(APD)–PROPERTY OF C^* –ALGEBRAS BY EXTENSIONS OF C^* –ALGEBRAS WITH (APD)

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(Communicated by David R. Larson)

ABSTRACT. A unital C^* –algebra $\mathcal{A}$ is said to have the (APD)–property if every nonzero element in $\mathcal{A}$ has the approximate polar decomposition. Let $\mathcal{J}$ be a closed ideal of $\mathcal{A}$. Suppose that $\mathcal{J}$ and $\mathcal{A}/\mathcal{J}$ have (APD). In this paper, we give a necessary and sufficient condition that makes $\mathcal{A}$ have (APD). Furthermore, we show that if $\text{RR}(\mathcal{J}) = 0$ and $\text{tsr}(\mathcal{A}/\mathcal{J}) = 1$ or $\mathcal{A}/\mathcal{J}$ is a simple purely infinite C^* –algebra, then $\mathcal{A}$ has (APD).

1. INTRODUCTION

It is well known that every bounded linear operator on a Hilbert space has the polar decomposition, and moreover every nonzero element in a Von Neumann algebra has the polar decomposition. This property also holds for certain AW^* –algebras (cf. [4]). For a C^* –algebra $\mathcal{A}$, $a \in \mathcal{A} \setminus \{0\}$ can be written as $a = v|a|$, where $|a| = (a^*a)^{1/2}$ and v is a partial isometry in $\mathcal{A}''$ which is not in $\mathcal{A}$ generally. But if a is well supported, i.e., there is a projection $p \in \mathcal{A}$ such that $a = ap$ and $|a|$ is invertible in $p\mathcal{A}p$ with the inverse a_0 (cf. [1, Definition 6.5.3]), then $v = aa_0$ is a partial isometry in $\mathcal{A}$ and $a = v|a|$, $a(a_0v)a = a$, that is, a has the polar decomposition in $\mathcal{A}$. This leads us to seek a suitable definition so that every nonzero element in a C^* –algebra has a kind of polar decomposition. An approximate version of the polar decomposition of an element is as follows.

Let $\mathcal{A}$ be a unital C^* –algebra and $a \in \mathcal{A} \setminus \{0\}$. We call a the approximate polar decomposition (for short (APD)) if for any $\epsilon > 0$, there is a partial isometry v in $\mathcal{A}$ such that $\|a - v|a|\| < \epsilon$. We will say $\mathcal{A}$ has the (APD)–property if every nonzero element in $\mathcal{A}$ has the (APD). If $\mathcal{A}$ is nonunital and $\tilde{\mathcal{A}}$ has the (APD)–property, we also say $\mathcal{A}$ has the (APD)–property, where $\tilde{\mathcal{A}}$ is the unitarization of $\mathcal{A}$.

For a C^* –algebra $\mathcal{A}$, let $\text{Ws}(\mathcal{A})$ denote the set of all well–supported elements in $\mathcal{A}$ and put $\text{Rg}(\mathcal{A}) = \{a \in \mathcal{A} \setminus \{0\} \mid a \in a\mathcal{A}a\}$. Denote by $\text{Pis}(\mathcal{A})$ the set of all partial isometries and set $\text{ext}(\mathcal{A}) = \{v \in \text{Pis}(\mathcal{A}) \mid (1 - v^*v)\mathcal{A}(1 - vv^*) = \{0\}\}$. Recall from [3] (resp. [6]) that a unital C^* –algebra $\mathcal{A}$ is said to be (resp. have) extremally rich (resp. (WS)–property) if $\mathcal{A}_q^{-1} = \{xvy \mid v \in \text{ext}(\mathcal{A}), x, y \text{ invertible in } \mathcal{A}\}$ (resp.

Received by the editors February 15, 2005 and, in revised form, July 15, 2005.

2000 *Mathematics Subject Classification.* Primary 46L05, 46L85, 46L80.

Key words and phrases. Approximate polar decomposition, generalized inverse, (WS)–property.

This research was supported by the Natural Science Foundation of China and the Foundation of CSC.

$\text{Ws}(\mathcal{A})$) is dense in $\mathcal{A}$. If $\mathcal{A}$ is nonunital and $\overline{\widetilde{\mathcal{A}}_q^{-1}}$ (resp. $\overline{\text{Ws}(\widetilde{\mathcal{A}})} = \widetilde{\mathcal{A}}$, then we call $\mathcal{A}$ extremally rich (resp. has (WS)).

