GLOBALIZATIONS OF PARTIAL ACTIONS ON NONUNITAL RINGS

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ABSTRACT. In this note we prove a criteria for the existence of a globalization for a given partial action of a group on an s -unital ring. If the globalization exists, it is unique in a natural sense. This extends the globalization theorem from Dokuchaev and Exel, 2005, obtained in the context of rings with 1.

1. INTRODUCTION

Partial actions of groups appeared in a systematic way in the theory of operator algebras in a process of a generalization of C^* -crossed products, and the main steps were made in [8], [16] and [9]. Since then important classes of C^* -algebras have been described as crossed products by partial actions, including the case of Cuntz-Krieger algebras [5], deeply investigated in [11] and [12] from the point of view of partial representations, a related concept which also appeared in the theory of operator algebras. The algebraic study of partial actions of groups on abstract algebras and the corresponding crossed products was begun in [6]. It was shown, in particular, that an essential part of elementary gradings on matrix algebras come from generalized crossed product structures. Further algebraic results involving partial actions on rings were obtained in [4], [7] and [13].

It seems that when a partial action on some structure is given, one of the most relevant problems is the question of the existence and uniqueness of a globalization, also called an enveloping action, i.e. of a global action, whose restriction to the original object gives the initial partial action. The study of this problem was initiated independently in [1] and [15]. Inspired by [15], further results on globalizations were obtained in [18]. In [6] a criteria of the existence of a globalization for a partial action on a unital ring was given. However, important examples of partial actions on rings occur in the nonunital context. For instance, by Gelfand's Theorem, any commutative C^* -algebra is of the form $C_0(X)$, the algebra of the complex-valued continuous functions on a locally compact Hausdorff space X which vanish at infinity. It is shown in [2] that the partial actions of a locally compact Hausdorff group G on X naturally correspond to the partial actions of G on $C_0(X)$. However, if X is not compact, then $C_0(X)$ is nonunital. Thus it is rather desirable to examine large classes of nonunital rings with respect to the globalization problem.

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If a partial action on a unital ring has a globalization, then it is unique in a canonical way [6], and the ring on which this globalization is defined is s -unital but not necessarily unital. Observe that in the theory of operator algebras there are highly important examples of s -unital rings. One is given by the algebra $C_c(X)$ of the complex-valued continuous functions on X with compact supports, where X is a noncompact locally compact Hausdorff space. Note that $C_c(X)$ has no idempotents, and it is dense in $C_0(X)$. Another example is the algebra of the finite-rank operators on a Hilbert space H . It is dense in the C^* -algebra of the compact operators on H . Thus it is natural to treat our problem in the s -unital context. This is the purpose of this paper in which we obtain a criteria to decide if a partial action on an s -unital ring has a globalization. We also prove that when a globalization of a partial action on an s -unital ring exists, it is unique and, moreover, the partial crossed product and the global one are Morita equivalent.

2. PRELIMINARIES

The first formal definition of a partial action was given by R. Exel in [10]. Thus while a global action of a group G on a set $\mathcal{X}$ is a collection of bijections $\mathcal{X} \rightarrow \mathcal{X}$ which agree with the group structure, a partial action of G on $\mathcal{X}$ is a family of bijections $\alpha_g : \mathcal{X}_{g^{-1}} \rightarrow \mathcal{X}_g$ between subsets of $\mathcal{X}$ such that $\alpha_g \circ \alpha_h$ is a restriction of α_{gh} . Usually one also requires that the identity element of G acts as the trivial bijection $\mathcal{X} \rightarrow \mathcal{X}$. When $\mathcal{X}$ has some additional structure, one has to specify this definition; we do it below for rings. By a ring we shall mean an associative non-necessarily unital ring.

Definition 2.1. Let G be a group with identity element 1 and let $\mathcal{A}$ be a ring. A partial action α of G on $\mathcal{A}$ is a collection of (two-sided) ideals $\mathcal{A}_g \subseteq \mathcal{A}$ ($g \in G$) and ring isomorphisms $\alpha_g : \mathcal{A}_{g^{-1}} \rightarrow \mathcal{A}_g$ such that

- (i) $\mathcal{A}_1 = \mathcal{A}$ and α_1 is the identity map of $\mathcal{A}$;
- (ii) $\mathcal{A}_{(gh)^{-1}} \supseteq \alpha_h^{-1}(\mathcal{A}_h \cap \mathcal{A}_{g^{-1}})$;
- (iii) $\alpha_g \circ \alpha_h(x) = \alpha_{gh}(x)$ for each $x \in \alpha_h^{-1}(\mathcal{A}_h \cap \mathcal{A}_{g^{-1}})$.

Conditions (ii) and (iii) mean that the function α_{gh} is an extension of the function $\alpha_g \circ \alpha_h$. Moreover, it is easily seen that (ii) can be replaced by a “stronger looking” condition (see [6]):

- (ii') $\alpha_g(\mathcal{A}_{g^{-1}} \cap \mathcal{A}_h) = \mathcal{A}_g \cap \mathcal{A}_{gh}$, for all $g, h \in G$.

Examples of partial actions can be obtained by restricting (global) actions to non-necessarily invariant (two-sided) ideals. Indeed, suppose that a group G acts on a ring $\mathcal{B}$ by automorphisms $\beta_g : \mathcal{B} \rightarrow \mathcal{B}$ and let $\mathcal{A}$ be an ideal of $\mathcal{B}$. Set $\mathcal{A}_g = \mathcal{A} \cap \beta_g(\mathcal{A})$ and let α_g be the restriction of β_g to $\mathcal{A}_{g^{-1}}$. It is easily verified that we have a partial action $\alpha = \{\alpha_g : \mathcal{A}_{g^{-1}} \rightarrow \mathcal{A}_g, g \in G\}$ of G on $\mathcal{A}$. One of the general problems is to determine the conditions under which a given partial action can be obtained as the restriction of a (global) action, and analyse the uniqueness of such a globalization up to isomorphism. This problem was solved in [6, Theorem 4.5] for partial actions on unital rings. Next we recall the precise meaning of a globalization.

