(\cdot, x_m)$. Clearly q_m is a continuous Δ -potential with harmonic support contained in the sphere $S(x_m, r_m)$, where r_m is the positive real number corresponding to $\alpha_m g(\cdot, x_m) = 1$ on $S(x_m, r_m)$ (a simple computation yields $r_m = (\alpha_m c_n)^{\frac{1}{n-2}}$), and so there is a Borel measure σ_m on $S(x_m, r_m)$ with total mass α_m such that $q_m(x) = \int_{S(x_m, r_m)} g(x, y) \sigma_m(dy)$. Let $\varphi_m \in C^\infty(\mathbb{R}^n)$, $\varphi_m \geq 0$, $\int \varphi_m(x) dx = 1$ and $\text{supp } \varphi_m \subset B(0, r_m)$. Consider the convolutions

$$V_m(x) = \sigma_m * \varphi_m(x) \equiv \int_{S(x_m, r_m)} \varphi_m(x - y) \sigma_m(dy)$$

and put

$$p_m(x) = \int_{\mathbb{R}^n} g(x, y) V_m(y) dy = \int_{B(0, r_m)} q_m(x - y) \varphi_m(y) dy.$$

Since $q_m(x_m) = 1$, $q_m \leq 2^{-m}$ on $B^c(x_m, 1)$ and $q_m \leq 1$ on $B(x_m, 1)$, then p_m satisfies the same properties. Moreover $p_m \in C^\infty(\mathbb{R}^n)$. Now let us take

$$V(x) = \sum_{m=1}^{\infty} V_m(x).$$

We have $V \in C^\infty(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} V(x) dx = \sum_{m=1}^{\infty} \int_{\mathbb{R}^n} V_m(x) dx = \sum_{m=1}^{\infty} \alpha_m < \infty.$$

Moreover, the potential $p(x) = \int_{\mathbb{R}^n} g(x, y) V(y) dy = \sum_{m=1}^{\infty} p_m(x) \in C^\infty(\mathbb{R}^n)$, $p \leq 2$,

and for every integer $m \geq 1$, $p(x_m) \geq p_m(x_m) = 1$. This shows that $p \in C_b(\mathbb{R}^n) \setminus C_0(\mathbb{R}^n)$ and so by Corollary 5.2, $V \in P_{\text{loc}}^\infty \setminus P^\infty$.

2) Let $V = V(x)$ be the function defined in 1). Choose a Borel measurable function $\varphi = \varphi(t)$ on $\mathbb{R}$ with $0 < \alpha \leq \varphi \leq \beta$. Here α and β are two constants. Clearly $\varphi \in L_{\text{loc}}^1(\mathbb{R}) \setminus L^1(\mathbb{R})$. Then the function $W(x, t) = V(x) \varphi(t) \in P_{\text{loc}}^\infty \setminus P^\infty$ with W does not belong to $L^1(\mathbb{R}^n \times \mathbb{R})$.

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