Let $\mathcal{A}$ be a unital C^* -algebra. Since $\mathcal{A}_q^{-1} \subset \text{Rg}(\mathcal{A}) = \text{Ws}(\mathcal{A})$ and $\mathcal{A}$ has (APD) iff $\mathcal{A}$ has (WS) (Proposition 2.1), it follows that if $\mathcal{A}$ is of real rank zero or extremally rich, $\mathcal{A}$ has (APD) (cf. [1, Proposition 6.5.4]). Moreover, by [6], when $\mathcal{A}$ has (APD), the real rank of $\mathcal{A}$ is no more than one, and if $\mathcal{J}$ is a hereditary C^* -subalgebra of $\mathcal{A}$, then $\mathcal{J}$ has (APD). In particular, if $\mathcal{J}$ is a closed ideal of $\mathcal{A}$, then $\mathcal{A}/\mathcal{J}$ also has (APD).

In order to enlarge this kind of C^* -algebra, we consider whether the (APD)-property can pass the extensions of C^* -algebras, that is, let $\mathcal{J}$ be a closed ideal of a unital C^* -algebra $\mathcal{A}$ and assume that $\mathcal{J}$ and $\mathcal{B} = \mathcal{A}/\mathcal{J}$ have (APD). Then under what conditions does $\mathcal{A}$ have (APD)? This is the first motivation to do this work. Let $\mathcal{J}$ be a nonunital separable simple C^* -algebra with real rank zero. It has been known that in many cases, the multiplier algebra $M(\mathcal{J})$ is not extremally rich even if $M(\mathcal{J})/\mathcal{J}$ is extremally rich (cf. [8]). This encourages us to seek a weaker condition than extremal richness on $M(\mathcal{J})$. The (APD)-property of $M(\mathcal{J})$ could be considered, and it will appear in another paper. But this consideration becomes our other motivation to do this work.

In what follows, $\mathcal{A}$ is a unital C^* -algebra, $\mathcal{J}$ is a closed ideal of $\mathcal{A}$ and let $\pi: \mathcal{A} \rightarrow \mathcal{A}/\mathcal{J} = \mathcal{B}$ be the canonical homomorphism. Assume that $\mathcal{J}$ and $\mathcal{B}$ have (APD). The following theorem gives a complete answer when $\mathcal{A}$ has (APD).

Theorem 1.1. *Assume that $\mathcal{J}$ and $\mathcal{B}$ have (APD). Then $\mathcal{A}$ has (APD) iff following two conditions are satisfied:*

- (1) *For each $v \in \text{Ais}(\mathcal{B})$ there is $w \in \text{Pis}(\mathcal{A})$ such that $\pi(w)v^*v = v$ and*
- (2) *$\text{Pis}(\mathcal{A}) + \mathcal{J} \subset \overline{\text{Ws}(\mathcal{A})}$,*

where $\text{Ais}(\mathcal{B}) = \{v \in \text{Pis}(\mathcal{B}) \mid b = v|b| \in \text{Ws}_0(\mathcal{B})\}$ and $\text{Ws}_0(\mathcal{B}) \subset \text{Ws}(\mathcal{B})$ such that $\text{Ws}_0(\mathcal{B})$ is dense in $\mathcal{B}$.

From Theorem 1.1, we can obtain an interesting result: if $\mathcal{J}$ is of real rank zero and $\mathcal{B}$ is of stable rank one or a simple purely infinite C^* -algebra, $\mathcal{A}$ has (APD).

2. PRELIMINARIES

Let $\mathcal{E}$ be a C^* -algebra. According to [5], $a \in \text{Rg}(\mathcal{E})$ iff 0 is an isolated point of $\sigma(a^*a)$ (or $0 \notin \sigma(a^*a)$ if $\mathcal{E}$ is unital) or iff $|a| \in \text{Rg}(\mathcal{E})$ or iff there is a unique element $d \in \mathcal{E}$ such that $ada = a$, $dad = d$, $(ad)^* = ad$, $(da)^* = da$. The element d is called the generalized inverse of a ; we denote it by a^+ . Consequently, if $a \in \text{Rg}(\mathcal{E})$ and put $v = a|a|^+$, $p = |a|^+|a|$, then $ap = a$, $a = v|a|$, $p|a| = |a|$, that is, a has the polar decomposition and $|a|$ is invertible in $p\mathcal{E}p$ with the inverse $|a|^+$. Thus, $\text{Ws}(\mathcal{E}) = \text{Rg}(\mathcal{E})$. Moreover, we have:

