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CALCULATION OF LEFSCHETZ AND NIELSEN NUMBERS IN HYPERSPACES FOR FRACTALS AND DYNAMICAL SYSTEMS

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Abstract. A simple argument is given as to why it is always trivial to calculate Lefschetz and Nielsen numbers for iterated function systems or dynamical systems in hyperspaces. The problem is reduced to a simple combinatorical situation on a finite set.

In the papers [3], [4] the question was raised whether the Lefschetz or even the Nielsen number might be used in the hyperspace $H(X)$ of all nonempty compact subsets of a metric space X to prove the existence of one or several fractals of a single- or multivalued iterated function system (IFS). This is also a question which arises naturally for discrete dynamical systems if studied from the viewpoint of hyperspaces [2], [17].

It is the purpose of this note to observe that it is possible to calculate these numbers and that this task actually is surprisingly trivial. In fact, we show that the situation is as simple as it can be: It can be completely understood by counting fixed points of a map of a finite set of typically very small cardinality. The reason for this is that hyperspaces are—despite of the fact that they lack any evident vector space structure—topologically as simple as one can expect. Moreover, this is not only true for the hyperspace $H(X)$ of nonempty compact subsets (endowed with the Hausdorff metric), but actually even for each growth hyperspace.

Recall that a subset $\mathcal{G} \subseteq H(X)$ is called a *growth hyperspace* of X if $\mathcal{G}$ is closed in $H(X)$ and has the following property: Whenever $A \in \mathcal{G}$ and $B \in H(X)$ are such that $A \subseteq B$ and each component of B meets A, then $B \in \mathcal{G}$.

Examples. (1) $\mathcal{G} = H(X)$ is a growth hyperspace.

(2) The family $H_1(X)$ of all connected compact nonempty subsets of X is a growth hyperspace. More generally, for each $n = 1, 2, \ldots$, the space $\mathcal{G} = H_n(X)$ consisting of all nonempty compact subsets with at most n components is a growth hyperspace. Note that the space of all nonempty

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compact subsets with at most countably many components is usually not a growth hyperspace, because it is not closed in $H(X)$, in general.

(3) The previous two examples can be combined: For each open $X_0 \subseteq X$ and each $n = 1, 2, \dots$, the space $\mathcal{G}$ consisting of all $A \in H(X)$ such that at most n components of A meet X_0 is a growth hyperspace. Note that if X_0 fails to be open, then $\mathcal{G}$ is usually not a growth hyperspace, because it is not closed in $H(X)$, in general.

(4) For each $X_0 \in H(X)$, the family $H^{X_0}(X)$ of all $A \in H(X)$ with $X_0 \subseteq A$ and such that each component of A meets X_0 is a growth hyperspace. It is the smallest growth hyperspace containing X_0.

(5) If $\mathcal{G}$ is a growth hyperspace and $X_0 \subseteq X$ is closed, then the family of all $A \in \mathcal{G}$ meeting X_0 is a growth hyperspace.

(6) If $\mathcal{G}$ is a growth hyperspace and $X_0 \subseteq X$ is arbitrary, then the family of all $A \in \mathcal{G}$ satisfying $X_0 \subseteq A$ is a growth hyperspace.

The reason for the strikingly "good" topological behaviour of growth hyperspaces is the following result which was implicitly proved in [8]. Before we can formulate it, we have to recall some terminology.

A metric space X is *locally continuum-connected* if for each neighbourhood U of each point $x \in X$ there is a neighbourhood $V \subseteq U$ of x such that each point of V is contained in a subcontinuum (i.e. a compact connected subset) of U which contains x.

A metric space X is called an ANR if, for each metric space Y, each closed subset $A \subseteq Y$ and each continuous map $f\colon A \to X$, there is a continuous extension $f\colon U \to X$, for some open set $U \subseteq Y$ with $U \supseteq A$; if one can even always choose $U = Y$, then X is called an AR. It is well known that an ANR is an AR if and only if it is contractible, i.e. homotopic to a one-point space. The following implications are easily seen for a metric space:

$$\begin{aligned} \text{AR} \implies \text{ANR} \implies \text{locally contractible} \implies \text{locally path-connected} \\ \implies \text{locally continuum-connected} \implies \text{locally connected.} \end{aligned}$$

The last two implications cannot be reversed in general (see [8]), but they can be reversed for complete metric spaces.

