WHITTAKER MODELS FOR $SL_2(F)$, F A p -ADIC FIELD

ROBERTO JOHNSON AND JOSÉ PANTOJA

(Communicated by Wen-Ching Winnie Li)

ABSTRACT. Let F be a non-archimedean local field with odd residual characteristic. Let G be the group $SL_2(F)$. We construct explicit Whittaker models for any supercuspidal representation of G with positive level.

INTRODUCTION

Let F be a non-Archimedean local field with residual characteristic an odd prime number p . Let G be the group of the invertible 2×2 matrices over F and let $\overline{G}$ be the subgroup of the 2×2 matrices over F with determinant equal to 1.

In this work we construct explicitly a Whittaker model for any supercuspidal representation of $\overline{G}$, of level greater than or equal to one (a representation which is obtained by restriction from a supercuspidal representation of G).

The paper is organized as follows. Section one is essentially devoted to gathering together some facts about the construction of supercuspidal representations of $G = GL_2(F)$ and their restriction to $\overline{G} = SL_2(F)$. Section two gives some preliminaries about Whittaker models. In section three we construct the hereditary order $\mathcal{A}$ which we use in section four to prove the main result.

1. CONSTRUCTION OF THE SUPERCUSPIDAL REPRESENTATIONS OF G AND $\overline{G}$

We denote the ring of integers in F by O_F , we fix a generator ϖ_F of the maximal ideal P_F of O_F and write $q_F = [O_F : P_F]$ for the order of the residue field $k = O_F/P_F$. For an element x in F , we denote by $\nu_F(x)$ its valuation. Furthermore, we set $U_F = O_F^\times$ and write U_F^n for $1 + P_F^n$, $n \geq 1$. If E is an extension field of F , $[E : F]$ denotes the degree of the extension, $e(E/F)$ its ramification degree and $f(E/F)$ the inertia degree. For more details see [Se].

By a smooth representation (Π, W) of G we mean a $\mathbf{C}$ -vector space W ($\mathbf{C}$ the field of complex numbers) together with a homomorphism Π of G into $\text{Aut}_{\mathbf{C}}(W)$ such that the stability subgroup in G , for any vector w in W , is open. A smooth representation of G is called supercuspidal if each of its coefficients is supported on a set which is compact-modulo-center; see [Cr] or [Cs] for more details.

Received by the editors March 21, 2005 and, in revised form, August 26, 2005.

2000 *Mathematics Subject Classification*. Primary 22D30.

The first author was partially supported by Pontificia Universidad Catolica de Valparaiso (Project DI 124.796/2004).

The second author was partially supported by FONDECYT (Project 1030907) and Pontificia Universidad Catolica de Valparaiso.

In what follows, $V = F^2$, $\mathcal{B} = \{e_1, e_2\}$ is the standard basis of V , and $\mathcal{A}_j$ ($j = 1, 2$) is the principal order in $A = M_2(F)$ with period $e(\mathcal{A}_j) = j$ ($j = 1, 2$) associated to the O_F -lattice chain $\mathcal{L}^j = \{L_i^j \mid i \in \mathbf{Z}\}$ ($j = 1, 2$) in V where

$$\begin{array}{ll} L_0^1 = O_F e_1 + O_F e_2, & L_0^2 = O_F e_1 + O_F e_2, \\ L_1^1 = P_F e_1 + P_F e_2, & L_1^2 = O_F e_1 + P_F e_2, \\ & L_2^2 = P_F e_1 + P_F e_2. \end{array} \quad \text{and}$$

So, $P_F L_i^j = L_{i+j}^j$ ($j = 1, 2$) for every integer i , and the Jacobson radical $\mathcal{P}_j$ of $\mathcal{A}_j$ is principal as both a left and a right ideal. Also, for any principal order $\mathcal{A}$ we set $\mathcal{P}$ for the Jacobson radical of $\mathcal{A}$. We can define a sequence of compact open subgroups of G by

$$\mathcal{U} = \mathcal{U}^0 = \mathcal{A}^\times$$

($\mathcal{A}^\times$ is the set of invertible elements of $\mathcal{A}$) and

$$(1.1) \quad \mathcal{U}^n = 1 + \mathcal{P}^n, \text{ for } n \geq 1.$$

The set

$$(1.2) \quad \mathcal{K}(A) = \{x \in G \mid x\mathcal{U}x^{-1} = \mathcal{U}\},$$

i.e. the normalizer of $\mathcal{U}$ in G , is a maximal open compact-modulo-center subgroup of G . See 1.3 in [B-F] for more details.