Definition 2.2. An action β of a group G on a ring $\mathcal{B}$ is said to be a globalization (or an enveloping action) for the partial action α of G on a ring $\mathcal{A}$ if there exists a

monomorphism $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ such that:

- (i) $\varphi(\mathcal{A})$ is an ideal in $\mathcal{B}$,
- (ii) $\mathcal{B} = \sum_{g \in G} \beta_g(\varphi(\mathcal{A}))$,
- (iii) $\varphi(\mathcal{A}_g) = \varphi(\mathcal{A}) \cap \beta_g(\varphi(\mathcal{A}))$,
- (iv) $\varphi \circ \alpha_g = \beta_g \circ \varphi$ on $\mathcal{A}_{g^{-1}}$.

Definition 2.3. We say that the globalization $(\beta, \mathcal{B})$ of a partial action $(\alpha, \mathcal{A})$ is unique if for any other globalization $(\beta', \mathcal{B}')$ of $(\alpha, \mathcal{A})$ there exists an isomorphism of rings $\phi : \mathcal{B} \rightarrow \mathcal{B}'$ such that

$$\beta'_g \circ \phi = \phi \circ \beta_g,$$

for every $g \in G$.

Given a partial action α of a group G on a ring $\mathcal{A}$, the skew group ring $\mathcal{A} *_{\alpha} G$ corresponding to α is the set of all finite formal sums $\{\sum_{g \in G} a_g u_g : a_g \in \mathcal{A}_g\}$, where u_g are symbols. The addition is defined in the obvious way, and the multiplication is determined by $(a_g u_g) \cdot (b_h u_h) = \alpha_g(\alpha_{g^{-1}}(a_g) b_h) u_{gh}$. Obviously, $\mathcal{A} \ni a \mapsto au_1 \in \mathcal{A} *_{\alpha} G$ is an embedding which permits us to identify $\mathcal{A}$ with $\mathcal{A}u_1$. The first question which naturally arises is whether or not $\mathcal{A} *_{\alpha} G$ is associative. It is shown in [6] that this is not always the case. However, if $\mathcal{A}$ is semiprime, $\mathcal{A} *_{\alpha} G$ is necessarily associative (see [6, Corollary 3.4]).

If $(\alpha, \mathcal{A})$ admits a globalization $(\beta, \mathcal{B})$, then the skew group ring $\mathcal{A} *_{\alpha} G$ has an embedding into $\mathcal{B} *_{\beta} G$. In particular, $\mathcal{A} *_{\alpha} G$ is associative.

We shall need some easy, but important for our purpose, results on s -unital rings. We recall that a left s -unital ring, by definition, is an associative ring $\mathcal{A}$ such that for any $x \in \mathcal{A}$ one has $x \in \mathcal{A}x$. We say that $e \in \mathcal{A}$ is a left unit for $x \in \mathcal{A}$ if $ex = x$. Of course every unital ring is s -unital. Other examples of s -unital rings are rings with local units (see e.g. [19]). It is interesting to observe that any C^* -algebra $\mathcal{A}$ is s -unital in an approximate sense: for each $a \in \mathcal{A}$ and arbitrary $\varepsilon > 0$ there exists $e \in \mathcal{A}$ with $\|a - ea\| < \varepsilon$ (see, for example, [17, Theorem 3.1.1]).

Lemma 2.4. Suppose that the ideals $\mathcal{I}$ and $\mathcal{J}$ of a ring $\mathcal{A}$ are left s -unital rings. Then $\mathcal{I} + \mathcal{J}$ and $\mathcal{I} \cap \mathcal{J}$ are also left s -unital and, moreover, $\mathcal{I} \cap \mathcal{J} = \mathcal{I}\mathcal{J}$. Furthermore, for arbitrary $a, b \in \mathcal{I}$ there exists $e \in \mathcal{I}$ such that $ea = a$ and $eb = b$.

Proof. Take $a \in \mathcal{I}, b \in \mathcal{J}$, and suppose that $x \in \mathcal{I}$ is a left unit for a and $y \in \mathcal{J}$ is a left unit for $xb - b \in \mathcal{J}$. Then it is easily seen that $x + y - yx$ is a left unit for $a + b$. Now take $a \in \mathcal{I} \cap \mathcal{J}$ and let $x \in \mathcal{I}, y \in \mathcal{J}$ be left units for a . Then obviously $z = xy \in \mathcal{I} \cap \mathcal{J}$ is a left unit for a . Moreover, $\mathcal{I} \cap \mathcal{J} \ni a = za \in \mathcal{I}\mathcal{J}$ implies $\mathcal{I} \cap \mathcal{J} \subseteq \mathcal{I}\mathcal{J}$. Finally, for arbitrary $a, b \in \mathcal{I}$ let $x, y \in \mathcal{I}$ be such that $xa = a$ and $y(xb - b) = xb - b$. Then it is easy to see that taking $e = x + y - yx$ one has $ea = a$ and $eb = b$. $\square$

Remark 2.5. It follows by Lemma 2.4 that a (nonnecessarily finite) sum of left s -unital ideals in a ring $\mathcal{A}$ is left s -unital. For it is enough to observe that any element belongs to a finite sum of left s -unital ideals. In particular, if $(\beta, \mathcal{B})$ is a globalization of a partial action of an s -unital ring $\mathcal{A}$, then $\mathcal{B}$ is left s -unital, too.

