PERSISTENCE OF FLOQUET INVARIANT TORI FOR A CLASS OF NON-CONSERVATIVE DYNAMICAL SYSTEMS

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ABSTRACT. In this paper we consider a class of non-conservative dynamical system with small perturbation. By the KAM method we prove existence of Floquet invariant tori under the weakest non-resonant conditions.

1. INTRODUCTION AND MAIN RESULTS

With the development of KAM theory, there are many results (called KAM theorems) about persistence of invariant tori of integrable systems under small perturbation for some conservative systems such as hamiltonian systems and reversible systems. More recently, the classical KAM theorems have been improved greatly in weaker non-degenerate conditions and fewer non-resonant conditions. The weakest non-degenerate condition is Rüssmann's non-degenerate condition (see [7, 11, 12]). Regarding weaker non-resonant conditions, in [13] we proved a KAM theorem for hamiltonian systems only under the first Melnikov's non-resonant condition (1.2). The result implies that the second Melnikov's non-resonant condition (1.3) is not necessary for the classical KAM theorems. We also obtained a similar result for reversible systems [14]. In this paper we want to generalize these results to a class of non-conservative systems.

Consider the following dynamical system:

$$(1.1) \quad \begin{cases} \dot{x} &= \xi + P_1(x, y; \xi), \\ \dot{y} &= \Omega(\xi)y + P_2(x, y; \xi), \end{cases}$$

where $(x, y) \in R^n/2\pi Z^n \times R^m$, $\xi \in O \subset R^n$ is a frequency parameter, Ω is a matrix depending on ξ , and P_1 and P_2 are small perturbations.

If $P_1 = 0, P_2 = 0$, then the system (1.1) is integrable and for each $\xi \in O$ has an invariant torus $T^n \times \{0\}$, on which there exists a linear flow $x = \xi t + x_0, y = 0$. We want to know whether the system (1.1) still has many invariant tori if P_1 and P_2 are sufficiently small.

Moser [1] studied a system of the form (1.1) by the rapidly convergent method, and the result can be applied to both hamiltonian systems and reversible systems. By the usual KAM method, Broer, Huitema and Sevryuk [2] proved that if ξ and

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the eigenvalues of Ω satisfy the following non-resonant conditions:

$$(1.2) \quad i\langle \xi, k \rangle - \Omega_j(\xi) \neq 0, \quad \forall k \in Z^n, |k| \leq M+1, 1 \leq j \leq m,$$

$$(1.3) \quad i\langle \xi, k \rangle + \Omega_i(\xi) - \Omega_j(\xi) \neq 0, \quad \forall |k| + |i-j| \neq 0, |k| \leq 2M+1,$$

where $i = \sqrt{-1}$, $\Omega_1(\xi), \Omega_2(\xi), \dots, \Omega_m(\xi)$ are the eigenvalues of Ω and $M = \max_{1 \leq j \leq m} \sup_{\xi \in O} |\partial \Omega_j / \partial \xi|$, then, if P_1 and P_2 are sufficiently small, most of the invariant tori can persist. In the recent paper [3], Bambusi considered non-hamiltonian perturbation of integrable systems and proved persistence of hyperbolic invariant tori.

By motivation of these works, we want to consider the persistence of invariant tori under a weaker non-resonant condition. We will show that if the first Melnikov's non-resonance condition (1.2) holds, then the system (1.1) persists many invariant tori when P_1 and P_2 are sufficiently small.

Before stating our results, we first give some notation and assumptions. Denote a complex neighborhood of $T^n \times \{0\}$ by

$$D(s, r) = \{(x, y) \in C^n / 2\pi Z^n \times C^m \mid |\operatorname{Im} x| \leq s, |y| \leq r\},$$

where $|\operatorname{Im} x| = \max_{1 \leq i \leq n} |\operatorname{Im} x_i|$, $|y| = \max_{1 \leq i \leq m} |y_i|$.

Let O be a bounded connected closed domain of R^n . Let $C^L(O)$ be the space of the L -th continuously differentiable function on O . For $f(\xi) \in C^L(O)$, define $\|f\|^L = \max_{|\alpha| \leq L} \sup_{\xi \in O} |\partial_\xi^\alpha f(\xi)|$.