Let $\mathcal{A}$ be any principal order and let $\mathcal{P}$ be the radical of $\mathcal{A}$. Let ψ be a smooth character of F with conductor P_F , i.e., $\ker \psi \not\supseteq O_F$. If n, r are integers satisfying $n \geq r$, then the map $b \mapsto \psi(\text{tr}(b?))$ induces an isomorphism of $\mathcal{P}^{-n}/\mathcal{P}^{-r}$ onto $(\mathcal{P}^{r+1}/\mathcal{P}^{n+1})^\wedge$ (the smooth dual of $\mathcal{P}^{r+1}/\mathcal{P}^{n+1}$). See, for example, [B].

If r, n are integers satisfying $n \geq r \geq [\frac{n}{2}] > 0$ (where $[x]$ denotes the greatest integer less than or equal to x), then $\mathcal{P}^{-n}/\mathcal{P}^{-r}$ is isomorphic to $(\mathcal{U}^{r+1}/\mathcal{U}^{n+1})^\wedge$ by $b + \mathcal{P}^{-r} \mapsto \psi_b$, where $\psi_b(x) = \psi(\text{tr}(b(x-1)))$.

An element b in $\mathcal{P}^{-n}/\mathcal{P}^{-r}$ which verifies that $E = F[b]$ is a field, that $\gcd(\nu_E(b), e(E/F)) = 1$, and such that $\bar{\beta} = \varpi_F^{-\nu_E(b)} b^{e(E/F)} + P_E$ generates the residue class field extension k_E/k_F , is called an E/F -minimal element of level $-n$. See, for example, [B-K].

It is known that (see for example [K2] or [K-P]) given an irreducible supercuspidal representation Π of G , there exists a principal order $\mathcal{A}$ and a character χ of $F^\times$ such that either

- (i) (representations of level zero) $e(\mathcal{A}) = 1$ and $\Pi \otimes (\chi \cdot \det)$ restricted to $\mathcal{U} = \mathcal{A}^\times$ contains a representation σ with $\mathcal{U}^1 < \ker \sigma$ such that σ , viewed as a representation of $GL_2(k_F)$, is cuspidal, or
- (ii) (representations of level n) there is an integer $n \geq 1$ and an E/F -minimal element b of level $-n$ such that $\Pi \otimes (\chi \cdot \det)$ restricted to $\mathcal{U}^n$ contains ψ_b .

Furthermore, if Π is an irreducible supercuspidal representation of G , then Π is equivalent to a representation of the form $\Pi^\sigma = \text{Ind}_{\mathcal{K}(A)}^G \sigma$, where σ is an irreducible representation of $\mathcal{K}(A)$, $\mathcal{A}$ being a principal order of A . (See [K2].)

More specifically, if n is odd we can construct $\sigma = \text{Ind}_{F[b]^\times \mathcal{U}^{\frac{n+1}{2}}}^{\mathcal{K}(A)} \lambda$, which is an irreducible representation of $\mathcal{K}(A)$, where $\lambda = \theta \psi_b$, and θ is a character of $F[b]^\times$ such that $\theta|_{F[b]^\times \cap \mathcal{U}^{\frac{n+1}{2}}} = \psi|_{b|F[b]^\times \cap \mathcal{U}^{\frac{n+1}{2}}}$. If n is even we can still associate a finite-dimensional representation λ of $F[b]^\times \mathcal{U}^{\frac{n}{2}}$ such that $\sigma = \text{Ind}_{F[b]^\times \mathcal{U}^{\frac{n}{2}}}^{\mathcal{K}(A)} \lambda$ is an

irreducible representation of $\mathcal{K}(A)$. So, in any case, Π is equivalent to a representation of the form

$$(1.3) \quad \Pi^\lambda = \text{Ind}_{F[b]^\times \mathcal{U}[\frac{n+1}{2}]} \lambda.$$