Proposition 2.1. *Let $\mathcal{E}$ be a unital C^* -algebra. Then $\mathcal{E}$ has (APD) iff $\mathcal{E}$ has (WS).*

Proof. If $\mathcal{E}$ has (APD), then for any $a \in \mathcal{E} \setminus \{0\}$ and any $\epsilon \in (0, 1)$, there is $v \in \text{Pis}(\mathcal{E})$ such that $\|a - v|a|\| < \frac{\epsilon}{2}$. Put $a_\epsilon = v(|a| + \frac{\epsilon}{2})$. Then $a_\epsilon \in \text{Rg}(\mathcal{E})$ ($a_\epsilon(|a| + \frac{\epsilon}{2})^{-1}v^*a_\epsilon = a_\epsilon$) and $\|a - a_\epsilon\| < \epsilon$, i.e., $\overline{\text{Ws}(\mathcal{E})} = \mathcal{A}$.

Now assume that $\mathcal{E}$ has (WS). Note that by [7, Lemma 2.5.11], there is $\delta > 0$ such that $\|x_1^{1/2} - x_2^{1/2}\| < \frac{\epsilon}{2}$ whenever $x_i \in \mathcal{E}_+$ with $\|x_i\| \leq 1$ and $\|x_1 - x_2\| < \delta$, $i = 1, 2$. Let $b \in \mathcal{E}$ with $\|b\| = \frac{1}{2}$. Choose $b_\epsilon \in \text{Ws}(\mathcal{E})$ such that $\|b - b_\epsilon\| < \min\left\{\frac{2\delta}{3}, \frac{\epsilon}{2}\right\}$.

Then $\|b_\epsilon\| \leq 1$ and $\| |b|^2 - |b_\epsilon|^2 \| < \delta$. It follows that $\| |b| - |b_\epsilon| \| < \frac{\epsilon}{2}$. Let $b_\epsilon = v_\epsilon |b_\epsilon|$ be the polar decomposition. We have $\|b - v_\epsilon |b|\| < \epsilon$. For any $a \in \mathcal{E} \setminus \{0\}$, put $b = \frac{a}{2\|a\|}$. Then $\|a - v_\epsilon |a|\| < 2\|a\|\epsilon$. So, $\mathcal{E}$ has (APD). $\square$

Let $\mathcal{E}$ be a C^* -algebra with unit 1. We write $U(\mathcal{E})$ to denote the unitary group and $U_0(\mathcal{E})$ to denote the connected component of 1 in $U(\mathcal{E})$. Recall that $\mathcal{E}$ is of stable rank one, denoted by $\text{tsr}(\mathcal{E}) = 1$ (resp. real rank zero, denoted by $\text{RR}(\mathcal{E}) = 0$), if every (resp. self-adjoint) element in $\mathcal{E}$ can be approximated by an invertible (resp. self-adjoint) element in $\mathcal{E}$. If $\mathcal{E}$ is nonunital and $\tilde{\mathcal{E}}$ is of stable rank one (resp. real rank zero), then we say $\text{tsr}(\mathcal{E}) = 1$ (resp. $\text{RR}(\mathcal{E}) = 0$).

Let p, q be two nonzero projections in a C^* -algebra $\mathcal{E}$ and $a, b \in \mathcal{E}$. Set

$$p\mathcal{E}q = \{pxq \mid x \in \mathcal{E}\}, \quad \text{Ws}(p\mathcal{E}q) = \{a \in p\mathcal{E}q \mid a^+ \text{ exists, } qa^+ = a^+p = a^+\}.$$

Write

$$\begin{aligned} a_{11} &= qap, \quad a_{12} = qa(1-p), \quad a_{21} = (1-q)ap, \quad a_{22} = (1-q)a(1-p), \\ b_{11} &= paq, \quad b_{12} = pb(1-q), \quad b_{21} = (1-p)bq, \quad b_{22} = (1-p)b(1-q). \end{aligned}$$

Then $a = a_{11} + a_{12} + a_{21} + a_{22}$, $b = b_{11} + b_{12} + b_{21} + b_{22}$. For convenience, we write a (resp. b) as a matrix $a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ (resp. $b = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$). It is easy to check that

$$ba = \begin{bmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} \end{bmatrix}.$$

Using the same method as in the proof of [7, Lemma 3.5.1], we have:

Lemma 2.2. *Let p, q be two nonzero projections in the C^* -algebra $\mathcal{E}$ and $a \in \mathcal{E}$. Put $x = paq$ and $\text{Her}(|x|) = \overline{|x|\mathcal{E}|x|}$. Let $x = v|x|$ be the polar decomposition of x in $\mathcal{E}''$. Then $pv = vq = v$, $bv^*v = b$ and $vb \in \mathcal{E}$ for every $b \in \text{Her}(|x|)$.*

The following corollary slightly improves [1, Proposition 6.5.4, Theorem 6.5.5].