Let $f(x, y; \xi)$ be analytic in (x, y) on $D(s, r)$ and let it belong to $C^L(O)$ in ξ . Then we have

$$f = \sum_{k, l} f_{k, l}(\xi) e^{i\langle k, x \rangle} y^l.$$

Define

$$\|f\|_{D(s, r)}^L = \sup_{|y| \leq r} \left| \sum_{k, l} \|f_{k, l}(\xi)\|^L e^{s|k|} y^l \right|.$$

Theorem 1.1. *Suppose $\Omega = \operatorname{diag}(\Omega_1(\xi), \Omega_2(\xi), \dots, \Omega_m(\xi))$ is analytic on O and the non-resonant condition (1.2) holds. Also suppose that P_1 and P_2 are analytic in $(x, y) \in D(s, r)$ and belong to $C^L(O)$ in ξ , where $L = m^2$. For sufficiently small $\alpha > 0$, there exists an $\epsilon > 0$ depending on Ω, α, O, n, m such that if*

$$(1.4) \quad \|P_1\|_{D(s, r)}^L \leq \epsilon, \quad \frac{1}{r} \|P_2\|_{D(s, r)}^L \leq \epsilon,$$

then there exists a non-empty subset O_ of O , and for each $\xi \in O_*$ there exists an analytic transformation¹*

$$\Phi_*(\cdot, \cdot; \xi) : D_{s/2, r/2} \rightarrow D_{s, r}$$

such that the system (1.1) is changed to

$$(1.5) \quad \dot{x} = \omega_* + P_{*1}, \quad \dot{y} = (\Omega + A_*)y + P_{*2},$$

where A_ is a constant matrix, and P_{*1} and P_{*2} satisfy*

$$P_{*1}(x, 0; \xi) = P_{*2}(x, 0; \xi) = 0, \quad \partial_y P_{*2}(x, y; \xi)|_{y=0} = 0.$$

Hence, $\forall \xi \in O_$, $\Phi_*(T^n \times \{0\}; \xi)$ is an invariant torus of the system (1.1) with the frequency ω_* . ω_* and A_* satisfy $\|\omega_*(\xi) - \xi\|^L \leq 2\epsilon$ and $\|A_*\|^L \leq 2\epsilon$. Moreover, we have $\operatorname{mes}(O \setminus O_*) \rightarrow 0$ as $\alpha \rightarrow 0$.*

¹ Φ_* is analytic in (x, y) and Whitney smooth in ξ .

The proof of Theorem 1.1 is based on the following theorem.

Theorem 1.2. *Let $\Omega(\xi) = \text{diag}(\Omega_1 I_{d_1}, \Omega_2 I_{d_2}, \dots, \Omega_m I_{d_m})$, where I_{d_j} is a d_j -unit matrix for $j = 1, 2, \dots, m$. Let $L = (\max_{1 \leq i \leq m} d_i)^2$. Suppose $\Omega_j(\xi)$ ($1 \leq j \leq m$) are analytic on O and satisfy the non-resonant conditions (1.2) and (1.3). Then the result of Theorem 1.1 holds true.*

Remark 1.3. In [1] Moser considered a modified system of (1.1) so that the frequency can be fixed in KAM iteration. In this case, the transformation is analytic with respect to the frequency parameter, and the invariant tori belong to the modified system. Here we directly consider the system (1.1). So we must adjust the frequency parameter for small divisor conditions in KAM iteration. Thus, we cannot obtain the analyticity of transformation with respect to frequency parameter. However, the persisting invariant tori are for the original system.

Remark 1.4. In Theorem 1.1, if all Ω_j have non-zero real parts, this corresponds to the case of hyperbolic invariant tori and can be dealt in the same way as in hamiltonian system (see [3, 4]). So, we are really interested in the case that Ω_j are pure imaginary numbers.

2. PROOF OF THEOREMS

The difference between Theorem 1.1 and Theorem 1.2 is in normal frequencies. In Theorem 1.1, only condition (1.2) is required, while in Theorem 1.2, both (1.2) and (1.3) are needed. We will first prove that the case of Theorem 1.1 can be reduced to the special case of Theorem 1.2. Thus, the proof of Theorem 1.1 is reduced to that of Theorem 1.2. Then, by KAM method we prove Theorem 1.2.

Proof of Theorem 1.1. Below we prove that under the assumptions of Theorem 1.1 the normal frequencies can be changed by a transformation such that the new normal frequencies satisfy both the non-resonant conditions (1.2) and (1.3). As in [13, 14], we divide $\Omega_1, \Omega_2, \dots, \Omega_m$ into some groups by a resonant relation. If there exists $k_{ij} \in Z^n$ such that $\Omega_j(\xi) - \Omega_i(\xi) = \sqrt{-1}\langle \xi, k_{ij} \rangle$, $\forall \xi \in O$, we say that Ω_j is resonant with Ω_i on O . Since the normal frequencies are analytic on the connected domain O , if Ω_i and Ω_j are not resonant at one point, then they are not resonant for almost every point on O .