Now, if Π is an irreducible supercuspidal representation of G satisfying (i), then $\Pi = \text{Ind}_{\mathcal{K}}^G \sigma$, where $\mathcal{K} = F^\times GL_2(O_F)$. We can apply Mackey's theorem (see [K1]) to get

$$\Pi|_{\overline{G}} = \bigoplus_{z \in G/\overline{G}\mathcal{K}} \text{Ind}_{z\mathcal{K}z^{-1}}^{\overline{G}} \sigma^z$$

with $\overline{\mathcal{K}} = \mathcal{K} \cap \overline{G}$. In this case we have that $G/\overline{G}\mathcal{K} \cong F^\times/F^{\times 2}U_F$ so that $G/\overline{G}\mathcal{K} = \{1, z_0\}$ and therefore

$$(1.4) \quad \Pi|_{\overline{G}} = \Pi_1^\sigma \oplus \Pi_{z_0}^\sigma,$$

where $\Pi_1^\sigma = \text{Ind}_{\overline{\mathcal{K}}}^{\overline{G}} \sigma$ and $\Pi_{z_0}^\sigma = \text{Ind}_{z_0\overline{\mathcal{K}}z_0^{-1}}^{\overline{G}} \sigma^{z_0}$ are supercuspidal representations of $\overline{G}$ of level zero.

On the other hand, if Π satisfies (ii), then $\Pi = \text{Ind}_H^G \lambda$, where $H = F[b]^\times \mathcal{U}[\frac{n+1}{2}]$. So, Mackey's theorem again gives us

$$\Pi|_{\overline{G}} = \bigoplus_{u \in G/\overline{G}H} \text{Ind}_{uHu^{-1}}^{\overline{G}} \lambda^u$$

with $\overline{H} = H \cap \overline{G}$. In this case we have that if E/F is ramified, then $G/\overline{G}H \cong F^\times/N_{E/F}(E^\times)$ if $[\frac{r+1}{2}] \geq d(E/F)$, with $d(E/F)$ the differential exponent of the extension E/F and $n = 2r - 1$. Since we are in the case where the residual characteristic of the field is odd, the above inequality holds, which implies that $G/\overline{G}H$ has order 2. In the case where E/F is unramified we still have that $G/\overline{G}H$ has order 2, so $G/\overline{G}H = \{1, u_0\}$. It follows that

$$(1.5) \quad \Pi|_{\overline{G}} = \Pi_1^\lambda \oplus \Pi_{u_0}^\lambda,$$

where $\Pi_1^\lambda = \text{Ind}_{\overline{H}}^{\overline{G}} \lambda$ and $\Pi_{u_0}^\lambda = \text{Ind}_{u_0\overline{H}u_0^{-1}}^{\overline{G}} \lambda^{u_0}$ are irreducible supercuspidal representations of $\overline{G}$ of level n . (For more details, see [K-P]. Also see [K-S] for the irreducibility of the 2 pieces.)

2. THE GENERAL SETTING

A character of $\overline{U}$, the unipotent radical of the standard Borel subgroup of $\overline{G}$, is a smooth homomorphism τ from $\overline{U}$ into the group of complex numbers of absolute value 1. Such character can be seen as a smooth homomorphism τ_F from the additive group of F into the group of complex numbers of module 1:

$$(2.1) \quad \tau \left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right) = \tau_F(b) \quad (b \in F).$$

The group $\overline{T}$ (the maximal torus of diagonal matrices in $\overline{G}$) acts on the set of the characters of $\overline{U}$ by conjugation, i.e. given a character τ of $\overline{U}$ we have $\tau^t(u) = \tau(tut^{-1})$ for $t \in \overline{T}$ and $u \in \overline{U}$.

The character τ of $\overline{U}$ (respectively U , the unipotent radical of the standard Borel subgroup of G) is called *generic* if $\tau^t = \tau$ for t in $\overline{T}$ (respectively T) implies that t is in the center $Z(\overline{G})$ (respectively $Z(G)$) of $\overline{G}$ (respectively G).

We then consider the set X of the nontrivial characters of U .