Corollary 2.3. *Let p, q be two nonzero projections in the C^* -algebra $\tilde{\mathcal{E}}$ and $a \in \mathcal{E}$. Assume that $x = paq \neq 0$ and $\text{RR}(\text{Her}(|x|)) = 0$. Then for any $\epsilon > 0$, there is $y \in \text{Ws}(p\mathcal{E}q)$ such that $\|x - y\| < \epsilon$.*

Proof. Let $x = v|x|$ be the polar decomposition in $\mathcal{E}''$ and put $p_x = v^*v$. Then by Lemma 2.2, $pv = vq = v$, $p_x b = bp_x = b$ and $vb \in \mathcal{E}$ for any $b \in \text{Her}(|x|)$.

Since $\text{RR}(\text{Her}(|x|)) = 0$, it follows from [2, Theorem 2.6] that there are $\lambda_1, \dots, \lambda_s \in \mathbb{R} \setminus \{0\}$ and mutually orthogonal projections $q_1, \dots, q_s$ in $\text{Her}(|x|)$ ($q_i \leq q$) such that $\| |x| - \sum_{i=1}^s \lambda_i q_i \| < \epsilon$. Set $y = v \left(\sum_{i=1}^s \lambda_i q_i \right)$, $y_0 = \left(\sum_{i=1}^s \lambda_i^{-1} q_i \right) v^*$. Then $y, y_0 \in \mathcal{E}$, $\|x - y\| < \epsilon$ and $y_0 = y^+$ and moreover, $py = yq = y$, $qy_0 = y_0p = y_0$, i.e., $y \in \text{Ws}(p\mathcal{E}q)$. $\square$

Proposition 2.4. *If $\mathcal{J}$ is of real rank zero, then $\text{Pis}(\mathcal{A}) + \mathcal{J} \subset \overline{\text{Ws}(\mathcal{A})}$.*

Proof. Let v be a partial isometry in $\mathcal{A}$. Put $p = v^*v$, $q = vv^*$ and

$$\begin{aligned} b_{11} &= q(v+k)p = qv(p+v^*k)p, \quad b_{12} = q(v+k)(1-p) = qk(1-p), \\ b_{21} &= (1-q)(v+k)p = (1-q)kp, \quad b_{22} = (1-q)(v+k)(1-p) = (1-q)k(1-p). \end{aligned}$$

Since for any nonzero projections $p_1, p_2 \in \mathcal{A}$ and any nonzero element $x \in p_1 \mathcal{J} p_2$, $\text{Her}(|x|)$ is a hereditary C^* -subalgebra of $\mathcal{J}$, it follows from [2, Corollary 2.8] and Corollary 2.3 that $\text{Ws}(p_1 \mathcal{J} p_2)$ is dense in $p_1 \mathcal{J} p_2$. Noting that $\text{RR}(p \mathcal{J} p) =$

$\text{RR}(\mathcal{J}) = 0$ and $\text{Her}(|p + v^*kp|)$ is a hereditary C^* -subalgebra of $p\widetilde{\mathcal{J}}p = \widetilde{p\mathcal{J}p}$, we have $\text{RR}(\text{Her}(|p + v^*kp|)) = 0$ by [2, Corollary 2.8], and hence there is $k_{11} \in \text{Ws}(p\widetilde{\mathcal{J}}p)$ such that $\|p + v^*k - k_{11}\| < \frac{\epsilon}{4}$ by Corollary 2.3. Put $a_{11} = vk_{11}$. Then $a_{11} \in \text{Ws}(q\mathcal{A}p)$ for $a_{11}^+ = k_{11}^+v^*$ satisfying $pa_{11}^+ = a_{11}^+q = a_{11}^+$. If we write

$$v + k = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \quad x_0 = \begin{bmatrix} a_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},$$

then $\|v + k - x_0\| < \frac{\epsilon}{4}$. Note that x_0 can be decomposed as

$$(2.1) \quad x_0 = \begin{bmatrix} q & 0 \\ b_{21}a_{11}^+ & 1 - q \end{bmatrix} \begin{bmatrix} a_{11} & (q - a_{11}a_{11}^+)b_{12} \\ b_{21}(p - a_{11}^+a_{11}) & b_{22} - b_{21}a_{11}^+b_{12} \end{bmatrix} \begin{bmatrix} p & a_{11}^+b_{12} \\ 0 & 1 - p \end{bmatrix}.$$