We re-arrange $\Omega_1, \Omega_2, \dots, \Omega_m$ to put together the resonant normal frequencies on O and denote them by

$$\Omega_1^1, \dots, \Omega_{d_1}^1, \Omega_1^2, \dots, \Omega_{d_2}^2, \dots, \Omega_1^{\bar{m}}, \dots, \Omega_{d_{\bar{m}}}^{\bar{m}}$$

such that

$$\Omega_j^i - \Omega_1^i = \sqrt{-1}\langle \xi, k_{ij} \rangle, \forall \xi \in O, \forall 1 \leq j \leq d_i, \forall 1 \leq i \leq \bar{m},$$

and

$$\Omega_1^i - \Omega_1^j \neq \sqrt{-1}\langle \xi, k \rangle, \forall i \neq j, \forall k \in Z^n.$$

We also re-arrange $y_1, y_2, \dots, y_m$ correspondingly and denote them by y_j^i . Let $y^i = (y_1^i, \dots, y_{d_i}^i)$, $1 \leq i \leq \bar{m}$, and $y = (y^1, y^2, \dots, y^{\bar{m}})$.

Now define a transformation by

$$x_+ = x, \quad y_{+j}^i = e^{\sqrt{-1}\langle x, k_{ij} \rangle} y_j^i, \quad 1 \leq i \leq \bar{m}, \quad 1 \leq j \leq d_i.$$

Under this transformation, the system (1.1) is changed to

$$\dot{x}_+ = \xi + P_{+1}, \quad \dot{y}_+ = \Omega_+(x)y_+ + P_{+2},$$

where

$$\Omega_+(\xi) = \text{diag}(\Omega_1^1 I_{d_1}, \Omega_1^2 I_{d_2}, \dots, \Omega_1^{\tilde{m}} I_{d_{\tilde{m}}}).$$

From the above we know that the resonant normal frequencies are changed to be multiple ones by the transformation, so that the non-resonant condition (1.3) holds. Obviously, the non-resonance condition (1.2) still holds after transformation. Thus, the non-resonant condition (1.2) and (1.3) both hold for the new normal frequencies Ω_+ .

Proof of Theorem 1.2. The proof of Theorem 1.2 is based on a KAM iteration, which divides into several steps.

A. Outline of the KAM step. At each step we consider the following system:

$$(2.1) \quad \begin{cases} \dot{x} &= \omega(\xi) + P_1(x, y; \xi), \\ \dot{y} &= (\Omega(\xi) + A(\xi))y + P_2(x, y; \xi), \end{cases}$$

where $A = \text{diag}(A_1, A_2, \dots, A_m)$ with A_j being a $d_j \times d_j$ matrix. Suppose P_1 and P_2 are analytic on $D(s, r)$ in (x, y) and belong to $C^L(O)$ in ξ . Moreover,

$$(2.2) \quad \|\omega - \xi\|^L \leq 2\epsilon_0, \quad \|A\|^L \leq 2\epsilon_0, \quad \|P_1\|_{D(s,r)}^L \leq \epsilon, \quad \|P_2\|_{D(s,r)}^L \leq r\epsilon.$$

We want to find a transformation near the identity map and change the system (2.1) to

$$(2.3) \quad \begin{cases} \dot{x} &= \omega_+(\xi) + P_{+1}(x, y; \xi), \\ \dot{y} &= (\Omega(\xi) + A_+(\xi))y + P_{+2}(x, y; \xi), \end{cases}$$

where P_{+1} and P_{+2} are much smaller perturbations, and ω_+ and A_+ are the corrections of ω and A , respectively.

B. Construction of a transformation. Define a transformation $\Phi : (x_+, y_+) \rightarrow (x, y)$ by

$$x = x_+ + h(x_+), \quad y = (I_{\tilde{m}} + a(x_+))y_+ + b(x_+),$$

where $h(x)$, $b(x)$ are n -, $\tilde{m}$ -dimensional vector functions, respectively, and $a(x)$ is an $\tilde{m} \times \tilde{m}$ -matrix function with $\tilde{m} = d_1 + d_2 + \dots + d_m$. Let $P_1^0 = P_1|_{y=0}$, $\omega_+ = \omega + [P_1^0]$, $P_2^0 = P_2|_{y=0}$, $P_2^1 = \partial_y P_2|_{y=0}$, where $[P_1^0]$ is the average of P_1^0 with respect to x on T^n . Define an operator ∂_ω by $\partial_\omega f = \sum_k i\langle k, \omega \rangle f_k e^{i\langle k, x \rangle}$, where $f = \sum_k f_k e^{i\langle k, x \rangle}$. If $f = (f_{ij})$ is a matrix, define $\partial_\omega f = (\partial_\omega f_{ij})$. Let $[P_2^1] = (p_{ij})_{1 \leq i, j \leq m}$ be the average of P_2^1 , where p_{ij} is a $d_i \times d_j$ matrix. Let $\hat{A} = \text{diag}(p_{11}, p_{22}, \dots, p_{mm})$ and $A_+ = A + \hat{A}$. Under the transformation Φ , the system (2.1) is changed to