Lemma 2.2. *The group T (the maximal torus of diagonal matrices in G) acts transitively on X by conjugation.*

Proof. Clearly the group $\widehat{U}$, of the characters of U , is isomorphic with the group $\widehat{F^+}$, of the characters of the additive group of F . For a fixed nontrivial character τ_0 in X and $\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}$ in T , we have that

$$\tau_0^{\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}} \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right) = \tau_0 \left(\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r^{-1} & 0 \\ 0 & s^{-1} \end{pmatrix} \right).$$

Then, via the obvious identifications, the above is equal to $\tau_0(rs^{-1}u)$. Now, if we denote by τ_l the character defined by $\tau_l(u) = \tau_0(lu)$, $l \in F^\times$ and $u \in F$, then the orbit of τ_0 , under the action of T , is the set $\{\tau_l \mid l \in F^\times\}$ that is the set X , which shows us that T acts transitively on X . A computation shows that the stabilizer subgroup of τ_0 under the action of T is the subgroup $Z(G)$, the center of G . $\square$

It follows that the group T acts transitively on the generic characters of $\overline{U}$. In the case of $\overline{G}$, the action of $\overline{T}$ is not transitive on the generic characters of $\overline{U}$.

Proposition 2.3. *The orbits of the generic characters of $\overline{U}$ under the action of $\overline{T}$ are in bijection with $T/\overline{T}Z(G)$.*

Proof. We know that T acts transitively on $X = \widehat{U} - \{1\}$. For this action the center acts trivially. If we fix a nontrivial character τ_0 of X , then any character τ of X is of the form

$$\tau = \tau_0^{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}},$$

where $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ is an element of T modulo $Z(G)$.

Since the F -orbit of τ is the set

$$\text{orb}(\tau) = \left\{ \tau^{\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}} \mid r \in F^\times \right\},$$

it follows that the function $\text{orb}(\tau) \mapsto \overline{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}} \in T/\overline{T}Z(G)$ is a bijection.

Corollary 2.4. *The $\overline{T}$ orbits of the generic characters of $\overline{U}$ are parametrized by $F^\times/(F^\times)^2U_F$.*

Proof. Since $F^\times = \langle \varpi \rangle U_F$, the function

$$\overline{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}} \longrightarrow \overline{ab}$$

is an isomorphism between $T/\overline{T}Z(G)$ and $F^\times/(F^\times)^2U$.

Let τ be a generic character of $\overline{U}$ of one of the $\overline{T}$ -orbits. Then there is an element t in T modulo $\overline{T}Z(G)$ such that $\tau(u) = \tau_0(t^{-1}ut)$ for $u \in \overline{U}$.

Since $\overline{U}$ is a closed subgroup of $\overline{G}$, we can construct the smooth induced representation (ρ, W) for the generic character τ of $\overline{U}$, i.e., W is the space of locally constant functions ϕ from $\overline{G}$ to $\mathbf{C}$ satisfying $\phi(ug) = \tau(u)\phi(g)$, for every $u \in \overline{U}$ and $g \in \overline{G}$, and the action is given by

$$(2.5) \quad [\rho(g)f](g') = f(g'g) \quad (g, g' \in \overline{G}; \quad f \in W).$$

We write

$$(2.6) \quad \rho = \text{Ind}_{\overline{U}}^{\overline{G}} \tau.$$

The representation ρ depends only on the class of t in T modulo $\overline{TZ}(G)$ and $\rho(g) = \rho_{\tau_0}(t^{-1}gt)$, g in $\overline{G}$, where $\rho_{\tau_0} = \text{Ind}_U^G \tau_0$, τ_0 is the “generic character” of U under the action by conjugation of T .

Definition 2.7. Let $(\overline{\Pi}, V)$ be a supercuspidal representation of $\overline{G}$. A subspace M of W is called a *Whittaker model for $(\overline{\Pi}, V)$ with respect to τ* if M is invariant under ρ and the restriction of ρ to M is equivalent to $\overline{\Pi}$.

A Whittaker model associated to a generic character τ of $\overline{U}$ can be seen as an intertwining operator from $\overline{\Pi}$ to ρ . Applying a result of F. Rodier [R] we have that

$$(2.8) \quad \text{Hom}_{\overline{G}}(\overline{\Pi}, \rho) \simeq \mathbf{C}.$$

We will have a Whittaker model for $\overline{G} = SL(2, F)$ for each $\rho = \rho(t + \overline{TZ}(G))$ for $t + \overline{TZ}(G)$ running in $T/\overline{TZ}(G)$.