Now put $b'_{12} = (q - a_{11}a_{11}^+)b_{12}$, $b'_{21} = b_{21}(p - a_{11}^+a_{11})$, $b'_{22} = b_{22} - b_{21}a_{11}^+b_{12}$. Then there exist $a_{12} \in \text{Ws}((q - a_{11}a_{11}^+)\mathcal{J}(1 - p))$, $a_{21} \in \text{Ws}((1 - q)\mathcal{J}(p - a_{11}^+a_{11}))$ such that $\|b'_{12} - a_{12}\|$ and $\|b'_{21} - a_{21}\|$ are all less than $\frac{\epsilon}{4\|1 + b_{21}a_{11}^+\|\|1 + a_{11}^+b_{12}\|}$. Put $x_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & b'_{22} \end{bmatrix}$. Then by (2.1),

$$(2.2) \quad \left\| x_0 - \begin{bmatrix} q & 0 \\ b_{21}a_{11}^+ & 1 - q \end{bmatrix} x_1 \begin{bmatrix} p & a_{11}^+b_{12} \\ 0 & 1 - p \end{bmatrix} \right\| < \frac{\epsilon}{2}.$$

Decompose x_1 as

$$x_1 = \begin{bmatrix} q & 0 \\ b'_{22}a_{12}^+ & 1 - q \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & d \end{bmatrix} \begin{bmatrix} p & a_{21}^+b'_{22}(1 - p - a_{12}^+a_{12}) \\ 0 & 1 - p \end{bmatrix},$$

where $d = (1 - q - a_{21}a_{21}^+)b'_{22}(1 - p - a_{12}^+a_{12})$. Pick

$$a_{22} \in \text{Ws}((1 - q - a_{21}a_{21}^+)\mathcal{J}(1 - p - a_{12}^+a_{12}))$$

such that

$$\|d - a_{22}\| < \frac{\epsilon}{4\|1 + b_{21}a_{11}^+\|\|1 + a_{11}^+b_{12}\|\|1 + b'_{22}a_{12}^+\|\|1 + a_{21}^+b'_{22}(1 - p - a_{12}^+a_{12})\|}.$$

Set $x_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and

$$x = \begin{bmatrix} q & 0 \\ b_{21}a_{11}^+ & 1 - q \end{bmatrix} \begin{bmatrix} q & 0 \\ b'_{22}a_{12}^+ & 1 - q \end{bmatrix} x_2 \begin{bmatrix} p & a_{21}^+b'_{22}(1 - p - a_{12}^+a_{12}) \\ 0 & 1 - p \end{bmatrix} \begin{bmatrix} p & a_{11}^+b_{12} \\ 0 & 1 - p \end{bmatrix}.$$

Then $\|v + k - x\| < \epsilon$ by (2.2). Noting that $a_{11}^*a_{12} = a_{11}^*(q - a_{11}a_{11}^+)a_{12} = 0$ and $a_{21}^*a_{22} = 0$, we get that

$$x_2^*x_2 = \begin{bmatrix} a_{11}^*a_{11} + a_{21}^*a_{21} & 0 \\ 0 & a_{12}^*a_{12} + a_{22}^*a_{22} \end{bmatrix}.$$

But we also have $(a_{11}^*a_{11})(a_{21}^*a_{21}) = 0$, $(a_{12}^*a_{12})(a_{22}^*a_{22}) = 0$, so

$$\sigma(x_2^*x_2) = \sigma(a_{11}^*a_{11}) \cup \sigma(a_{12}^*a_{12}) \cup \sigma(a_{21}^*a_{21}) \cup \sigma(a_{22}^*a_{22}).$$

This indicates that $x_2 \in \text{Ws}(\mathcal{A})$. Therefore, $x \in \text{Ws}(\mathcal{A})$. $\square$

The following lemma comes from [7, Lemma 2.5.1, Lemma 2.5.4].