Theorem 1. *Let X be a metric space, and let $\mathcal{G} \subseteq H(X)$ be some growth hyperspace of X with $\mathcal{G} \supseteq H_1(X)$. Then the following statements are equivalent:*

(1) *X is locally continuum-connected.*
(2) *$\mathcal{G}$ is locally path-connected.*
(3) *$\mathcal{G}$ is an ANR.*
(4) *$\mathcal{G}$ divides into disjoint open components, each being an AR.*

Moreover, if one of these statements holds, then $\mathcal{G}$ is an AR if and only if X is connected.

Without the hypothesis $\mathcal{G} \supseteq H_1(X)$ it is still true that the first of the above statements implies the other three.

Proof. The last statement is a special case of [8, Theorem 1.6]. Moreover, it is shown in the proof of that theorem that if $\mathcal{G}$ is locally path-connected and $\mathcal{G} \supseteq H_1(X)$, then X is locally continuum-connected. It is also shown in that proof that if X is locally continuum-connected, then $\mathcal{G}$ is an ANR which is n-connected for all $n \geq 1$. In particular, $\mathcal{G}$ is locally path-connected, and so its path-components are open and

coincide with the components. Since open subsets of ANRs are ANRs, it follows that these path-components are ANRs which have in all dimensions the homotopy of a one-point space. By [9, Theorem 12.4], these components are ARs. Conversely, if $\mathcal{G}$ is a union of open ARs, then $\mathcal{G}$ is locally path-connected. □

If one wants to apply any topological methods for $\mathcal{G}$ such as fixed point index, Lefschetz or Nielsen number, then the least that one has to require is that $\mathcal{G}$ is locally path-connected (for most methods, one even has to require that $\mathcal{G}$ is an ANR). Hence, in the following, we will restrict our attention to this situation, i.e. we will require that X is locally continuum-connected. But in this case, $\mathcal{G}$ is only a union of strictly separated ARs. Recall that a Lefschetz number is defined for continuous and compact (or at least condensing) self-maps of an ANR; see [10] or [15]. Since ARs have the homotopy and homology of a one-point space, and since continuous maps send components into connected sets, it is immediately clear that, for each continuous self-map $F\colon \mathcal{G} \to \mathcal{G}$ for which a Lefschetz number is defined, this number is simply the number of components of $\mathcal{G}$ which are mapped into itself. It is straightforward to verify that this number is the Nielsen number of F in the Wecken sense (defined for the same class of homotopies under which the Lefschetz number is stable); see [6]. Moreover, if F is compact, then since the components of $\mathcal{G}$ are contractible and strictly separated, this Nielsen number has the Wecken property for compact homotopies, i.e. it is the minimal number of fixed points of each map homotopic to F by a compact homotopy.

Thus, for all further considerations, we only have to study the components of $\mathcal{G}$. But these relate to the components of X in a rather evident way as we will see in Lemma 2. To formulate this result, we introduce a further notation:

For $K \subseteq X$, we denote by X_K the family of all components of X which meet K.

Lemma 1. *Let X be a locally connected metric space. Then, for each $K \in H(X)$, the set X_K is finite. Moreover, for each connected topological space C and each continuous map $\gamma\colon C \to H(X)$, the finite set $X_{\gamma(t)}$ is independent of $t \in C$.*

Proof. Since X is locally connected, all components are open (and closed). By the compactness, K is covered by finitely many of the open components of X, and thus K only meets finitely many of these components. Since γ is continuous with respect to the Hausdorff distance, the multivalued map $\gamma\colon C \to H(X)$ is upper and lower semicontinuous, and so, for each open and closed set $M \subseteq X$, the large and small preimages $\gamma^-(M) := \{t \in C : \gamma(t) \subseteq M\}$ and $\gamma^+(M) := \{t \in C : \gamma(t) \cap M \neq \emptyset\}$ are open and closed. In particular, fixing $t_0 \in C$ and putting $K := \gamma(t_0)$, we find that the sets $\gamma^-(\bigcup X_K)$ and $\gamma^+(D)$ (for $D \in X_K$) are open and closed. Since these sets contain t_0, and C is connected, we have $\gamma^-(\bigcup X_K) = C$ and $\gamma^+(D) = C$, for each $D \in X_K$, i.e. $X_{\gamma(t)} = X_K$, for each $t \in C$. □

Lemma 2. *Let $\mathcal{G}$ be some growth hyperspace of a locally continuum-connected metric space X, and $K_0 \in \mathcal{G}$. Then the connected component of K_0 in $\mathcal{G}$ is*

$$\begin{aligned}\mathcal{G}_{K_0} &:= \{K \in \mathcal{G} : X_K = X_{K_0}\}\\ &= \{K \in \mathcal{G} : K \textit{ and } K_0 \textit{ meet the same components of } X\}\,.\end{aligned}$$

Proof. Since $\mathcal{G}$ is locally path-connected by Theorem 1, the component of K_0 in $\mathcal{G}$ is path-connected. Since Lemma 1 implies that each path in $\mathcal{G}$ starting from K_0 remains in $\mathcal{G}_{K_0}$, we only have to show that $\mathcal{G}_{K_0}$ is path-connected.