3. MAIN LEMMAS

Let b be an element in $A = M_2(F)$ with irreducible characteristic polynomial $p(x) = a_0 + a_1x + x^2$ over F . In what follows, we pick b as the companion matrix of the monic polynomial $p(x) \in F[x]$, i.e.,

$$(3.1) \quad b = \begin{bmatrix} 0 & -a_0 \\ 1 & -a_1 \end{bmatrix}.$$

So the ring $E = F[b]$, which is a subring of A , is a field extension of F of degree 2. From now on, we assume that $\nu_E(b) = -n$ and that b is E/F -minimal.

Set $\mathbf{A} = \text{End}_F(E)$ and $\mathbf{G} = \mathbf{A}^\times$. We denote by $\mathbf{Q}$ the stabilizer subgroup of 1_E in $\mathbf{G}$. Then, as in [J], we have

Lemma 3.2. *With notations as above, the following statements hold:*

- (1) $E^\times \cap \mathbf{Q} = \{1\}$.
- (2) $\mathbf{G} = E^\times \mathbf{Q}$.
- (3) Let $\mathcal{A}_o$ be the principal order in $\mathbf{A}$ associated to the O_F -lattice chain $\mathcal{L}_o = \{P_E^i \mid i \in \mathbf{Z}\}$ in E and $\mathcal{U}_o^n = 1 + \mathcal{P}_o^n$, for any $n \geq 1$, with $\mathcal{P}_o = \text{rad}(\mathcal{A}_o)$. Then: $\mathcal{U}_o^n = (E^\times \cap \mathcal{U}_o^n)(\mathbf{Q} \cap \mathcal{U}_o^n)$.
- (4) $E^\times \mathcal{U}_o^n = E^\times (\mathbf{Q} \cap \mathcal{U}_o^n)$ with $E^\times \cap (\mathbf{Q} \cap \mathcal{U}_o^n) = \{1\}$.

Let $\tilde{\mathcal{B}} = \{1, b\}$, which is a basis of E as an F -vector space. If $\varphi : E \longrightarrow V$ is the coordinate map, then φ induces an algebra isomorphism $\Phi : \text{End}_F(E) \longrightarrow A$, defined by $\Phi(T) = \varphi T \varphi^{-1}$, for every $T \in \text{End}_F(E)$.

Proposition 3.3. *With $\mathcal{A}_o$ as above, put $\mathcal{A} = \Phi(\mathcal{A}_o)$ and $\mathcal{L} = \varphi(\mathcal{L}_o)$. Then:*

- (1) $\mathcal{L}$ is an O_F -lattice chain in V and $\mathcal{A}$ is the principal order in A associated to $\mathcal{L}$.
- (2) $E^\times \subset \mathcal{K}(A)$, where $\mathcal{K}(A)$ is defined as in (1.2).

Since b is E/F -minimal we have that there exists a basis $\mathcal{B}' = \{1_E, \varpi_F^z b\}$ ($z \in \mathbf{Z}$), for the F -vector space E which is a basis for the uniform O_F -lattice chain $\mathcal{L}_o = \{P_E^i \mid i \in \mathbf{Z}\}$ such that

$$e(E/F) = 1, \begin{cases} O_E = O_F 1_E + O_F \varpi_F^{1-n} b, \\ P_E = P_F 1_E + P_F \varpi_F^{1-n} b, \end{cases}$$

or

$$e(E/F) = 2, \begin{cases} O_E = O_F 1_E + O_F \varpi_F^{\frac{1-n}{2}} b, \\ P_E = O_F 1_E + P_F \varpi_F^{\frac{1-n}{2}} b, \\ P_E^2 = P_F 1_E + P_F \varpi_F^{\frac{1-n}{2}} b. \end{cases}$$

So $\varphi(\mathcal{B}')$ is a basis for the O_F -lattice chain $\mathcal{L} = \varphi(\mathcal{L}_o)$. Since $\varphi(\varpi_F^z b) = \varpi_F^z \varphi(b) = \varpi_F^z e_2$, we have that φ is a monomial matrix and also that $\varphi(1_E) = e_1$.

Lemma 3.4. *Let $\mathcal{A}$ be the principal order defined in Proposition 3.3, $\mathcal{P} = \text{rad}(\mathcal{A})$. Then,*

$$U \cap E^\times \mathcal{U}^n = U \cap \mathcal{U}^n.$$

Proof. See, for example, [J].