$$(2.4) \quad \begin{cases} \dot{x} = \omega_+ + (I_n + \mathcal{D}_x h)^{-1}(P_1^0 - [P_1^0] - \partial_\omega h) + P_{+1}, \\ \dot{y} = (\Omega + A_+)y + (I_{\tilde{m}} + a)^{-1}[(\Omega + A)b - \partial_\omega b + P_2^0 \\ \quad + (\Omega + A)ay - a(\Omega + A)y - \partial_\omega ay + (P_2^1 - \hat{A})y] + P_{+2}, \end{cases}$$

where

$$(2.5) \quad P_{+1} = (I_n + \mathcal{D}_x h)^{-1} \left(P_1 - P_1^0 - \mathcal{D}_x h[P_1^0] + \int_0^1 \langle \nabla_z P_1(z + t\delta z), \delta z \rangle dt \right)$$

and

$$(2.6) \quad \begin{aligned} P_{+2} = & -(I_{\bar{m}} + a)^{-1} (\partial_{P_{+1}+[P_1^0]} ay + \partial_{P_{+1}+[P_1^0]} b) \\ & + (I_{\bar{m}} + a)^{-1} (P_2 - P_2^0 - P_2^1 y - a \hat{A} y \\ & + \int_0^1 \langle \nabla_z P_2(z + t \delta z), \delta z \rangle dt), \end{aligned}$$

where $\mathcal{D}_x h$ is the Jacobian matrix of h with respect to x , $z = (x, y)$, and $\delta z = (h, ay + b)$. Note that in (2.4) we have already used the variables (x, y) in place of (x_+, y_+) for simplifying notations.

If we find h, a, b such that

$$(2.7) \quad -\partial_\omega h + P_1^0 - [P_1^0] = 0,$$

$$(2.8) \quad (\Omega + A)b - \partial_\omega b + P_2^0 = 0,$$

$$(2.9) \quad (\Omega + A)a - a(\Omega + A) - \partial_\omega a + P_2^1 - \hat{A} = 0,$$

then the system (2.4) becomes (2.3).

C. Solving linear equations. Now we solve the linear equations (2.7), (2.8) and (2.9). Let $h = \sum_k h_k e^{i\langle k, x \rangle}$ and $P_1^0 = \sum_k P_{1k}^0 e^{i\langle k, x \rangle}$. By (2.7) we have

$$h_k = P_{1k}^0 / i\langle k, \omega \rangle, \quad k \neq 0.$$

It is easy to see that $\|P_1^0\|_{D(s,r)}^L \leq \epsilon$. By Cauchy's estimates we have $\|P_{1k}^0\|^L \leq \epsilon e^{-s|k|}$. There exists a subset O_+^0 of O such that for $\xi \in O_+^0$ we have

$$|\langle \omega(\xi), k \rangle| \geq \frac{\alpha}{|k|^\tau}, \quad \forall k \neq 0.$$

Thus, for $\xi \in O_+^0$, we have $\|h_k\|^L \leq |\alpha|^{-L-1} |k|^{(1+L)\tau+L+1} \epsilon$, and so

$$(2.10) \quad \|h\|_{D(s-\rho,r)}^L \leq c\epsilon \alpha^{-L-1} \rho^{-\kappa} \quad \text{with } \kappa = (1+L)\tau + L + n + 1.$$

Note that h should be defined on O_+^0 . By the Whitney's extension theorem, we can extend it to the initial domain O . So, for simplicity we do not concern ourselves with the domain of the parameter in all estimates of the KAM steps. However, in our problem it only makes sense for $\xi \in O_+^0$.