Thus, let $K_1, K_2 \in \mathcal{G}_{K_0}$. For each $D \in X_{K_0}$, we find by [8, Lemma 1.3] a continuum C_D containing $(K_1 \cup K_2) \cap D$. Then $C := \bigcup_{D \in X_{K_0}} C_D \in H(X)$ has only finitely many components, because X_{K_0} is finite by Lemma 1, and each of these components meets K_i, for $i = 1, 2$. By [8, Lemma 1.1] (see also [11]), there exist continuous paths $\gamma_i \colon [0,1] \to H^{K_i}(X)$ with $\gamma_i(0) = K_i$ and $\gamma_i(1) = C$ for $i = 1, 2$. Since $K_i \in \mathcal{G}$ and $\mathcal{G}$ is a growth hyperspace, we have $\gamma_i([0,1]) \subseteq H^{K_i}(X) \subseteq \mathcal{G}$. In view of Lemma 1, we even have $\gamma_i([0,1]) \subseteq \mathcal{G}_{K_0}$. Hence, each of the sets K_1 and K_2 can be connected with C by a continuous path in $\mathcal{G}_{K_0}$. Thus, $\mathcal{G}_{K_0}$ is path-connected, as claimed. □

Now let a finite family of continuous maps $f_1, \dots, f_n \colon X \to H(X)$ be given. Then $f_1, \dots, f_n$ induce (cf. [1], [2]) a Hutchinson-Barnsley operator $F \colon H(X) \to H(X)$ by the formula

$$F(K) := f_1(K) \cup \dots \cup f_n(K) \qquad (K \in H(X)). \tag{1}$$

We are interested in sets $K \in H(X)$ satisfying $F(K) = K$, i.e. we are interested in fixed points of F.

In the classical situation considered by Hutchinson and Barnsley, the maps $f_1, \dots, f_n$ are only single-valued contractions, the so-called fractal set K is unique and it can be obtained by the method of successive approximations. We will however consider the more general situation of not necessarily single-valued or contracting maps $f_1, \dots, f_n$, as e.g. in [1], [3], [4], [12], [13, 16]. Nevertheless, observe that we require that the maps f_k are continuous with respect to the Hausdorff distance (which is equivalent to the upper and lower semicontinuity of the multivalued functions $f_k \colon X \to H(X)$; cf. e.g. [5, (I.3.64)]).

The map F defined in the above way is continuous [3].

Proposition 1. *The function $x \mapsto X_{F(\{x\})}$ is constant on the components of X.*

Proof. Defining $i \colon X \to H(X)$ by $i(x) := \{x\}$, we have $F(\{x\}) = (F \circ i)(x)$. Since F and i are continuous, the claim follows from Lemma 1. □

We will assume that F has a nonempty compact (sub)invariant set $K_\infty \subseteq H(X)$ containing the core of F, i.e.

$$F(K_\infty) \subseteq K_\infty \quad \text{and} \quad \bigcap_{n=1}^{\infty} F^n(H(X)) \subseteq K_\infty. \tag{2}$$

This condition is satisfied, in particular, if F or some iterate of F has a relatively compact range. More generally, this holds if the range of some iterate of F has a complete closure and a bounded diameter and if F is strictly condensing with respect to the Hausdorff measure of noncompactness (or some equivalent set function). Note in this connection that F is compact or strictly condensing with respect to the Hausdorff measure of noncompactness if each of the generating functions $f_1, \dots, f_k$ has this property; see [4] or [12], respectively.

Let X_∞ be the family of all components of X which meet the compact set $\bigcup K_\infty$. Then X_∞ is finite (Lemma 1). Moreover, F induces a map $F_\infty \colon H(X_\infty) \to H(X_\infty)$ by the formula

$$F_\infty(U) := \bigcup_{D \in U} X_{F(D)}. \tag{3}$$

In view of Proposition 1, the function F_∞ may be easily calculated by picking for each $D \in X_\infty$ some $x_D \in D$; then

$$F_\infty(U) = \bigcup_{D \in U} X_{F(\{x_D\})}. \tag{4}$$

The function F_∞ on the finite set $H(X_\infty)$ contains all necessary information to calculate the Lefschetz and Nielsen numbers of F.