4. CONSTRUCTION OF WHITTAKER MODELS

We now construct Whittaker models for the irreducible supercuspidal representations of $\overline{G}$ of level n , for any integer number $n \geq 1$.

Proposition 4.1. *Let $\mathcal{A}$ be the principal order in A defined in Proposition 3.3 and let $\mathcal{P}$ be its Jacobson radical. Assume that $n > r \geq [\frac{n}{2}]$ and that ψ_b is the character of $\mathcal{U}^{r+1}/\mathcal{U}^{n+1}$ with b as in (3.1) and τ is the character of $\overline{U}$ defined as in (2.1) with $\tau_F = \psi$. Then*

$$\psi_{b|\mathcal{U}^{r+1} \cap \overline{U}}^z = \tau_{|\mathcal{U}^{r+1} \cap \overline{U}},$$

where $z \in G/\overline{GH}$.

Proof. We have two possibilities.

The first one is $H = F^\times GL_2(O_F)$ and so $\det : G \longrightarrow F^\times / F^{\times 2} U_F$ is a group epimorphism with kernel $\overline{GH}$. Picking $z_0 = \begin{pmatrix} 1 & 0 \\ 0 & \varpi_F \end{pmatrix}$ we have $\det z_0 = \varpi_F$ and $\varpi_F \notin F^{\times 2} U_F$, therefore $G/\overline{GH} = \{\overline{Id}, \overline{z_0}\}$. Now, we have that $z_0 u z_0^{-1} \in U$ for every $u \in U$ and a computation show that

$$\psi_{b|\mathcal{U}^{r+1} \cap \overline{U}}^z = \tau_{|\mathcal{U}^{r+1} \cap \overline{U}}, \quad \text{for } z \in G/\overline{GH}.$$

The other one is $H = F[b]^\times \mathcal{U}^{[\frac{n+1}{2}]}$ and then $\det : G \longrightarrow F^\times / N_{E/F}(E)$, with $E = F[b]$ a group epimorphism with kernel $\overline{GH}$. We know (see [Se]) there exists u in $F^\times$ such that $u \notin N_{E/F}(E)$ and so $z_0 = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$ is not an element of $N_{E/F}(E)$. Therefore $G/\overline{GH} = \{\overline{Id}, \overline{z_0}\}$ and similarly as above we get the same equality.

Proposition 4.2. *Let $\overline{\Pi}_z^\lambda$ be an irreducible supercuspidal representation of $\overline{G}$ as in (1.5). Let τ be the generic character of $\overline{U}$ defined as in (2.1) with $\tau_F = \psi$. Then*

$$\text{Hom}_{\overline{U} \cap \mathcal{U}^{[\frac{n+1}{2}]}}(\lambda^z, \tau) \neq 0.$$

Proof. First of all we pick the principal order $\mathcal{A}$ defined in Proposition 3.3, b as in (3.1) and $z \in G/\overline{GH} = \{\overline{Id}, \overline{z_0}\}$ being, in each case, $\overline{z_0}$ as in the above proposition.

- (i) If n is odd, then $\lambda_{|\mathcal{U}^{\frac{n+1}{2}}}^z$ is ψ_b^z , and from Proposition 4.1 we have $\psi_{b|\overline{U} \cap \mathcal{U}^{\frac{n+1}{2}}}^z = \tau_{|\overline{U} \cap \mathcal{U}^{\frac{n+1}{2}}}$ in both cases.

(ii) If n is even, we can choose H containing $\overline{U} \cap \mathcal{U}^{\frac{n}{2}}$ because

$$\left[(\overline{U} \cap \mathcal{U}^{\frac{n}{2}}) - 1/(P_{F[b]}^{\frac{n}{2}} + \mathcal{P}^{\frac{n}{2}+1}) \right]$$