For homomorphisms of left $\mathcal{A}$ -modules we shall use the right-hand side notation, i.e. we write $x \mapsto x\gamma$ for $\gamma : {}_{\mathcal{A}}M \rightarrow {}_{\mathcal{A}}N$, while for homomorphisms of right $\mathcal{A}$ -modules $\gamma : M_{\mathcal{A}} \rightarrow N_{\mathcal{A}}$, we use the usual notation: $x \mapsto \gamma x$. Accordingly,

composition of left module homomorphisms is read from left to right, i.e. $x(\gamma_1\gamma_2) = (x\gamma_1)\gamma_2$, while composition of right module homomorphisms is read in the usual right to left way. We recall that given a ring $\mathcal{A}$, the multiplier ring $M(\mathcal{A})$ of $\mathcal{A}$ is the set

$$M(\mathcal{A}) = \{(R, L) \in \text{End}({}_{\mathcal{A}}\mathcal{A}) \times \text{End}(\mathcal{A}_{\mathcal{A}}) : (aR)b = a(Lb) \text{ for all } a, b \in \mathcal{A}\}$$

with component-wise addition and multiplication (for more details see [3] or [6]). For a multiplier $\gamma = (R, L) \in M(\mathcal{A})$ and $a \in \mathcal{A}$ we set $a\gamma = aR$ and $\gamma a = La$. Thus one always has $(a\gamma)b = a(\gamma b)$ ($a, b \in \mathcal{A}$). An element a in a ring $\mathcal{A}$ obviously determines the multiplier $(R_a, L_a) \in M(\mathcal{A})$, where $xR_a = xa$ and $L_ax = ax$ ($x \in \mathcal{A}$). If $\mathcal{I}$ is an ideal in $\mathcal{A}$, then this multiplier evidently restricts to one of $\mathcal{I}$ which shall be denoted by the same pair of symbols (R_a, L_a) . The first (resp. second) components of the elements of $M(\mathcal{A})$ are called right (resp. left) multipliers of $\mathcal{A}$.

Lemma 2.6. *Let $\mathcal{A}$ be a ring.*

- (i) *Suppose that $\mathcal{I}$ is an ideal in $\mathcal{A}$ and $\mathcal{I}$ is left s -unital. If $\phi, \psi : \mathcal{A} \rightarrow \mathcal{I}$ are left $\mathcal{A}$ -module homomorphisms such that ϕ and ψ coincide on $\mathcal{I}$, then $\phi = \psi$.*
- (ii) *Suppose that $\mathcal{A}$ is left s -unital. If $\gamma, \gamma' \in M(\mathcal{A})$ and $a\gamma = a\gamma'$ for all $a \in \mathcal{A}$, then $\gamma = \gamma'$.*

Proof. (i) Given an arbitrary $x \in \mathcal{A}$ let $e \in \mathcal{I}$ be a left unit for $\phi(x) - \psi(x)$, that is, $e(\phi(x) - \psi(x)) = \phi(x) - \psi(x)$. Because ϕ and ψ are left $\mathcal{A}$ -module maps, we have $\phi(x) - \psi(x) = e(\phi(x) - \psi(x)) = \phi(ex) - \psi(ex) = 0$.

(ii) For arbitrary $a, b \in \mathcal{A}$ we have $a(\gamma'b) = (a\gamma')b = (a\gamma)b = a(\gamma b)$, so that $\mathcal{A}(\gamma'b - \gamma b) = 0$, and because $\mathcal{A}$ is left s -unital, it follows that $\gamma b = \gamma'b$. $\square$

3. THE GLOBALIZATION THEOREM

The next result extends the globalization theorem obtained in [6] for partial actions on rings with unity.

Theorem 3.1. *Let α be a partial action of a group G on a left s -unital ring $\mathcal{A}$. Then α admits a globalization if and only if the following two conditions are satisfied:*

- (i) *$\mathcal{A}_g$ is a left s -unital ring for every $g \in G$.*
- (ii) *For each $g \in G$ and $a \in \mathcal{A}$ there exists a multiplier $\gamma_g(a) \in M(\mathcal{A})$ such that $\mathcal{A}\gamma_g(a) \subseteq \mathcal{A}_g$ and $\gamma_g(a)$, restricted to $\mathcal{A}_g$ as a right multiplier, is $\alpha_{g^{-1}}R_a\alpha_g$.*

Moreover, if a globalization exists, it is unique (in the sense of Definition 2.3), and the ring under the global action is left s -unital.

Proof. The “only if” part. Suppose that an action β of G on a ring $\mathcal{B}$ is a globalization of α . Then $\mathcal{A}$ can be considered as an ideal in $\mathcal{B}$ and $\mathcal{A}_g = \mathcal{A} \cap \beta_g(\mathcal{A})$. Since the intersection of two left s -unital ideals is left s -unital, (i) follows. We see that $\alpha_{g^{-1}}R_a\alpha_g$ maps $x \in \mathcal{A}_g$ to $\alpha_g(\alpha_{g^{-1}}(x)a) = x\beta_g(a)$, and for any $y \in \mathcal{A}$ one has $y\beta_g(a) \in \mathcal{A} \cap \beta_g(\mathcal{A}) = \mathcal{A}_g$, so that we may take $\gamma_g(a) = (R_{\beta_g(a)}, L_{\beta_g(a)}) \in M(\mathcal{A})$.

The “if” part. Let $\mathcal{F} = \mathcal{F}(G, M(\mathcal{A}))$ be the ring of all functions from G to $M(\mathcal{A})$, that is, the Cartesian product of the copies of $M(\mathcal{A})$ indexed by the elements of G . For convenience the element $f(g)$ will also be denoted by $f|_g$ ($f \in \mathcal{F}, g \in G$).