Lemma 2.5. *Let p be a projection in a C^* -algebra $\mathcal{E}$.*

- (1) *If there is a projection $q \in \mathcal{E}$ such that $\|p - q\| < 1$, then there is a unitary element $u \in \mathcal{E}$ such that $q = upu^*$ and $\|1 - u\| < \sqrt{2}\|p - q\|$.*
- (2) *If there is an $a \in \mathcal{E}_{sa}$ such that $\|a - p\| < \frac{1}{2}$, then there is a projection q in the C^* -subalgebra generated by a such that $\|p - q\| < 2\|a - p\|$.*

3. PROOF OF THEOREM 1.1

Proof of Theorem 1.1. ($\Leftarrow$) Let $x \in \mathcal{A}$. Then for any $\epsilon > 0$, there is $y \in \text{Ws}_0(\mathcal{B})$ such that $\|\pi(x) - y\| < \frac{\epsilon}{3}$. Let $y = v|y|$, $v \in \text{Ais}(\mathcal{B})$. By hypotheses, we have a partial isometry $w \in \mathcal{A}$ such that $\pi(w)v^*v = v$. Pick $a \in \mathcal{A}_+$ with $\pi(a) = |y|$. Noting that $v^*v|y| = |y|$, we have $\pi(wa) = \pi(w)v^*v|y| = v|y| = y$. Put $x_0 = w(a + \frac{\epsilon}{3})$. Then $\|\pi(x - x_0)\| < \frac{2\epsilon}{3}$ and $x_0 \in \text{Ws}(\mathcal{A})$. Thus, we can find an element $k \in \mathcal{J}$ such that $\|x - x_0 - k\| < \frac{2\epsilon}{3}$. Applying Condition (2) to $w + k(a + \frac{\epsilon}{3})^{-1}$, we obtain $z_\epsilon \in \text{Ws}(\mathcal{A})$ such that

$$\|w + k(a + \frac{\epsilon}{3})^{-1} - z_\epsilon\| < \frac{\epsilon}{3\|a + \frac{\epsilon}{3}\|}.$$

Put $x_\epsilon = z_\epsilon(a + \frac{\epsilon}{3})$. Then $x_\epsilon \in \text{Ws}(\mathcal{A})$ and $\|x - x_\epsilon\| < \epsilon$.

($\Rightarrow$) Let v be in $\text{Ais}(\mathcal{B})$. Then there is $x \in \text{Ws}(\mathcal{A})$ such that $\|\pi(x) - v\| < \frac{1}{100}$. Let $x = u|x|$ be the polar decomposition and set $p' = \pi(u^*u)$, $q' = \pi(ux)$, $p = v^*v$, $q = vv^*$, $b = \pi(|x|)$. Simple computation shows that

$$(3.1) \quad \|p - b^2\| < \frac{3}{100}, \quad \|(1 - p')p\| < \frac{3}{100}, \quad \|q - \pi(u)p\pi(u^*)\| < \frac{6}{100}.$$

Note that $\|b\| < 2$. Thus, from (3.1), we get that

$$\|pb - bp\| < \frac{6\|b\|}{100} < \frac{12}{100}, \quad \|pbpbp - p\| < \frac{27}{100},$$

so that $\|pbp - p\| < \frac{27}{100}$ by spectral theorem. Furthermore, $\|bp - p\| < \frac{39}{100}$ and $\|v - \pi(u)p\| = \|(v - \pi(u)b)p - \pi(u)(p - bp)\| < \frac{40}{100}$.

Now by (3.1), $\|p - p'pp'\| < \frac{6}{100}$. Thus, by Lemma 2.5(2), there are a projection $p'_0 \in p'\mathcal{B}p'$ and a unitary element $v_0 \in U_0(\mathcal{B})$ such that

$$\|p - p'_0\| < 2\|p - p'pp'\| < \frac{12}{100}, \quad \|1 - v_0\| < \frac{12\sqrt{2}}{100} \text{ and } p = v_0^*p'_0v_0.$$

Choose $u_0 \in U_0(\mathcal{A})$ such that $\pi(u_0) = v_0$. Put $w_0 = uu_0$. Then $w_0 \in \text{Pis}(\mathcal{A})$ and $p \leq v_0^*p'_0v_0 = \pi(w_0^*w_0)$. From (3.1), we get that

$$\begin{aligned} \|q - \pi(w_0)p\pi(w_0^*)\| &= \|q - \pi(u)p'_0\pi(u^*)\| \\ &\leq \|q - \pi(u)p\pi(u^*)\| + \|p - p'_0\| < \frac{18}{100}. \end{aligned}$$