is an isotropic subspace of $\mathcal{P}^{\frac{n}{2}}/(P_{F[b]}^{\frac{n}{2}} + \mathcal{P}^{\frac{n}{2}+1})$ with respect to the O_F/P_F -nondegenerate alternating form $\langle x, y \rangle = \text{tr}(b(xy - yx))$. So, $\psi_b^z = \tau$ on $\mathcal{U}^{\frac{n}{2}+1} \cap (\overline{U} \cap \mathcal{U}^{\frac{n}{2}})$ in both cases, and we can extend ψ_b^z from $\mathcal{U}^{\frac{n}{2}+1}$ to $\psi_b^z \tau$ on $\mathcal{U}^{\frac{n}{2}+1}(\overline{U} \cap \mathcal{U}^{\frac{n}{2}})$. Now we can take χ on $U_{F[b]}^1 H$ as an extension (from $\mathcal{U}^{\frac{n+1}{2}}(\overline{U} \cap \mathcal{U}^{\frac{n}{2}})$) of $\psi_b^z \tau$. Hence $\chi|_{\overline{U} \cap \mathcal{U}^{\frac{n}{2}}} = \tau|_{\overline{U} \cap \mathcal{U}^{\frac{n}{2}}}$, and since χ is in $\lambda_{U_{F[b]}^1 H}^z$ we have that $\text{Hom}_{\overline{U} \cap \mathcal{U}^{\frac{n}{2}}}(\lambda^z, \tau) \neq 0$.

Proposition 4.3. *Let $\overline{\Pi}_z^\lambda$ be an irreducible supercuspidal representation of $\overline{G}$ as above. Let τ be the generic character of $\overline{U}$ defined as in (2.1) with $\tau_F = \psi$. Let ρ be the smooth representation given in (2.5). Then*

$$\text{Hom}_{\overline{G}}(\overline{\Pi}_z^\lambda, \rho) \simeq \text{Hom}_{\overline{U} \cap F[b] \times \mathcal{U}[\frac{n+1}{2}]}(\lambda^z, \tau)$$

with λ^z as in the above proposition.

Proof. From (1.5) and (2.6) we have that

$$\text{Hom}_{\overline{G}}(\overline{\Pi}_z^\lambda, \rho) \simeq \text{Hom}_{\overline{G}}(\text{Ind}_z^{\overline{G}} \left[\overline{G} \cap F[b] \times \mathcal{U}[\frac{n+1}{2}] \right]_{z^{-1}} \lambda^z, \text{Ind}_{\overline{U}}^{\overline{G}} \tau).$$

On the other hand, by Mackey's theorem (see [K1]) we have that

$$\text{Hom}_{\overline{G}}(\text{Ind}_z^{\overline{G}} \left[\overline{G} \cap F[b] \times \mathcal{U}[\frac{n+1}{2}] \right]_{z^{-1}} \lambda^z, \text{Ind}_{\overline{U}}^{\overline{G}} \tau)$$

is isomorphic to

$$\prod_{x \in \overline{U} \backslash \overline{G}/z \left[\overline{G} \cap F[b] \times \mathcal{U}[\frac{n+1}{2}] \right]_{z^{-1}}} \text{Hom}_{x^{-1} \overline{U} x \cap z \left[\overline{G} \cap F[b] \times \mathcal{U}[\frac{n+1}{2}] \right]_{z^{-1}}}(\lambda^z, \tau^x).$$

But since $\text{Hom}_{\overline{G}}(\overline{\Pi}_z^\lambda, \rho) \simeq \mathbf{C}$ (see (2.8)), $\overline{U} \cap F[b] \times \mathcal{U}[\frac{n+1}{2}] = \overline{U} \cap \mathcal{U}[\frac{n+1}{2}]$ (Lemma 3.4) and $\text{Hom}_{\overline{U} \cap \mathcal{U}[\frac{n+1}{2}]}(\lambda^z, \tau) \neq 0$, the result follows.

We can now describe Whittaker models for the supercuspidal representations of $\overline{G}$.