Theorem 2. *Let the metric space X be locally continuum-connected, and let $f_1, \dots, f_n \colon X \to H(X)$ be continuous. Let F be defined by* (1)*, let $K_\infty \in H(X)$ satisfy* (2)*, and let the corresponding function $F_\infty \colon H(X_\infty) \to H(X_\infty)$ be defined by* (3)*, i.e. by* (4)*. Then for each growth hyperspace $\mathcal{G}$ over X satisfying*

$$F(\mathcal{G}) \subseteq \mathcal{G}, \tag{5}$$

the following holds: If the Lefschetz number of $F \colon \mathcal{G} \to \mathcal{G}$ is defined such that the Lefschetz theorem holds, then the Lefschetz and Nielsen numbers (corresponding to the class of admissible homotopies for the Lefschetz number) are the same, and they represent the cardinality of the set

$$\{U \in H(X_\infty) : F_\infty(U) = U \text{ and there is some } K \in \mathcal{G} \text{ with } U = X_K\}. \tag{6}$$

If $\mathcal{G} = H(X)$, then this set is always nonempty, i.e. the Lefschetz and Nielsen number is nonzero.

Thus, roughly speaking, we just have to consider the finitely many components X_∞ of X, and we have to determine for some point of each of these components to which components the image under F belongs: This gives us the induced function F_∞ in view of (4). The required number is then just the number of those nonempty invariant sets under F_∞ for which the corresponding sets are admissible for the choice of the hyperspace $\mathcal{G}$.

We point out that if F is not condensing, then the Lefschetz number need not be defined. Moreover, even for condensing maps on ANRs (even on ARs) it is an open problem whether the Lefschetz fixed point theorem holds; it is only known to hold if either the map or the ANR has some additional properties. More precisely, the following conditions are known to be sufficient for the Lefschetz fixed point theorem of a continuous self-map F:

(1) F has a compact attractor, and in addition either
   (a) F is locally compact [5, (I.6.21)], or
   (b) F is condensing and defined on an open subset of a Banach or Fréchet space; see [15] or [5, (I.7.1)], respectively.
(2) F is strictly condensing and defined on a finite union of closed convex sets in a Banach space; see [15].
(3) F is condensing and defined on a so-called *special ANR*, i.e. on a retract of some neighbourhood in a Fréchet space such that the retraction does not increase a measure of noncompactness; see [5, (I.7.3)].

Unfortunately, although the hyperspace $\mathcal{G}$ has some "good" properties from the topological point of view by Theorem 1, it is unknown whether it has such good properties from the metric viewpoint which could give some of the above additional properties automatically.

Nevertheless, if some of these additional properties hold such that the Lefschetz number is defined (e.g. if the maps $f_1, \dots, f_n$ and thus F are compact), then Theorem 2 of course implies a corresponding existence result and even a multiplicity result, since the Nielsen number is a lower bound for the number of fixed points. Of course, since this number only stems from different components of the underlying space X, this multiplicity observation is actually rather trivial (cf. the example at the end of the paper).

For the case $\mathcal{G} = H(X)$, an existence result can be obtained more easily by observing that the map F is monotone (isotone) and thus by applying the Knaster-Tarski theorem or the ultimate range principle of Sadovskiĭ. This was carried out in [4], [13, 16]. This program might probably also be carried out e.g. when $\mathcal{G} \subseteq H(X)$ is the hyperspace of all nonempty compact connected subsets of X, but the proof is then more cumbersome. Note that (5) restricts us for this choice of $\mathcal{G}$ to the case of a single function $n = 1$ with connected values $f_1(x)$; however, this is a rather natural setting even for single-valued f_1 from the viewpoint of dynamical systems [2], [17].

Proof of Theorem 2. In view of the remarks near the beginning of the paper, we have to count how many components of $\mathcal{G}$ are mapped by F into itself. Of course, these can only be components meeting the core of F. By Lemma 2, the components of $\mathcal{G}$ are in a one-to-one correspondence with the set $\{X_K : K \in \mathcal{G}\}$, and so the components of $\mathcal{G}$ meeting the core of F are in a one-to-one correspondence with the subset consisting of those X_K ($K \in \mathcal{G}$) which are contained in X_∞. The component of some $K \in \mathcal{G}$ in $\mathcal{G}$ is mapped by F into itself if and only if the induced map sends the corresponding set $U = X_K$ into itself, i.e. if and only if $F_\infty(U) = U$.

