NOTES ON BRAIDZEL SURFACES FOR LINKS

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ABSTRACT. As a generalization of pretzel surfaces, L. Rudolph has introduced a notion of braidzel surfaces in his study of the quasipositivity for pretzel surfaces. In this paper, we show that any oriented link has a braidzel surface. We also introduce a new geometric numerical invariant of links with respect to their braidzel surface and study relationships among them and other “genus” for links.

1. INTRODUCTION

A *link* is a smooth 1-manifold embedded in the 3-sphere S^3 , and a *knot* is a link with one connected component. It is well known that any link is the boundary of a compact, connected surface, called a *spanning surface* for the link. If a spanning surface for a link is orientable, then we call it a *Seifert surface* for the link. For n integers $t_1, t_2, \dots, t_n$, a spanning surface for a link is said to be a *pretzel surface*, denoted by $P(t_1, t_2, \dots, t_n)$, if it consists of two disks joined by a $|t_i|$ half-twisted, unknotted vertical band b_i for $i = 1, 2, \dots, n$ as in Figure 1(a). If t_i is positive (resp. negative), then b_i is right-handed (resp. left-handed). Note that a pretzel surface $P(t_1, t_2, \dots, t_n)$ is orientable if and only if all t_i 's are odd or even. A link with a pretzel surface is called a *pretzel link*. Figure 1(b) illustrates a schematic diagram of a spine of a pretzel surface.

FIGURE 1.

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We can regard the cores of the bands of a pretzel surface as the n -string trivial braid; see Figure 1(b). From such a point of view, L. Rudolph has introduced a notion of *braidzel surfaces* as a generalization of pretzel surfaces in [5] on his study of the quasipositivity for pretzel surfaces. Let β be an n -string braid and $t_1, t_2, \dots, t_n$ integers. A spanning surface for a link is said to be a braidzel surface, denoted by $B(\beta; t_1, t_2, \dots, t_n)$, if it consists of two disks joined by n bands $b_1, b_2, \dots, b_n$ such that: (1) the cores of the bands form an n -string braid β ; (2) each b_i ($1 \leq i \leq n$) is $|t_i|$ half-twisted. If t_i is positive (resp. negative), then b_i is right-handed (resp. left-handed). More precisely, we call it an n -braidzel surface. See Figure 2, which illustrates $B(\beta; 3, -3, 2, 4, -4)$, where $\beta = \sigma_1^2 \sigma_3 \sigma_4 \sigma_3 \sigma_2 \sigma_1^{-1} \sigma_2 \sigma_4^{-2}$ is written by Artin's standard generators. Similar to the case of a pretzel surface, a braidzel surface $B(\beta; t_1, t_2, \dots, t_n)$ is orientable if and only if all t_i 's are odd or even.

FIGURE 2.

By direct deformation, we see that any 2-braidzel surface is ambient isotopic to a twisted annulus or a twisted Möbius band. Moreover we see that any 3-braidzel surface is ambient isotopic to a pretzel surface.

We see that any 2-bridge link and any torus link has a braidzel surface by direct deformation. Moreover a twisted double of a trefoil knot and the connected-sum of a trefoil knot and its copy also have braidzel surfaces, though any pretzel knot is simple. So the question can be proposed naturally. *What kinds of links have braidzel surfaces?* In this paper, we give the complete answer to this question.

Theorem 1.1. *Any oriented link has an oriented braidzel surface.*

By considering the connected-sum of an orientable braidzel surface and the unknotted, untwisted Möbius band and some modifications, we have a corollary.

Corollary 1.2. *Any unoriented link has a non-orientable braidzel surface.* $\square$

This paper is organized as follows. In Section 2, we will prove Theorem 1.1. In Section 3, we will introduce a new geometric numerical invariant of links with respect to their braidzel surface, the *braidzel genus*. We will study relationships among them and other “genus” for links.

For the definition of standard terms in knot theory, we refer to [2] and [4].

2. PROOF OF THEOREM 1.1

In this section, we prove our main theorem, Theorem 1.1. Hereafter we assume that all links are oriented unless otherwise stated.

A Seifert surface for a link L is said to be *canonical* if it is obtained from a connected diagram of L by applying Seifert's algorithm. Theorem 1.1 immediately follows from the next theorem.