We define a (global) action β of G on $\mathcal{F}$ by the formula

$$\beta_g(f)|_h = f|_{g^{-1}h} \quad (g, h \in G, f \in \mathcal{F}).$$

For $g \in G$ and $a \in \mathcal{A}$ the element $\gamma_g(a) \in M(\mathcal{A})$ given by (ii) is uniquely determined in view of Lemma 2.6. Hence we can define a map $\varphi : \mathcal{A} \rightarrow \mathcal{F}$ by

setting

$$\varphi(a)|_g = \gamma_{g^{-1}}(a), \quad g \in G.$$

Because $\mathcal{A}$ is left s -unital, φ is injective. Taking $a, b \in \mathcal{A}$ and $x \in \mathcal{A}_g$ one has

$$(x\gamma_g(a))\gamma_g(b) = ((x)\alpha_{g^{-1}}R_a\alpha_g)\alpha_{g^{-1}}R_b\alpha_g = (x)\alpha_{g^{-1}}R_{ab}\alpha_g = x\gamma_g(ab).$$

Thus by (i) of Lemma 2.6, $\gamma_g(a)\gamma_g(b)$ and $\gamma_g(ab)$ coincide as right multipliers and (ii) of Lemma 2.6 implies $\gamma_g(a)\gamma_g(b) = \gamma_g(ab)$. It follows that φ is multiplicative, and a similar argument shows that it is additive. We conclude that φ is a ring monomorphism.

Let $\mathcal{B} = \sum_{g \in G} \beta_g(\varphi(\mathcal{A}))$. We want to show that the restriction of β to $\mathcal{B}$ is a globalization for α . We denote this restriction by the same symbol β . We need to check that the axioms (i)-(iv) of Definition 2.2 hold. Axiom (ii) is clear.

We first check axiom (iv), that is,

$$(1) \quad \beta_g(\varphi(a)) = \varphi(\alpha_g(a)), \quad \text{for any } g \in G \text{ and } a \in \mathcal{A}_{g^{-1}}.$$

Let $h \in G$, $\phi = \beta_g(\varphi(a))|_h = \varphi(a)|_{g^{-1}h} = \gamma_{h^{-1}g}(a)$ and $\psi = \varphi(\alpha_g(a))|_h = \gamma_{h^{-1}}(\alpha_g(a))$ be seen as right multipliers of $\mathcal{A}$. We have to show that $\phi = \psi$. By Lemma 2.6 it is enough to show that $\mathcal{A}\phi, \mathcal{A}\psi \subseteq \mathcal{I}$ and ϕ and ψ coincide on $\mathcal{I}$, where $\mathcal{I} = \mathcal{A}_{h^{-1}} \cap \mathcal{A}_{h^{-1}g}$, an s -unital ideal of $\mathcal{A}$.

Taking $x \in \mathcal{A}$, let $e \in \mathcal{A}_{h^{-1}g}$ be a left unit for $x\gamma_{h^{-1}g}(a)$. Then we see that

$$\begin{aligned} x\phi &= x\gamma_{h^{-1}g}(a) = e(x\gamma_{h^{-1}g}(a)) = (ex) \cdot \gamma_{h^{-1}g}(a) \\ &= \alpha_{h^{-1}g}(\alpha_{g^{-1}h}(ex)a) \in \alpha_{h^{-1}g}(\mathcal{A}_{g^{-1}h} \cap \mathcal{A}_{g^{-1}}) = \mathcal{I}. \end{aligned}$$

Similarly, if $f \in \mathcal{A}_{h^{-1}}$ is a left unit for $x\gamma_{h^{-1}}(\alpha_g(a))$, we have

$$\begin{aligned} x\psi &= x\gamma_{h^{-1}}(\alpha_g(a)) = f(x\gamma_{h^{-1}}(\alpha_g(a))) = (fx) \cdot \gamma_{h^{-1}}(\alpha_g(a)) \\ &= \alpha_{h^{-1}}(\alpha_h(fx)\alpha_g(a)) \in \alpha_{h^{-1}}(\mathcal{A}_h \cap \mathcal{A}_g) = \mathcal{I}. \end{aligned}$$

This shows that $\mathcal{A}\phi, \mathcal{A}\psi \subseteq \mathcal{I}$.

For $x \in \mathcal{I}$ using (iii) of Definition 2.1 one has

$$\begin{aligned} x\phi &= x\gamma_{h^{-1}g}(a) = \alpha_{h^{-1}g}(\alpha_{g^{-1}h}(x)a) = (\alpha_{h^{-1}} \circ \alpha_g)((\alpha_{g^{-1}} \circ \alpha_h)(x)a) \\ &= \alpha_{h^{-1}}(\alpha_h(x)\alpha_g(a)) = x\gamma_{h^{-1}}(\alpha_g(a)) = x\psi. \end{aligned}$$

This finished the proof of (1). For future use we set the following equality which is exactly the equality $\phi = \psi$ that we just proved:

$$(2) \quad \gamma_{h^{-1}g}(a) = \gamma_{h^{-1}}(\alpha_g(a)), \quad \text{for any } g, h \in G, a \in \mathcal{A}_{g^{-1}}.$$

Next we deal with axiom (iii), that is, we show

$$(3) \quad \varphi(\mathcal{A}_g) = \varphi(\mathcal{A}) \cap \beta_g(\varphi(\mathcal{A})),$$

for all $g \in G$. An element from the right-hand side can be written as $\varphi(a) = \beta_g(\varphi(b))$ for some $a, b \in \mathcal{A}$. For $h \in G$ we see that $\varphi(a)|_h = \gamma_{h^{-1}}(a)$ and $\beta_g(\varphi(b))|_h = \varphi(b)|_{g^{-1}h} = \gamma_{h^{-1}g}(b)$. Thus

$$\gamma_{h^{-1}}(a) = \gamma_{h^{-1}g}(b),$$

and taking $h = 1$ we obtain $a = \gamma_g(b) \in \mathcal{A}_g$. This proves the inclusion $\varphi(\mathcal{A}_g) \supseteq \varphi(\mathcal{A}) \cap \beta_g(\varphi(\mathcal{A}))$.