To see that the set (6) is nonempty, observe that the decreasing sequence $C_n := F_\infty^n(H(X_\infty))$ must stabilize after at most $|H(X_\infty)|$ steps. This argument was also observed in [16] and is nothing else than a trivial special case of the ultimate range principle of Sadovskiĭ; see [13]. □

In an important special case the map F has a compact attractor.

Proposition 2. *Assume that X is complete and locally continuum-connected. Furthermore, assume that some of the continuous functions $f_1, \dots, f_n \colon X \to H(X)$ are contractions and that the range of the remaining functions is contained in a compact subset of X.*

Then the map F has a nonempty compact (sub)invariant attractor which contains the core of F. Moreover, F has an extension to a continuous map defined on an open subset of a Banach space with the same range as F and which thus also has a nonempty compact attractor.

Proof. Let $G \colon H(X) \to H(X)$ be the map induced by the union of the contractions, and let $K_0 \in H(X)$ be a set containing the range of the remaining functions. One part of the proof of the classical Hutchinson-Barnsley theorem implies that G is a contraction in the complete space $H(X)$ and thus has a unique fixed point $M \in H(X)$. The infinite union

$$K := M \cup K_0 \cup \bigcup_{n=1}^{\infty} G^n(K_0)$$

is compact. Indeed, if $x_n \in K$, then either all x_n are contained in a finite union of the above compact sets or there is a subsequence $x_{n_j} \in G^{k_j}(K_0)$ where $k_j \to \infty$.

Since $G^{k_j}(K_0) \to M$ with respect to the Hausdorff distance, the latter implies by a diagonal argument that x_{n_j} has a subsequence convergent to some element of $M \subseteq K$. Considering the compact hyperspace $H(K)$ as a subset of $H(X)$, we show that $H(K)$ is a (sub)invariant attractor of F.

Since $G(K) \subseteq G(M) \cup K = M \cup K = K$, we have $F(K) \subseteq K_0 \cup G(K) \subseteq K$, and so $H(K)$ is (sub)invariant. Moreover, an induction implies for each $S \in H(X)$ that $F^n(S) \subseteq K \cup G^n(S)$. Indeed, the case $n = 1$ follows from $F(S) \subseteq K_0 \cup G(S) \subseteq K \cup G(S)$, and since $H(K)$ is (sub)invariant, the induction hypothesis implies $F^{n+1}(S) = F(F^n(S)) \subseteq F(K \cup G^n(S)) = F(K) \cup F(G^n(S)) \subseteq K \cup K_0 \cup G(G^n(S)) = K \cup G^{n+1}(S)$. This completes the induction step.

To see that $H(K)$ is an attractor, let $U \subseteq H(X)$ be some neighborhood of $H(K)$. Since $H(K)$ is compact, there is some $\varepsilon > 0$ such that U contains all points of $H(X)$ which have from $H(K)$ at most the distance ε. Since each sequence of successive approximations (for G) converges to M, we find, for each $S \in H(X)$, some n_S such that $G^n(S)$ has from M at most the Hausdorff distance ε, for all $n \geq n_S$. We conclude in view of the above inclusion $F^n(S) \subseteq K \cup G^n(S)$ that all points of $F^n(S)$ have from $K \cup M = K$ at most distance ε if $n \geq n_S$. Hence, $F^n(S)$ has at most the Hausdorff distance ε from an appropriate compact subset of K, i.e. $F^n(S) \in U$, for each $n \geq n_S$. Hence, $H(K)$ is a (nonempty compact) attractor.

For the second claim note that by the Arens-Eells embedding theorem [7], we may assume that $H(X)$ is a subset of a Banach space Z. Since $H(X)$ is complete, it is closed, and since $H(X)$ is an ANR, we find a retraction ρ of some neighborhood of $H(X)$ onto $H(X)$; hence, $F \circ \rho$ is a required extension. □

Corollary 1. *Under the hypothesis of Proposition* 2*, the map F has a fixed point in $H(X)$, i.e. there is a nonempty compact set $K \subseteq X$ with $F(K) = K$.*

Proof. Proposition 2 implies in particular the existence of a nonempty compact subinvariant set. Hence, the existence of a minimal invariant set in $H(X)$ follows from the Knaster-Tarski theorem; see [13]. □

In contrast to general results concerning condensing IFS in [13] or [4], we do not have to assume in Corollary 1 that the range of the functions f_k has a bounded diameter. On the other hand, in this particular case of condensing IFS, it can also be deduced from our observation in [4, Proposition 3].