For the equation (2.8), write $b = (b_1, \dots, b_m)$, where b_j is a d_j -dimensional vector. Let $f = P_2(x, 0; \xi) = P_2^0$. It follows easily that

$$\|f\|_{D(s,r)}^L \leq \|P_2\|_{D(s,r)}^L \leq r\epsilon.$$

In the same way, write $f = (f_1, \dots, f_m)$. The equation (2.8) is equivalent to

$$\partial_\omega b_j - (\Omega_j I_{d_j} + A_j) b_j = f_j, \quad j = 1, 2, \dots, m.$$

Let $b_j = \sum_{k \in \mathbb{Z}^n} b_{j,k} e^{i\langle k, x \rangle}$ and $f_j = \sum_{k \in \mathbb{Z}^n} f_{j,k} e^{i\langle k, x \rangle}$. Then we have

$$(2.11) \quad [(i\langle k, \omega \rangle - \Omega_j) I_{d_j} - A_j] b_{j,k} = f_{j,k}.$$

Denote

$$E = (i\langle k, \omega \rangle - \Omega_j) I_{d_j} - A_j.$$

By assumption (2.2) and Lemmas 3.1 and 3.2, there is $K > 0$ such that for sufficiently small ϵ_0 , there exists a subset O_+^1 of O such that, for all $\xi \in O_+^1$ and

$|k| \geq K$, the matrix E is non-singular. Moreover, we have

$$\|E^{-1}\|^L \leq c(|k| + 1)^{\tau'} \alpha^{-L-1},$$

where $\tau' = (1 + L)\tau + L$ and $\tau > (n + 1)L$.

Let

$$O^1 = \{\xi \mid |\mathbf{i}\langle k, \omega(\xi) \rangle + \Omega_j(\xi)| \geq \alpha_0, \forall |k| \leq K, \forall j = 1, 2, \dots, m\},$$

where $\alpha_0 \geq \alpha$. So if ϵ_0 is sufficiently small, for $\xi \in O^1$, E^{-1} exists for all $|k| \leq K$, and the above estimate for E^{-1} also holds.

Thus, $\forall \xi \in O_+^1 \cap O^1$, we can solve the equation (2.8) for $b_{j,k}$ with

$$\|b_{j,k}\|^L \leq c(|k| + 1)^{\tau'} \alpha^{-L-1} \|f_{j,k}\|^L.$$

Therefore, we have

$$(2.12) \quad \frac{1}{r} \|b\|_{D(s-\rho, r)}^L \leq \frac{c\epsilon}{\alpha^{L+1} \rho^\kappa},$$

where $\kappa = \tau' + n - 1$.

Now we consider the last equation, (2.9). Write $a = (a_{ij})_{1 \leq i, j \leq m}$ and $g = P_2^1 - \hat{A} = (g_{ij})_{1 \leq i, j \leq m}$, where a_{ij} and g_{ij} are $d_i \times d_j$ -matrices.

Then (2.9) becomes

$$\partial_\omega a_{ij} + a_{ij}(\Omega_j I_{d_j} + A_j) - (\Omega_i I_{d_i} + A_i)a_{ij} = g_{ij}.$$

In the same way as above we have

$$\mathbf{i}\langle k, \omega \rangle a_{ij,k} - (\Omega_i + A_i)a_{ij,k} + a_{ij,k}(\Omega_j + A_j) = g_{ij,k},$$

where $a_{ij,k}$ and $g_{ij,k}$ are the Fourier coefficients of a_{ij} and g_{ij} , respectively. The above matrix equations for $a_{ij,k}$ are equivalent to the following linear equations for the elements of the matrix $a_{ij,k}$:

$$[(\mathbf{i}\langle k, \omega \rangle + \Omega_i - \Omega_j)I_{d_i d_j} + O(\epsilon_0)]\{a_{ij,k}\} = \{g_{ij,k}\},$$

where $\{a_{ij,k}\}$ and $\{g_{ij,k}\}$ are vectors whose components are the elements of the matrices $a_{ij,k}$ and $g_{ij,k}$, respectively, and $O(\epsilon_0)$ is a $d_i d_j$ -order matrix depending on A_i and A_j . By the assumption (2.2), we have $\|O(\epsilon_0)\|^L \leq c\epsilon_0$.

Let $E = (\mathbf{i}\langle k, \omega \rangle + \Omega_i - \Omega_j)I_{d_i d_j} + O(\epsilon_0)$. By the assumption (2.2) and Lemmas 3.1 and 3.2, there is a $K > 0$ such that for $|k| \geq K$, there exists a subset O_+^2 of O such that, for $\xi \in O_+^2$, the matrix E is non-singular. Moreover, we have

$$\|E^{-1}\|^L \leq c(|k| + 1)^{\tau'} \alpha^{-L-1},$$

where $\tau' = (1 + L)\tau + L$ and $\tau > (n + 1)L$.