For the converse inclusion take $a \in \mathcal{A}_g$ and set $b = \alpha_{g^{-1}}(a)$. Then

$$\varphi(a)|_h = \varphi(\alpha_g(b))|_h = \gamma_{h^{-1}}(\alpha_g(b)) = \gamma_{h^{-1}g}(b),$$

in view of (2). On the other hand, $\gamma_{h^{-1}g}(b) = \beta_g(\varphi(b))|_h$, by the definition of φ and β . Consequently, $\varphi(a) = \beta_g(\varphi(b))$. This proves that $\varphi(\mathcal{A}_g) \subseteq \varphi(\mathcal{A}) \cap \beta_g(\varphi(\mathcal{A}))$ and (3) follows.

To prove that $(\beta, \mathcal{B})$ is a globalization of $(\alpha, \mathcal{A})$, it only remains to show that $\varphi(\mathcal{A})$ is an ideal in $\mathcal{B}$. For this we need two properties of $\gamma_g(a)$. The first one is

$$(4) \quad \mathcal{A}_h \gamma_g(a) \subseteq \mathcal{A}_g \cap \mathcal{A}_h$$

for any $g \in G, a \in \mathcal{A}$. Indeed, for $x \in \mathcal{A}_h$ let $e \in \mathcal{A}_g$ be a left unit for $x\gamma_g(a)$. Then $x\gamma_g(a) = e(x\gamma_g(a)) = \alpha_g(\alpha_g^{-1}(ex)a) \in \mathcal{A}_g \cap \mathcal{A}_h$. The second property is

$$(5) \quad \mathcal{A} \gamma_g(a) \subseteq \mathcal{A}_g \cap \mathcal{A}_{gh}$$

for all $g \in G$ and $a \in \mathcal{A}_h$. Take $x \in \mathcal{A}$ and let $e \in \mathcal{A}_g$ be a left unit for $x\gamma_g(a)$. Then as above $x\gamma_g(a) = \alpha_g(\alpha_g^{-1}(ex)a)$ which lies in $\mathcal{A}_g \cap \mathcal{A}_{gh}$ as $\alpha_g^{-1}(ex)a \in \mathcal{A}_{g^{-1}} \cap \mathcal{A}_h$.

We first check that $\varphi(\mathcal{A})$ is a right ideal in $\mathcal{B}$. Since $\mathcal{B}$ is the sum of the $\beta_g(\varphi(\mathcal{A}))$'s, it is enough to show that $\varphi(\mathcal{A})\beta_g(\varphi(\mathcal{A})) \subseteq \varphi(\mathcal{A})$ for all $g \in G$. We do so by proving the following formula:

$$(6) \quad \varphi(a)\beta_g(\varphi(b)) = \varphi(a\gamma_g(b))$$

for any $a, b \in \mathcal{A}$ and $g \in G$. Indeed, taking $h \in G$ we set $\phi = \varphi(a)|_h \beta_g(\varphi(b))|_h = \varphi(a)|_h \varphi(b)|_{g^{-1}h} = \gamma_{h^{-1}}(a)\gamma_{h^{-1}g}(b)$ and $\psi = \varphi(a\gamma_g(b))|_h = \gamma_{h^{-1}}(a\gamma_g(b))$. We have to show that $\phi = \psi$. Let $\mathcal{I} = \mathcal{A}_{h^{-1}} \cap \mathcal{A}_{h^{-1}g}$, which is a left s -unital ideal of $\mathcal{A}$. Using Lemma 2.6, it is enough to show that $\mathcal{A}\phi, \mathcal{A}\psi \subseteq \mathcal{I}$, and ϕ and ψ coincides as right multipliers on $\mathcal{I}$. The first is a straightforward consequence of (4) and (5). For the second let $z \in \mathcal{I}$. Then

$$\begin{aligned} z\phi &= z\gamma_{h^{-1}}(a)\gamma_{h^{-1}g}(b) = \alpha_{h^{-1}g}((\alpha_{g^{-1}h} \circ \alpha_{h^{-1}})(\alpha_h(z)a)b) \\ &= (\alpha_{h^{-1}} \circ \alpha_g)((\alpha_{g^{-1}} \circ \alpha_h \circ \alpha_{h^{-1}})(\alpha_h(z)a)b) = \alpha_{h^{-1}}(\alpha_g(\alpha_{g^{-1}}(\alpha_h(z)a)b)) \\ &= \alpha_{h^{-1}}((\alpha_h(z)a) \cdot \gamma_g(b)) = \alpha_{h^{-1}}(\alpha_h(z)(a\gamma_g(b))) = z\gamma_{h^{-1}}(a\gamma_g(b)) = z\psi. \end{aligned}$$

This shows that $\varphi(\mathcal{A})$ is a right ideal in $\mathcal{B}$.

Observe that so far we have not used the fact that $\gamma_g(a)$ is a left multiplier. However we need this to show that $\varphi(\mathcal{A})$ is a left ideal in $\mathcal{B}$. For that we prove the following formula:

$$(7) \quad \beta_g(\varphi(a))\varphi(b) = \varphi(\gamma_g(a) \cdot b)$$

for any $a, b \in \mathcal{A}$ and $g \in G$, in which $\gamma_g(a)$ acts on b as a left multiplier. Let $h \in G$ and set $\phi = \beta_g(\varphi(a))|_h \varphi(b)|_h = \gamma_{h^{-1}g}(a)\gamma_{h^{-1}}(b)$ and $\psi = \varphi(\gamma_g(a)b)|_h = \gamma_{h^{-1}}(\gamma_g(a)b)$. Again we have to prove $\phi = \psi$, and by (4) and (5) both ϕ and ψ map $\mathcal{A}$ to $\mathcal{I} = \mathcal{A}_{h^{-1}} \cap \mathcal{A}_{h^{-1}g}$, as right multipliers. Thus in view of Lemma 2.6 it is enough to see that they coincide on $\mathcal{I}$. Taking arbitrary $z \in \mathcal{I}$ we have