Since $\pi(w_0)p\pi(w_0^*)$ is a projection, it follows from Lemma 2.5(1) that there is $v_1 \in U_0(\mathcal{B})$ with $\|1 - v_1\| < \frac{18\sqrt{2}}{100}$ such that $q = v_1\pi(w_0)p\pi(w_0^*)v_1^*$. Pick $u_1 \in U(\mathcal{A})$ with $\pi(u_1) = v_1$ and set $w_1 = u_1w_0$. Then $w_1 \in \text{Pis}(\mathcal{A})$ and $q = \pi(w_1)p\pi(w_1^*)$. Therefore, $q\pi(w_1) = \pi(w_1)p$ and $v^*\pi(w_1)$ is unitary in $p\mathcal{B}p$. Since

$$\begin{aligned} \|p - v^*\pi(w_1)p\| &\leq \|v - q\pi(w_1)p\| = \|v - \pi(w_1)p\| \\ &\leq \|v - \pi(u)p\| + \|(v_1 - 1)\pi(u)v_0p\| + \|\pi(u)(v_0 - 1)p\| \\ &< \frac{40}{100} + \frac{12\sqrt{2}}{100} + \frac{18\sqrt{2}}{100} < 1, \end{aligned}$$

we have $v^*\pi(w_1)p + 1 - p \in U_0(\mathcal{B})$. Consequently, there is $u_2 \in U(\mathcal{A})$ such that $\pi(u_2) = v^*\pi(w_1)p + 1 - p$. So, $p\pi(u_2) = \pi(u_2)p$. Put $w = w_1u_2^*$. Then $w \in \text{Pis}(\mathcal{A})$ and $v = \pi(w)p$. This proves Condition (1). Condition (2) is obvious since $\overline{\text{Ws}(\mathcal{A})} = \mathcal{A}$. The proof is finished. $\square$

Corollary 3.1. *Suppose that $\text{RR}(\mathcal{J}) = 0$. If $\mathcal{B}$ satisfies one of the following conditions, then $\mathcal{A}$ has (APD):*

- (1) $\mathcal{B}$ has (APD) and every projection in $\mathcal{B}$ can be lifted to a projection in $\mathcal{A}$;
- (2) the set of all one-sided invertible elements in $\mathcal{B}$ is dense in $\mathcal{B}$, especially, $\text{tsr}(\mathcal{B}) = 1$ or $\mathcal{B}$ is simple purely infinite;
- (3) $\mathcal{B}$ is extremally rich and $\text{ext}(\mathcal{B}) \subset \pi(\text{Pis}(\mathcal{A}))$.

Proof. (1) Let $v \in \text{Ais}(\mathcal{B})$. It is well known that if v^*v or vv^* can be lifted to a projection in $\mathcal{A}$, then there is $w \in \text{Pis}(\mathcal{A})$ such that $\pi(w) = v$. Therefore $\mathcal{A}$ has (APD) by Theorem 1.1 and Proposition 2.4.

(2) Since the isometry (or coisometry or unitary) in $\mathcal{B}$ can be lifted to a partial isometry in $\mathcal{A}$ and $\text{Ais}(\mathcal{B})$ consists of isometries or coisometries in $\mathcal{B}$ by assumption, it follows from Theorem 1.1 and Proposition 2.4 that $\mathcal{A}$ has (APD).

If $\text{tsr}(\mathcal{B}) = 1$, then $\text{Ais}(\mathcal{B}) = U(\mathcal{B})$; if $\mathcal{B}$ is a simple purely infinite C^* -algebra, then the set of all one-sided invertible elements is dense in $\mathcal{B}$ by [9, Theorem 4.5, Theorem 3.3]. Thus, $\mathcal{A}$ has (APD).

(3) Let $b \in \text{Ws}_0(\mathcal{B})$ and $b = v|b|$ be the polar decomposition. Since $\overline{\mathcal{B}_q^{-1}} = B$, it follows from [3, Proposition 2.6] that there is $u \in \text{ext}(\mathcal{B})$ such that $b = u|b|$. The assertion follows from the proof of Theorem 1.1 and Proposition 2.4. $\square$

Corollary 3.1 shows that if $\text{RR}(\mathcal{J}) = 0$ and $\text{tsr}(\mathcal{B}) = 1$, then $\mathcal{A}$ has (APD). A natural problem is when $\mathcal{A}$ is extremally rich. We will use K -theory data to describe the extremal richness of $\mathcal{A}$ in Proposition 3.2. Some K -theory notations used in the following may be found in [1] or [7].