Let

$$O^2 = \{\xi \mid |\mathbf{i}\langle k, \xi \rangle + \Omega_i(\xi) - \Omega_j(\xi)| \geq \alpha_0, \quad i \neq j, \quad \forall |k| \leq K\}.$$

By analyticity of $\Omega_j(\xi)$ and condition (1.2), if α_0 is sufficiently small, the measure of the set $O \setminus O^2$ is small. So for sufficiently small ϵ_0 , E^{-1} exists, and the above estimate of E^{-1} still holds.

Thus, for $i \neq j$ or $k \neq 0$ and $\forall \xi \in O_+^2 \cap O^2$, the coefficient matrix of the linear equation system for $\{a_{ij,k}\}$ is non-singular, and so we find $a_{ij,k}$ with

$$\|a_{ij,k}\|^L \leq c(|k| + 1)^{\tau'} \alpha^{-L-1} \|g_{ij,k}\|^L.$$

If $i = j$ and $k = 0$, by the choice of $\hat{A}$, it follows that $g_{jj,0} = 0$ and so we let $a_{jj,0} = 0$. It follows that

$$(2.13) \quad \|a(x)\|_{D(s-\rho, r)}^L \leq \frac{c\epsilon}{\alpha^{L+1}\rho^\kappa}.$$

Let $O_+ = O_+^0 \cap O_+^1 \cap O_+^2 \cap O^1 \cap O^2$. For $\xi \in O_+$, we can obtain solutions of the equations (2.7), (2.8) and (2.9), h, b and a with the estimates (2.11), (2.12) and (2.13) holding.

By Cauchy's estimate it follows that

$$\|\mathcal{D}_x h\|_{D(s-2\rho, r)}^L, \quad \frac{1}{r} \|\mathcal{D}_x b\|_{D(s-2\rho, r)}^L, \quad \|\mathcal{D}_x a\|_{D(s-2\rho, r)}^L \leq \frac{c\epsilon}{\alpha^{L+1}\rho^{\kappa+1}}.$$

Moreover, by Lemma 3.1, for $\tau \geq (n+1)L$, we have

$$(2.14) \quad \text{mes}(O \setminus (O_+^0 \cup O_+^1 \cup O_+^2)) \leq c\alpha^{\frac{1}{L}} \sum_{k \neq 0} (|k| + 1)^{\frac{L-\tau}{L}} \leq c\alpha^{\frac{1}{L}}.$$

By analyticity of the normal frequencies (see [8]), it easily follows that

$$(2.15) \quad \text{mes}(O \setminus O^1 \cup O^2) \rightarrow 0, \quad \text{as } \alpha_0 \rightarrow 0.$$

By the above discussion, if

$$\xi \in O_+ \quad \text{and} \quad \frac{c\epsilon}{\alpha^{L+1}\rho^\kappa} \leq \eta < \rho < \frac{1}{8},$$

we have a transformation

$$\Phi : (x_+, y_+) \in D(s - 3\rho, \eta r) \rightarrow (x, y) \in D(s - \rho, 2\eta r) \subset D(s, r)$$

such that

$$(2.16) \quad \begin{cases} \|x - x_+\|_{D(s-\rho, r)}^L, & \frac{1}{r} \|y - y_+\|_{D(s-\rho, r)}^L \leq \frac{c\epsilon}{\alpha^{L+1}\rho^\kappa}, \\ \|\mathcal{D}_{x_+} x - I_n\|_{D(s-2\rho, r)}^L, & \frac{1}{r} \|\mathcal{D}_{x_+} y\|_{D(s-2\rho, r)}^L \leq \frac{c\epsilon}{\alpha^{L+1}\rho^{\kappa+1}}, \\ \|\mathcal{D}_{y_+} y - I_{\tilde{m}}\|_{D(s-2\rho, r)}^L & \leq \frac{c\epsilon}{\alpha^{L+1}\rho^\kappa}. \end{cases}$$

Moreover, it follows easily that

$$(2.17) \quad \|\omega_+ - \omega\|^L \leq \epsilon, \quad \|A_+ - A\|^L \leq \epsilon.$$

D. Estimates for new error terms. Let

$$\eta = \left(\frac{\epsilon}{\alpha^{L+1}\rho^{\kappa+1}} \right)^{\frac{1}{2}}, \quad r_+ = r\eta, \quad s_+ = s - 3\rho.$$

We will prove

$$(2.18) \quad \|P_{+1}\|_{D(s_+, r_+)}^L \leq \epsilon_+, \quad \|P_{+2}\|_{D(s_+, r_+)}^L \leq \epsilon_+ r_+$$

with $\epsilon_+ = c\epsilon\eta$. Since P_{+1} is simpler, we only consider P_{+2} .