Theorem 2.1. *Any canonical Seifert surface for L is ambient isotopic to a braidzel surface.*

A diagram $\tilde{L}$ is said to be *flat* if any Seifert circle in $\tilde{L}$ is innermost. A canonical Seifert surface obtained from a flat diagram is called a *flat Seifert surface*. M. Hirasawa [1] showed that any canonical Seifert surface is ambient isotopic to a flat Seifert surface. Therefore in fact we prove the following.

Proposition 2.2. *Any flat Seifert surface for L is ambient isotopic to a braidzel surface.*

Let G be a connected, finite plane graph on the 2-sphere S^2 , and v_0 a given vertex of G . We denote by $V(G)$ the set of vertices of G and by $|V(G)|$ the number of vertices of G . We define the three moves for G as indicated in Figure 3 as follows: The move M1 is defined as changing each multi-edge into one edge (Figure 3(a)).

Let $v (\neq v_0)$ be a vertex of degree two and e, e' edges incident to v . Assume that e and e' has no common vertex except v . Then the move M2 is defined as changing e, e' and v into one edge (Figure 3(b)).

Let e_0 be an edge incident to v_0 and another vertex v , and e an edge incident to v and another vertex $v' (\neq v_0)$. Let a be an arc on S^2 connecting v and v' . If the interior of a domain bounded by $v_0 \cup e_0 \cup v \cup e \cup v' \cup a$ has no intersection with G , that is, the domain is innermost, then we replace the edge e by a . We can regard this operation as moving the boundary of e on v to v_0 along e_0 . The move M3 is defined as this operation (Figure 3(c)).

FIGURE 3.

Let G_0 be the tree of two vertices.

Lemma 2.3. *Assume that G has no loop and no cut-vertex. Then independent of the choice of v_0 , G can be deformed into G_0 by a finite sequence of the moves M1, M2 and M3 without deleting v_0 .*

Proof. We prove Lemma 2.3 by induction on $|V(G)|$. Note that each of the moves M1, M2 and M3 does not make a loop in the resulting graph after applying them. Furthermore each of the moves M1 and M2 also does not make a cut-vertex.

Suppose $|V(G)| = 2$. By applying the move M1, we may assume that G has no multi-edge. Then obviously Lemma 2.3 holds. Suppose $|V(G)| = 3$. Then we see that G is a triangle graph after applying the move M1. By applying the move M2 to a vertex except v_0 , we obtain a graph consisting of two vertices connected by two edges. Then Lemma 2.3 holds by the move M1.

Assume that Lemma 2.3 holds for any G with $|V(G)| < n$ for a positive integer n (≥ 4). Now suppose $|V(G)| = n$. By applying the move M1, we may assume G has no multi-edge. If there exists a vertex of degree two except v_0 , then by applying the move M2, we obtain a new plane graph G' with $|V(G')| < |V(G)|$. Thus Lemma 2.3 holds by induction assumption.

Assume that there exists no vertex of degree two in $V(G) \setminus \{v_0\}$. Let $v_1, v_2, \dots, v_k$ be the vertices incident to v_0 . Since if $k = 1$, then v_1 is a cut-vertex, we may assume $2 \leq k \leq n - 1$. Let e be the edge incident to v_0 and v_i for some i ($1 \leq i \leq k$). Let A_i be a simple arc on S^2 connecting v_0 and v_i , where $A_i \cap G = \{v_0, v_i\}$. Then $A_i \cup v_0 \cup e \cup v_i$ forms a simple loop C_i on S^2 . Let D_i and D'_i be the two disks of $S^2 - \text{int}N(C_i)$, where $N(C_i)$ is a tubular neighborhood of C_i in S^2 . If $D_i \cap G \neq \emptyset$ and $D'_i \cap G \neq \emptyset$, then we call such a simple loop a *cut-loop* for G . See Figure 4. If there exists a cut-loop C_i , then applying the moves M3 with respect to v_i makes v_0 a cut-vertex. Conversely if applying the move M3 to G makes v_0 a cut-vertex, then there exists a cut-loop for G on S^2 .

FIGURE 4.

Assume that there exist several cut-loops for G on S^2 . We denote the innermost one by C_i , and two disks of $S^2 - \text{int}N(C_i)$ by D_i and D'_i , again. If there exists no vertex in D_i , then v_0 and v_i are connected by a multi-edge. If there exists no vertex incident to v_0 in D_i , then v_i is a cut-vertex. These are contradictions. Hence there exists an integer $j \in \{1, 2, \dots, i - 1, i + 1, \dots, k\}$ such that vertex v_j is incident to v_0 in D_i . Then since C_i is innermost, the move M3 with respect to v_j does not make v_0 a cut-vertex.