Let $\mathcal{E}$ be a C^* -algebra and let τ be a quasitrace on $\mathcal{E}$. Let τ_n be a quasitrace on the matrix algebra $M_n(\mathcal{E})$ given by $\tau_n((a_{ij})_{n \times n}) = \sum_{i=1}^n \tau(a_{ii})$. Thus τ induces a homomorphism $\tau_*: K_0(\mathcal{E}) \rightarrow \mathbb{R}$ defined by $\tau_*(x) = \tau_n(p) - \tau_m(q)$, for $x = [p] - [q] \in K_0(\mathcal{E})$, where p (resp. q) is a projection in $M_n(\mathcal{E})$ (resp. $M_m(\mathcal{E})$).

Proposition 3.2. *Let $\mathcal{J}$ be a σ -unital and simple C^* -algebra with $\text{RR}(\mathcal{J}) = 0$ and $\text{tsr}(\mathcal{J}) = \text{tsr}(\mathcal{A}/\mathcal{J}) = 1$. Suppose that there is a quasitrace τ on $\mathcal{J}$ such that for any projections p_1, p_2 in $\mathcal{J}$, if $\tau(p_1) < \tau(p_2)$, then p_1 is equivalent to a projection $q_2 \leq p_2$ in $\mathcal{J}$. Then $\mathcal{A}$ is extremally rich iff $\text{Ker } \tau_* \cap \text{Im } \partial = \{0\}$, where $\partial: K(\mathcal{A}/\mathcal{J}) \rightarrow K_0(\mathcal{J})$ is the index map.*

Proof. By Corollary 3.1(2), $\mathcal{A}$ has (APD). Since $\text{tsr}(\mathcal{J}) = \text{tsr}(\mathcal{A}/\mathcal{J}) = 1$, it follows from [3, Corollary 6.3] that $\mathcal{A}$ is extremally rich iff for each $v \in U(\mathcal{A}/\mathcal{J})$, there is $w \in \text{ext}(\mathcal{A})$ such that $\pi(w) = v$.

Now suppose that $\text{Ker } \tau_* \cap \text{Im } \partial = \{0\}$. Let v be in $U(\mathcal{A}/\mathcal{J})$. Then there is $v \in \text{Pis}(\mathcal{A})$ such that $\pi(v) = u$ and $\partial([u]) = [1 - v^*v] - [1 - vv^*]$. So if $\tau(1 - v^*v) = \tau(1 - vv^*)$, then $[1 - v^*v] = [1 - vv^*]$. Since $\text{tsr}(\mathcal{J}) = 1$, we can find $z_0 \in \mathcal{J}$ such that $1 - v^*v = z_0^*z_0$, $1 - vv^* = z_0z_0^*$. Put $w = u + z_0$. Then $w \in U(\mathcal{A})$ and $\pi(w) = u$. If $\tau(1 - v^*v) > \tau(1 - vv^*)$, there is $z_1 \in \mathcal{J}$ such that $z_1z_1^* = 1 - vv^*$, $z_1^*z_1 \leq 1 - v^*v$ by the assumption of τ . Put $w_1 = v + z_1$. Then w_1 is a coisometry in $\mathcal{A}$ and $\pi(w_1) = u$; similarly, if $\tau(1 - v^*v) < \tau(1 - vv^*)$, we can find an isometry w_2 in $\mathcal{A}$ such that $\pi(w_2) = v$. Therefore, $\mathcal{A}$ is extremally rich.

Conversely, if $\text{Ker } \tau_* \cap \text{Im } \partial \neq \{0\}$, then there is $y \in K_1(\mathcal{A}/\mathcal{J})$ such that $\partial(y) \neq 0$ and $\tau_*(\partial(y)) = 0$. $\text{tsr}(\mathcal{A}/\mathcal{J}) = 1$ implies that there is $u \in U(\mathcal{A}/\mathcal{J})$ such that $x = \partial(y) = \partial([u])$ (cf. [7, Theorem 3.1.11] or [10, Corollary 2.2]). Let $w \in \text{Pis}(\mathcal{A})$

such that $\pi(w) = u$. Then $x = [1 - w^*w] - [1 - ww^*]$ in $K_0(\mathcal{J})$ and $\tau(1 - w^*w) = \tau(1 - ww^*)$. Consequently, $w^*w \neq 1$ and $ww^* \neq 1$. Since $\mathcal{J}$ is simple, there exist $k_1, \dots, k_n \in \mathcal{J}$ such that $1 - ww^* = \sum_{i=1}^n k_i^*(1 - w^*w)k_i$ by [7, Lemma 3.3.6]. Clearly, $w \notin \text{ext}(\mathcal{A})$, so $\mathcal{A}$ is not extremally rich. $\square$

ACKNOWLEDGMENT

The author would like to thank Professor Larson and the referee for their helpful suggestions.

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