Remark. In Corollary 1 (and the first part of Proposition 2) we need not require that the functions f_k are continuous, provided we consider the closure in the definition of F, i.e. when we replace (1) by

$$F(K) := \overline{f_1(K)} \cup \cdots \cup \overline{f_n(K)} \qquad (K \in H(X)).$$

Nevertheless, despite Proposition 2, we cannot claim that the Lefschetz theorems [15] or [5, (I.7.1)] hold for F, because we do not know whether the extension can be chosen such that it is condensing. This is obviously true if $H(X)$ is a special ANR (see above), but we do not know the latter either. On the other hand, the following holds.

Corollary 2. *Under the hypothesis of Proposition* 2*, if X is locally compact and $\mathcal{G} := H(X)$, then all hypotheses of Theorem* 2 *are satisfied and the Lefschetz number of F is defined; for K_∞ one may choose the compact attractor of Proposition* 2*.*

In particular, the (nonzero) number of fixed points of $F_\infty\colon H(X_\infty) \to H(X_\infty)$ *is the Lefschetz and (corresponding) Nielsen number of* F*, and it is a lower bound for the number of fixed points of* $F\colon H(X) \to H(X)$*.*

Proof. Since X is locally compact, then so is $H(X)$ (see e.g. [14]), and so the Lefschetz and corresponding Nielsen number is well defined; see [5, (I.6.21)] and [6]. Note that the attractor K_∞ automatically contains the core of F, because each element S of the core must be contained in each neighborhood containing K_∞; since $H(X)$ is regular and K_∞ is closed, this implies $S \in K_\infty$. □

Example. Let X be a locally compact and locally connected metric space, consisting of two connected components X_1 and X_2. By Lemma 2, the hyperspace $H(X)$ consists of three components, namely $C_1 := H(X_1)$, $C_2 := H(X_2)$, and $C_3 := H(X) \setminus (C_1 \cup C_2)$.

Assume that some of the functions $f_1, \dots, f_n\colon X \to H(X)$ are contractions and that the remaining ones are continuous and have their range contained in a compact subset of X. Then the Lefschetz number of the function (1) is defined on $\mathcal{G} := H(X)$, and it is either 1, 2 or 3: Indeed, for $j = 1, 2$, let $i_j \in \{1, 2, 3\}$ be such that $f_1(x) \cup \dots \cup f_n(x) \in C_j$ for some (and thus all) $x \in X_i$. The situation is completely described by the following cases:

(1) $(i_1, i_2) = (1, 2)$: We have $F(C_i) \subseteq C_i$ for all $i = 1, 2, 3$, and so the Lefschetz and Nielsen number is 3.

It is not surprising that in this case there are indeed three fixed points of F: Since $F(C_i) \subseteq C_i$ for $i = 1, 2$, we find $A_i \in H(X_i)$ satisfying $F(A_i) = A_i$; then $A_3 := A_1 \cup A_2 \in C_3$ satisfies $F(A_3) = A_3$.

(2) $i_1 = i_2$: We have $F(H(X)) \subseteq C_{i_1}$, and so the Lefschetz and Nielsen number is 1.

(3) $(i_1, i_2) = (2, 1)$ or $(i_1, i_2) = (2, 3)$ or $(i_1, i_2) = (3, 1)$: We have $F(C_i) \subseteq C_i$ if and only if $i = 3$, and so the Lefschetz and Nielsen number is 1.

(4) $(i_1, i_2) = (1, 3)$ or $(i_1, i_2) = (3, 2)$: We have $F(C_i) \subseteq C_i$ if and only if $i \in \{i_1, i_2\}$. Hence, the Lefschetz and Nielsen number is 2.

If some of the functions f_k are not self-maps, and so are not F, then one can still define the fixed point index on open subsets $U \subseteq H(X)$ of ANR-hyperspaces $H(X)$, provided the boundary ∂U of U is fixed point free with respect to F (see [1], [2], [4], [17]). Although this requirement can be expressed in locally compact spaces in terms of isolating neighbourhoods as in the Conley index theory for discrete (multivalued) dynamical systems (cf. [2], [17]), the computation of such an index can be, unlike the Lefschetz or Nielsen number for self-maps, a difficult task. On the other hand, one can use the Lefschetz number as the normalization property for such an index.

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