Suppose that the degree of v_j is $m + 1$. Let e_0 be the edge connecting v_j and v_0 . For the edges incident to v_j except e_0 , we denote them by $e_1, e_2, \dots, e_m$ counterclockwise around v_j from e_0 . By applying the move M3 with respect to v_j , we move the boundary of each e_i on v_j to v_0 along e_0 for $i = 1$ to $m - 1$. Then since the degree of v_j is two in the resulting graph, we can apply the move M2 to e_0, e_m and v_j . See Figure 5. Therefore we obtain a new plane graph G' with

$|V(G')| < |V(G)|$. If there exists no cut-loop for G , then we also obtain G' with $|V(G')| < |V(G)|$ by applying the move M3 with respect to any vertex incident to v_0 . The conclusion follows from induction assumption. $\square$

FIGURE 5.

Translation. Before proving Proposition 2.2, we translate this argument for a plane graph into that for a spanning surface for a link. We regard each vertex as a disk and each edge as several bands whose cores form a braid as in Figure 6(a). We call such a set of bands a β -braiding-bands-set when the cores of the bands form a braid β . Let D_{v_0} be the disk corresponding to v_0 . For only the disk D_{v_0} , the cores of all bands in the braiding-bands-sets incident to D_{v_0} form one braid β as in Figure 6(b). Then we see that a graph G in Lemma 2.3 presents a spanning surface for a link. Note that the graph G_0 presents a braidzel surface.

FIGURE 6.

The moves M1, M2 and M3 for a graph are corresponding to those for a spanning surface for a link depicted in Figure 7.

The move M1 is realized by the following: If there exist two disks connected by a β_1 -braiding-bands-set and a β_2 -braiding-bands-set, then we present them with one braiding-bands-set.

The move M2 is realized by the following deformation of a surface: Let D be a disk corresponding to a vertex of degree two and D' (resp. D'') a disk connected with D by a β' -braiding-bands-set (resp. a β'' -braiding-bands-set). Let $b'_1, b'_2, \dots, b'_m$ (resp. $b''_1, b''_2, \dots, b''_n$) be the elements of the β' -braiding-bands-set (resp. the β'' -braiding-bands-set) according to Figure 8(a). First we thicken the band b'_1 (resp. b''_1) and thin bands $b'_2, \dots, b'_m$ (resp. $b''_2, \dots, b''_n$). See Figure 8(b).

FIGURE 7.

Second, we deform D as follows: Let a'_i (resp. a''_i) be a “short” subarc of ∂D between b'_i and b'_{i+1} for $i = 1, 2, \dots, m-1$ (resp. b''_i and b''_{i+1} for $i = 1, 2, \dots, n-1$). By pushing the arcs $a'_1, a'_2, \dots, a'_{m-1}$ (resp. $a''_1, a''_2, \dots, a''_{n-1}$) into the thickened band b'_1 (resp. the thickened band b''_1) until they reach $\partial D'$ (resp. $\partial D''$), then we see that D' and D'' are connected by a new braiding-bands-set, since D can be regarded as being deleted.

FIGURE 8.

Let B_1 be a β_1 -braiding-bands-set incident to the disk D_{v_0} and a disk D_v . Let B_2 be a β_2 -braiding-bands-set incident to D_v and another disk $D_{v'} (\neq D_{v_0})$. The move M3 is realized by sliding the bands in B_2 on D_v to D_{v_0} along the band in B_1 nearest to B_2 on ∂D_v . See Figure 7(c).

Note that these moves for a spanning surface preserves the orientability. We also note that this translation completely recovers Lemma 2.3, since the edges incident to v_0 are not moved by the move M3.

Proof of Proposition 2.2. Let $\tilde{L}$ be a reduced, connected flat diagram of a link L and S a flat Seifert surface obtained from $\tilde{L}$. A flat Seifert surface S consists of Seifert disks and half-twisted bands connecting them and is on S^2 except small neighborhoods of crossing points of half-twisted bands. Let ℓ be a simple loop on S^2 meeting $\tilde{L}$ at two points. If any such a loop ℓ bounds a disk intersecting $\tilde{L}$ in a simple arc, then we say $\tilde{L}$ is *prime*. Otherwise, ℓ is called a *separating loop*. (Assume that ℓ meets $\tilde{L}$ in a Seifert circle in $\tilde{L}$.)