Theorem 4.4. *Let $\overline{\Pi}_z^\lambda$ be an irreducible supercuspidal representation of $\overline{G}$ and λ^z the representation of $z \left[\overline{G} \cap F[b] \times \mathcal{U}[\frac{n+1}{2}] \right]_{z^{-1}}$ as above. Let τ be the generic character of $\overline{U}$ such that τ is a constituent of $\lambda_{U \cap \mathcal{U}[\frac{n+1}{2}]}^z$. Then, the Whittaker vector of $\overline{\Pi}_z^\lambda$ for an element f in $\text{Ind}_z^{\overline{G}} \left[\overline{G} \cap F[b] \times \mathcal{U}[\frac{n+1}{2}] \right]_{z^{-1}} \lambda^z$ is given by*

$$[A_{\overline{\Pi}_z^\lambda}(f)](x) = \sum_{w \in z \left[\overline{G} \cap F[b] \times \mathcal{U}[\frac{n+1}{2}] \right]_{z^{-1}} \backslash \overline{G}} S_{\lambda^z}(xw^{-1})f(w) \quad (x \in \overline{G}),$$

where

$$S_{\lambda^z}(x) = \begin{cases} \tau(u)\lambda^z(h), & \text{if } x = uh \in \overline{U}z \left[\overline{G} \cap F[b]^\times \mathcal{U}^{\left[\frac{n+1}{2}\right]} \right] z^{-1}, \\ 0, & \text{if } x \notin \overline{U}z \left[\overline{G} \cap F[b]^\times \mathcal{U}^{\left[\frac{n+1}{2}\right]} \right] z^{-1}. \end{cases}$$

Proof. The result follows directly from the results above using the same arguments as in [K1].

ACKNOWLEDGMENTS

We thank Anne-Marie Aubert for encouraging us to write this work and for her always valuable comments.

REFERENCES

- [B] C.J. Bushnell, *Hereditary orders, Gauss sums and supercuspidal representations of $GL_N(F)$* , J. Fur Reine und Angew. Math. **375/376**, (1987), 184-210. MR0882297 (88e:22024)
- [B-F] C.J. Bushnell and A. Fröhlich, *Non abelian congruence, Gauss sums and p -adic simple algebras*, Proc. London Math. Soc. **50**, (1985), 207-264. MR0772712 (86g:11071)
- [B-K] C.J. Bushnell and P. C. Kutzko, *The admissible dual of $GL(N)$ via compact open subgroups*, Annals of Mathematics Studies, 1993, Princeton University Press. MR1204652 (94h:22007)
- [Cr] P. Cartier, *Representations of p -adic groups: a survey*, Proc. of Symposia in Pure Math., Amer. Math. Soc. **23**, (1979), 111-155. MR0546593 (81e:22029)
- [Cs] W. Casselman, *Introduction to the theory of admissible representations of p -adic reductive groups*, Preprint.
- [J] R. Johnson, *Some Whittaker Models for $GL_N(F)$* , Geometry and representation theory of real and p -adic groups, Progress in Mathematics, v. 158, Birkhäuser, 1998. MR1486140 (98k:22071)
- [K1] P. C. Kutzko, *Mackey's theorem for nonunitary representations*, Proc. Amer. Math. Soc. **64**, (1977), 173-175. MR0442145 (56:533)
- [K2] P. C. Kutzko, *On the supercuspidal representations of GL_2 , I, II*, Amer. J. Math. **100**, (1978), 43-60, 705-716. MR0507253 (58:22411a), MR0507254 (58:22411b)
- [K-P] P. C. Kutzko and J. Pantoja, *The restriction to SL_2 of a supercuspidal representation of GL_2* , Compositio Mathematica **79**, (1991), 139-155. MR1117337 (92d:22027)
- [K-S] P. C. Kutzko and P. Sally, *All supercuspidal representations of SL_1 over a p -adic field are induced*, Representation Theory of Reductive Groups, Proc. Univ. of Utah Conf. 1982, Birkhäuser. MR733814 (85i:22029)
- [R] F. Rodier, *Whittaker models for admissible representations of reductive p -adic split groups*, Proc. of Symposia in Pure Math., Amer. Math. Soc. **26**, (1973), 425-430. MR0354942 (50:7419)
- [Se] J. P. Serre, *Corps Locaux*, Hermann, Paris (1962). MR0150130 (27:133)

INSTITUTO DE MATEMÁTICAS, PONTIFICIA UNIVERSIDAD CATÓLICA DE VALPARAÍSO, CASILLA 4059, VALPARAÍSO, CHILE
E-mail address: rjohnson@ucv.cl

INSTITUTO DE MATEMÁTICAS, PONTIFICIA UNIVERSIDAD CATÓLICA DE VALPARAÍSO, CASILLA 4059, VALPARAÍSO, CHILE
E-mail address: jpantoja@ucv.cl