$$\begin{aligned} z\phi &= z\gamma_{h^{-1}g}(a)\gamma_{h^{-1}}(b) = \alpha_{h^{-1}}((\alpha_h \circ \alpha_{h^{-1}g})(\alpha_{g^{-1}h}(z)a)b) \\ &= \alpha_{h^{-1}}((\alpha_h \circ \alpha_{h^{-1}} \circ \alpha_g)((\alpha_{g^{-1}} \circ \alpha_h)(z)a)b) = \alpha_{h^{-1}}(\alpha_g((\alpha_{g^{-1}} \circ \alpha_h)(z)a)b) \\ &= \alpha_{h^{-1}}((\alpha_h(z) \cdot \gamma_g(a))b) = \alpha_{h^{-1}}(\alpha_h(z)(\gamma_g(a) \cdot b)) = z\gamma_{h^{-1}}(\gamma_g(a)b) = z\psi. \end{aligned}$$

The above shows that $(\alpha, \mathcal{A})$ has a globalization if and only if conditions (i) and (ii) hold. That this globalization is left s -unital was already pointed out in Remark 2.5. Now we prove the uniqueness of the globalization. Suppose that $(\beta, \mathcal{B})$ and $(\beta', \mathcal{B}')$ are two globalizations of $(\alpha, \mathcal{A})$, and let $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ and $\varphi' : \mathcal{A} \rightarrow \mathcal{B}'$ be the corresponding embeddings. It follows from the definition of a globalization that $\mathcal{B}'$ is the sum of the ideals $\beta'_g(\varphi'(\mathcal{A}))$, $g \in G$. Thus an element of $\mathcal{B}'$ can

be written as a finite sum $\sum_i \beta'_{g_i}(\varphi'(a_i))$ with $g_i \in G$ and $a_i \in \mathcal{A}$. We want to show that the map $\phi : \mathcal{B}' \rightarrow \mathcal{B}$ given by $\beta'_g(\varphi'(a)) \mapsto \beta_g(\varphi(a))$, $g \in G, a \in \mathcal{A}$, is well defined. For that suppose $\sum_{i=1}^s \beta'_{g_i}(\varphi'(a_i)) = 0$, and we need to be sure that $\sum_{i=1}^s \beta_{g_i}(\varphi(a_i)) = 0$.

For all $h \in G$ and $a \in \mathcal{A}$ we have $\sum_i \beta'_h(\varphi'(a))\beta'_{g_i}(\varphi'(a_i)) = 0$, and applying $\beta'_{h^{-1}}$ we obtain $\sum_i \varphi'(a)\beta'_{h^{-1}g_i}(\varphi'(a_i)) = 0$. Observe that

$$\varphi'(a)\beta'_{h^{-1}g_i}(\varphi'(a_i)) \in \varphi'(\mathcal{A}) \cap \beta'_{h^{-1}g_i}(\varphi'(\mathcal{A})) = \varphi'(\mathcal{A}_{h^{-1}g_i}),$$

and thus $\varphi'(a)\beta'_{h^{-1}g_i}(\varphi'(a_i)) = \varphi'(b'_{i,a})$ with $b'_{i,a} \in \mathcal{A}_{h^{-1}g_i}$. Similarly

$$\varphi(a)\beta_{h^{-1}g_i}(\varphi(a_i)) = \varphi(b_{i,a})$$

with $b_{i,a} \in \mathcal{A}_{h^{-1}g_i}$. Let $e_{i,a} \in \mathcal{A}_{h^{-1}g_i}$ be a left unit for $b'_{i,a}$ and $b_{i,a}$. It exists by Lemma 2.4. Then

$$\varphi'(a)\beta'_{h^{-1}g_i}(\varphi'(a_i)) = \varphi'(e_{i,a})\varphi'(a)\beta'_{h^{-1}g_i}(\varphi'(a_i)) = \varphi'(e_{i,a}a)\beta'_{h^{-1}g_i}(\varphi'(a_i)).$$

Since $e_{i,a}a \in \mathcal{A}_{h^{-1}g_i}$ we can write $e_{i,a}a = \alpha_{h^{-1}g_i}(\tilde{e}_{i,a})$ with $\tilde{e}_{i,a} \in \mathcal{A}_{g_i^{-1}h}$. Hence

$$\begin{aligned} \varphi'(a)\beta'_{h^{-1}g_i}(\varphi'(a_i)) &= \varphi'(\alpha_{h^{-1}g_i}(\tilde{e}_{i,a}))\beta'_{h^{-1}g_i}(\varphi'(a_i)) \\ &= \beta'_{h^{-1}g_i}(\varphi'(\tilde{e}_{i,a}))\beta'_{h^{-1}g_i}(\varphi'(a_i)) \\ &= \beta'_{h^{-1}g_i}(\varphi'(\tilde{e}_{i,a}a_i)) = \varphi'(\alpha_{h^{-1}g_i}(\tilde{e}_{i,a}a_i)). \end{aligned}$$

Consequently,

$$0 = \sum_i \varphi'(a)\beta'_{h^{-1}g_i}(\varphi'(a_i)) = \varphi'(\sum_i \alpha_{h^{-1}g_i}(\tilde{e}_{i,a}a_i)).$$

This yields that $\sum_i \alpha_{h^{-1}g_i}(\tilde{e}_{i,a}a_i) = 0$, as φ' is a monomorphism. Because $e_{i,a}$ is also a left unit for $b_{i,a}$, we similarly have

$$0 = \varphi(\sum_i \alpha_{h^{-1}g_i}(\tilde{e}_{i,a}a_i)) = \sum_i \varphi(a)\beta_{h^{-1}g_i}(\varphi(a_i)).$$

Applying β_h we come to $\beta_h(\varphi(a))\sum_i \beta_{g_i}(\varphi(a_i)) = 0$. Since this holds for any $h \in G$, we see that $\mathcal{B}(\sum_i \beta_{g_i}(\varphi(a_i))) = 0$. Finally, because $\mathcal{B}$ is left s -unital, we obtain $\sum_i \beta_{g_i}(\varphi(a_i)) = 0$, as desired.