Assume that $\tilde{L}$ is prime. We regard each Seifert disk and each half-twisted band of S as a vertex and an edge, respectively. Then we obtain a graph G , which is called a *Seifert graph*, from S . A half-twisted band can be regarded as a β -braiding-bands-set where β is the 1-string trivial braid. Then since $\tilde{L}$ is flat and prime, we

see that G satisfies the assumption of Lemma 2.3. By Lemma 2.3 and its translation mentioned above, G can be deformed into G_0 . Hence we obtain a braidzel surface which is ambient isotopic to S .

Assume that $\tilde{L}$ is not prime. Let $\tilde{L}_1, \tilde{L}_2, \dots, \tilde{L}_n$ be n subdiagrams of $\tilde{L}$ as prime factors and $S_1, S_2, \dots, S_n$ subsurfaces of S obtained from $\tilde{L}_1, \tilde{L}_2, \dots, \tilde{L}_n$, respectively. Then there exist $n - 1$ separating loops on S^2 . By deforming $\tilde{L}$ with S , we may assume that the separating loops are concentric. Note that this deformation preserves the flatness of $\tilde{L}$, hence of S . We denote by ℓ_i the i -th separating loop from the innermost one; see Figure 9(a). (Each rectangle in Figure 9(a) contains S_i .) Let D_i be a Seifert disk of S such that $\ell_i \cap S \subset D_i$ ($1 \leq i \leq n - 1$). Let D_n be a Seifert disk in S_n as in Figure 9(a). To be precise, we consider S as $S_1 \#_{D_1} S_2 \#_{D_2} \cdots \#_{D_{n-1}} S_n$, where $S_i \#_{D_i} S_{i+1}$ means the connected sum of S_i and S_{i+1} along D_i . We deform S into a braidzel surface as follows: First, we designate D_1 as the disk corresponding to the vertex v_0 in Lemma 2.3. Second, we deform S_1 into a braidzel surface B_1 as in the translation of the proof of Lemma 2.3. Then we see that it can be done in the disk bounded by ℓ_1 on S^2 which contains S_1 . Therefore we can regard S as $B_1 \#_{D_1} S_2 \#_{D_2} \cdots \#_{D_{n-1}} S_n$; see Figure 9(b). Next we slide B_1 so that it is connected with S_n along D_n , and denote by D'_1 the other disk of B_1 . Then we have $S = S_2 \#_{D_2} \cdots \#_{D_{n-1}} S_n \#_{D_n} B_1$; see Figure 9(c). Similarly, we deform S_2 into a braidzel surface B_2 in the disk bounded by ℓ_2 on S^2 which contains S_2 . We also slide B_2 so that it is connected with B_1 along D'_1 and denote by D'_2 the other disk of B_2 . Then we obtain $S = S_3 \#_{D_3} \cdots \#_{D_{n-1}} S_n \#_{D_n} B_1 \#_{D'_1} B_2$; see Figure 9(d). By proceeding with this procedure, we have $S = B_1 \#_{D_1} B_2 \#_{D'_2} \cdots \#_{D'_{n-1}} B_n$. Finally, we obtain a braidzel surface by applying $n - 2$ times of the deformation in the translation of the move M2. The proof is completed. $\square$

FIGURE 9.

3. BRAIDZEL GENUS FOR LINKS

By our main theorem, we can define a new geometric numerical invariant for links with respect to their braidzel surfaces.

$g_f(L_m) = (1 - \mu(L_m) + B_1(S_m))/2 = 0$, where $\mu(L_m)$ is the number of components of L and $B_1(S_m)$ is the first Betti number of S_m . It is known that any oriented braidzel surface of the first Betti number two is a 3-braidzel surface, hence a 3-strand pretzel surface. The bridge index of a 3-strand pretzel link is at most three. The bridge index of L_m is, however, at least four, since $\mu(L_m) = 3$ and one of the components of L_m is a trefoil knot. Thus the first Betti number of any oriented braidzel surface for L_m is greater than two, hence $g_b(L_m) \geq 1$. Note that we see $g_b(L_m) = 1$, since L_m has the braidzel surface as in Figure 11(b), where the rectangle means $|m|$ full-twists of two bands. $\square$

FIGURE 11.

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