By (2.13), we suppose

$$\|a\|_{D(s-\rho, r)}^L \leq c\epsilon/\alpha^{L+1}\rho^\kappa \leq \frac{1}{2}.$$

By the Neumann series, it follows that $(I_{\tilde{m}} + a)^{-1}$ exists and we have

$$\|(I_{\tilde{m}} + a)^{-1}\|_{D(s-\rho, r)}^L \leq 1 + \|a\|_{D(s-\rho, r)}^L + (\|a\|_{D(s-\rho, r)}^L)^2 + \cdots \leq 2.$$

Since $\|ay + b\|_{D(s-\rho, r)}^L \leq c\epsilon r/\alpha^{L+1}\rho^\kappa$, so

$$\|\partial_{P_{+1}+[P_1^0]}(ay + b)\|_{D(s-3\rho, r)}^L \leq c\epsilon^2\eta r/\alpha^{L+1}\rho^{\kappa+1}.$$

Again

$$\|P_2 - P_0^0 - P_2^1 y\|_{D(s, \eta r)}^L \leq \eta^2 \|P_2 - P_0^0 - P_2^1\|_{D(s, r)}^L \leq \eta^2 \epsilon r$$

and

$$\|a \hat{A} y\|_{D(s, \eta r)}^L \leq \|a\|_{D(s, \eta r)}^L \|\hat{A} y\|_{D(s, \eta r)}^L \leq c \epsilon^2 \eta r / \alpha^{L+1} \rho^\kappa.$$

Moreover, $\forall t \in [0, 1]$ we have

$$\|\langle \nabla_z P_2(z + t \delta z), \delta z \rangle\|_{D(s-3\rho, \eta r)}^L \leq c \epsilon^2 r / \alpha^{L+1} \rho^{\kappa+1}.$$

Thus, by (2.6) and combining all the above estimates we have

$$\|P_{+2}\|_{D(s_+, r_+)}^L \leq c \eta^2 \epsilon r = \epsilon_+ r_+.$$

Thus, the second inequality of (2.18) holds.

E. Convergence of iteration. Below we make KAM iteration. For given s, ϵ, r in the theorem, we define some sequences inductively:

$$\begin{aligned} \epsilon_0 &= \epsilon, \quad r_0 = r, \quad s_0 = s, \quad \alpha_0 = \alpha, \quad \rho_0 = \frac{s_0}{12}, \quad \eta_0 = \left(\frac{\epsilon_0}{\alpha_0^{L+1} \rho_0^{\kappa+1}}\right)^{\frac{1}{2}}, \\ \epsilon_{\nu+1} &= c \eta_\nu \epsilon_\nu, \quad r_{\nu+1} = \eta_\nu r_\nu, \quad s_{\nu+1} = s_\nu - 3\rho_\nu, \quad \rho_{\nu+1} = \frac{1}{2} \rho_\nu, \\ \alpha_{\nu+1} &= \frac{\alpha_\nu}{2}, \quad \eta_\nu = \left(\frac{\epsilon_\nu}{\alpha_\nu^{L+1} \rho_\nu^{\kappa+1}}\right)^{\frac{1}{2}}, \quad D_\nu = D(s_\nu, r_\nu), \quad \nu = 0, 1, \dots \end{aligned}$$

At the ν -th step, we have a subset $O_\nu \subset O$. For $\xi \in O_\nu$, there exist transformations $\Phi_\nu : (x_{\nu+1}, y_{\nu+1}) \rightarrow (x_\nu, y_\nu)$, satisfying

$$(2.19) \quad \|x_{\nu+1} - x_\nu\|_{D_\nu}^L \leq c \eta_\nu^2, \quad \|y_{\nu+1} - y_\nu\|_{D_\nu}^L \leq c r_\nu \eta_\nu^2.$$

Let $\Phi^{\nu+1} = \Phi^\nu \circ \Phi_{\nu+1}$ with $\Phi^0 = Id$. Let $P_1^0 = P_1$, $P_2^0 = P_2$, $\omega^0 = \xi$, $A^0 = 0$. Then by the transformations Φ^ν the system (1.1) is changed to

$$(2.20) \quad \dot{x} = \omega^\nu(\xi) + P_1^\nu(x, y; \xi), \quad \dot{y} = (\Omega(\xi) + A^\nu(\xi))y + P_2^\nu(x, y; \xi).$$