Thus $\phi : \mathcal{B}' \rightarrow \mathcal{B}$ is a well-defined map, and by symmetry $\beta_g(\varphi(a)) \mapsto \beta'_g(\varphi'(a))$, $g \in G, a \in \mathcal{A}$, also determines a well-defined map $\phi' : \mathcal{B} \rightarrow \mathcal{B}'$. Obviously, $\phi' \circ \phi = \phi \circ \phi' = 1$. It is easily verified that ϕ preserves products. By taking $g, h \in G$ and $a, b \in \mathcal{A}$, we have $\beta_g(\varphi(a))\beta_h(\varphi(b)) = \beta_h(\beta_{h^{-1}g}(\varphi(a))\varphi(b)) = \beta_h(\varphi(x))$ and $\beta'_g(\varphi'(a))\beta'_h(\varphi'(b)) = \beta'_h(\varphi'(y))$ for some $x, y \in \mathcal{A}_{h^{-1}g}$. Let $e \in \mathcal{A}_{h^{-1}g}$ be a left unit for x and y . Then $\beta'_g(\varphi'(a))\beta'_h(\varphi'(b)) = \beta'_h(\varphi'(e)\beta'_{h^{-1}g}(\varphi'(a))\varphi'(b)) = \beta'_h(\varphi'(\alpha_{h^{-1}g}(\alpha_{h^{-1}g}^{-1}(e)a)b))$ which is mapped by ϕ to $\beta_h(\varphi(\alpha_{h^{-1}g}(\alpha_{h^{-1}g}^{-1}(e)a)b)) = \beta_g(\varphi(a))\beta_h(\varphi(b))$. Consequently, ϕ is an isomorphism of rings. It is easily seen that for all $g \in G$ one has $\beta_g \circ \phi = \phi \circ \beta'_g$, and this yields the uniqueness of the globalization in the sense of Definition 2.3. $\square$

We know by Theorem 4.5 of [6] that a partial action α of a group G on a unital ring $\mathcal{A}$ admits a globalization if and only if each $\mathcal{A}_g$ is a unital ring. Thus we have the following.

Corollary 3.2. *Suppose that α is a partial action of a group G on a unital ring $\mathcal{A}$ such that $\mathcal{A}_g$ is nonunital for some $g \in G$. Then there exists $h \in G$ such that either $\mathcal{A}_h$ is not s -unital or for some $a \in \mathcal{A}$ no element of $M(\mathcal{A})$ extends, as a right multiplier, the left $\mathcal{A}$ -module map $\mathcal{A}_g \ni x \mapsto \alpha_g(\alpha_{g^{-1}}(x)a) \in \mathcal{A}_g$ and maps $\mathcal{A}$ into $\mathcal{A}_g$ (as a right multiplier).*

Remark 3.3. Observe that when dealing with Theorem 4.5 of [6] one treats $\{0\}$ as a unital ring, thus we do not require $1 \neq 0$ in our definition of a unital ring.

The next example shows that the second alternative in Corollary 3.2 really occurs. This implies that condition (ii) of Theorem 3.1 cannot be omitted.

Example 3.4. Let $\mathcal{A}$ be a ring with 1 and let $\mathcal{I}$ be an ideal in $\mathcal{A}$ which is an s -unital but not a unital ring. Let G be the cyclic group of order 2 with generator g and let α_1 be the identity map $\mathcal{A} \mapsto \mathcal{A}$ and α_g the identity map $\mathcal{I} \mapsto \mathcal{I}$. Evidently we have a partial action α of G on $\mathcal{A}$ which is not globalizable by Theorem 4.5 of [6]. Hence by Corollary 3.2 the left $\mathcal{A}$ -module map $\mathcal{I} \ni x \mapsto \alpha_g(\alpha_{g^{-1}}(x)a) = xa \in \mathcal{I}$ cannot be extended to a multiplier γ of $\mathcal{A}$ such that $\mathcal{A}\gamma \subseteq \mathcal{A}_g$. For a more concrete example take as $\mathcal{A}$ the ring of row and column finite matrices over a field. This ring evidently has 1, and the subset $\mathcal{I}$ of the matrices with a finite number of nonzero entries is an ideal which is both left and right s -unital but is not unital.

Given a globalizable partial action α of a group G on a unital ring $\mathcal{A}$, we know by Theorem 3.1 that the ring $\mathcal{B}$ under the global action is left s -unital. If G is finite, then $\mathcal{B}$ has 1 by Lemma 4.4 of [6]. However for an infinite G the ring $\mathcal{B}$ is not necessarily unital, as shown in the next example.

Example 3.5. Let G be an infinite group and let $\mathcal{A}$ be a ring with 1. Let α_1 be the trivial map $\mathcal{A} \mapsto \mathcal{A}$ and set $\mathcal{A}_g = 0$ for each $g \neq 1$. Then we evidently have a globalizable partial action α of G on $\mathcal{A}$ (see Remark 3.3). It is easily seen that the ring under the globalized action in the proof of Theorem 3.1 is $\mathcal{B} = \mathcal{A}^{(|G|)}$, the direct sum of $|G|$ copies of $\mathcal{A}$, indexed by the elements of G , on which G acts by the permutation of indices. Clearly, $\mathcal{B}$ is not unital.

4. MORITA EQUIVALENCE

Morita's fundamental results were extended in [14] to the case of idempotent rings, which permits us to easily adapt the proof of [6, Theorem 5.4] to our situation. Since $\mathcal{A}$ and $\mathcal{B}$ are left s -unital, by Lemma 2.4, the skew group rings $\mathcal{A} *_\alpha G$ and $\mathcal{B} *_\beta G$ are also left s -unital. The notion of the Morita context for idempotent rings is the same as for rings with 1, i.e. a *Morita context* is a six-tuple $(\mathcal{R}, \mathcal{R}', M, N, \tau, \tau')$, where:

- (a) $\mathcal{R}$ and $\mathcal{R}'$ are rings,
- (b) M is an $\mathcal{R}$ - $\mathcal{R}'$ -bimodule,
- (c) N is an $\mathcal{R}'$ - $\mathcal{R}$ -bimodule,
- (d) $\tau : M \otimes_{\mathcal{R}'} N \rightarrow \mathcal{R}$ is a bimodule map,
- (e) $\tau' : N \otimes_{\mathcal{R}} M \rightarrow \mathcal{R}'$ is a bimodule map,

such that

$$\tau(m \otimes n) m' = m \tau'(n \otimes m'), \quad \forall m, m' \in M, n \in N,$$

and

$$\tau'(n \otimes m) n' = n \tau(m \otimes n'), \quad \forall n, n' \in N, m \in M.$$

Given a Morita context $(\mathcal{R}, \mathcal{R}', M, N, \tau, \tau')$ with R and R' idempotent rings, and τ and τ' onto, the categories of $\mathcal{R}$ -modules and of $\mathcal{R}'$ -modules are equivalent, that is, $\mathcal{R}$ and $\mathcal{R}'$ are *Morita equivalent* [14].

We have the following.

Theorem 4.1. *Let α be a partial action of a group G on a left s -unital ring $\mathcal{A}$ and suppose that $(\beta, \mathcal{B})$ is a globalization for $(\alpha, \mathcal{A})$. Then $\mathcal{A} *_{\alpha} G$ and $\mathcal{B} *_{\beta} G$ are Morita equivalent.*

Proof. Set $\mathcal{R} = \mathcal{A} *_{\alpha} G$ and $\mathcal{R}' = \mathcal{B} *_{\beta} G$ and consider $M, N \subseteq \mathcal{B} *_{\beta} G$ given by

$$M = \left\{ \sum_{g \in G} c_g u_g : c_g \in \mathcal{A} \right\} \quad \text{and} \quad N = \left\{ \sum_{g \in G} c_g u_g : c_g \in \beta_g(\mathcal{A}) \right\}.$$

We view $\mathcal{R}$ as a subring of $\mathcal{R}'$. The proofs of Propositions 5.1 and 5.2 of [6] do not use the fact that the ring has 1 and hold for arbitrary (associative) rings. According to these propositions M is an $\mathcal{R}$ - $\mathcal{R}'$ -bimodule and N is a $\mathcal{R}'$ - $\mathcal{R}$ -bimodule. Let $\tau : M \otimes_{\mathcal{R}'} N \rightarrow \mathcal{R}'$ and $\tau' : N \otimes_{\mathcal{R}} M \rightarrow \mathcal{R}'$ be given by $\tau(m \otimes n) = mn$ and $\tau'(n \otimes m) = nm$. Obviously τ is an $\mathcal{R}$ -bimodule map and τ' is an $\mathcal{R}'$ -bimodule map. Thus to show that $\mathcal{R}$ and $\mathcal{R}'$ are Morita equivalent, it is enough to prove that $\tau(M \otimes_{\mathcal{R}'} N) = \mathcal{R}$ and $\tau'(N \otimes_{\mathcal{R}} M) = \mathcal{R}'$. The inclusions $\tau(M \otimes_{\mathcal{R}'} N) \subseteq \mathcal{R}$ and $\tau'(N \otimes_{\mathcal{R}} M) \subseteq \mathcal{R}'$ are clear.

If $g \in G, a \in \mathcal{A}_g$ and $e \in \mathcal{A}_g$ is a left unit for a , then $eu_1 \in M, au_g \in N$ because $a \in \alpha_g(\mathcal{A}_{g^{-1}}) \subseteq \beta_g(\mathcal{A})$, and

$$au_g = eu_1 \cdot au_g = \tau(eu_1 \otimes au_g) \in \tau(M \otimes_{\mathcal{R}'} N).$$

This shows that $\tau(M \otimes_{\mathcal{R}'} N) \supseteq \mathcal{R}$.

Finally, let $g, h \in G$ and $a \in \mathcal{A}$ be arbitrary, and let $e \in \mathcal{A}$ be a left unit for a . Then $\beta_g(e)u_g \in N, au_{g^{-1}h} \in M$ and

$$\beta_g(a)u_h = \beta_g(ea)u_h = \beta_g(e)u_g \cdot au_{g^{-1}h} = \tau'(\beta_g(e)u_g \otimes au_{g^{-1}h}),$$

and we have $\tau'(N \otimes_{\mathcal{R}} M) \supseteq \mathcal{R}'$. □

Remark 4.2. Take the partial action $(\alpha, \mathcal{A})$ of Example 3.5 and let $(\beta, \mathcal{B})$ be its globalization. By Theorem 4.1, $\mathcal{A} \cong \mathcal{A} *_{\alpha} G$ is Morita equivalent to $\mathcal{B} *_{\beta} G$. If G is infinite cyclic, then the latter is isomorphic to the skew Laurent polynomial ring associated to the action of G on $\mathcal{B} = \mathcal{A}^{|G|}$. In fact, it can be shown that for every group G , $\mathcal{B} *_{\beta} G$ is isomorphic to the ring of $|G| \times |G|$ matrices over $\mathcal{A}$ with finitely many nonzero entries.

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