Moreover, we have

$$(2.21) \quad \|\omega^{\nu+1} - \omega^\nu\|^L \leq \epsilon_\nu, \quad \|A^{\nu+1} - A^\nu\|^L \leq \epsilon_\nu,$$

$$(2.22) \quad \|P_1^\nu\|_{D_\nu}^L \leq \epsilon_\nu, \quad \|P_2^\nu\|_{D_\nu}^L \leq r_\nu \epsilon_\nu.$$

By the definitions of η_ν , ϵ_ν , α_ν , ρ_ν , we have $\eta_{\nu+1} \leq c^{\frac{1}{2}}(\eta_\nu)^{\frac{3}{2}}$, and so $c \eta_{\nu+1} \leq (c \eta_\nu)^{\frac{3}{2}}$. If $c \eta_0 \leq 2^{-1}$, then $\eta_{\nu+1} \leq c^{-1} 2^{-(\frac{3}{2})^\nu}$. Thus, we have $\epsilon_{\nu+1} \leq 2^{-1} \epsilon_\nu$, and so $\epsilon_\nu \leq 2^{-\nu} \epsilon_0$. It is easy to see that the assumption (2.2) holds by (2.21).

Let $O_* = \bigcap_{\nu \geq 1} O_\nu$. By the above estimates we can prove that Φ^ν is convergent to a transformation Φ_* for $(x, y; \xi) \in D_*(s/2, r/2) \times O_*$. The proof is the same as in the case of hamiltonian systems and reversible systems, so we omit the details and refer the reader to [5, 6]. Thus, it is easy to see that for $\nu \rightarrow \infty$ we get $\omega^\nu \rightarrow \omega^*$, $A^\nu \rightarrow A^*$, $P_1^\nu \rightarrow P_1^*$, $P_2^\nu \rightarrow P_2^*$. It follows easily that $\|\omega^* - \xi\|^L \leq 2\epsilon_0$, $\|A^*\|^L \leq 2\epsilon_0$. By (2.22) we have

$$P_1^*(x, y; \xi)|_{y=0} = 0, \quad P_2^*(x, y; \xi)|_{y=0} = 0, \quad \text{and} \quad \partial_y P_2^*(x, y; \xi)|_{y=0} = 0.$$

Thus, for $\xi \in O_*$, by the transformation Φ_* the system (1.1) is changed to the form of (1.5).

F. Measure estimate. Now we estimate the measure of the set O_* to prove it is non-empty. By the above discussion, $O_\nu = O_\nu^0 \cap O_\nu^1 \cap O_\nu^2 \cap O^1 \cap O^2$. Let $O_*^j = \bigcap_{\nu \geq 1} O_\nu^j$. Then $O_* = O_*^0 \cap O_*^1 \cap O_*^2 \cap O^1 \cap O^2$. By (2.14), we have $\text{mes}(O \setminus (O_\nu^0 \cup O_\nu^1 \cup O_\nu^2)) \leq c(\alpha_\nu)^{\frac{1}{L}}$. It follows that

$$\text{mes}(O \setminus (O_*^0 \cup O_*^1 \cup O_*^2)) \leq c\alpha^{\frac{1}{L}}.$$

Thus, it follows that

$$\text{mes}(O \setminus O_*) \leq \text{mes}(O \setminus O^1) + \text{mes}(O \setminus O^2) + c\alpha^{\frac{1}{L}} \rightarrow 0 \quad (\text{as } \alpha \rightarrow 0).$$

3. APPENDIX

In this section, we give some lemmas, which are used to control the measure of parameters such that the equations (2.7), (2.8) and (2.9) are solvable. These results are actually Lemmas A.1 and A.2 in [13], so we refer to the reference for their proofs.

Lemma 3.1. *Suppose that $\|\omega(\xi) - \xi\|^L \leq \epsilon$ and $f_j(\xi)$ are L -th continuously differentiable functions with $\|f_j(\xi)\|^L \leq M$ ($1 \leq j \leq L$). Let $P(\xi) = (p_{ij}(\xi))$ be an $L \times L$ -matrix, and L -th continuously differentiable with respect to $\xi \in O$ with $\|P\|^L = \max_{|\beta| \leq L} \max_{1 \leq i, j \leq L} \sup_{\xi \in O} \left| \frac{\partial^\beta p_{ij}}{\partial \xi^\beta} \right| \leq \epsilon$. Let $D = \langle \omega, k \rangle I + \text{diag}(f_1(\xi), f_2(\xi), \dots, f_L(\xi)) + P(\xi)$ with I being the unit matrix, and let $\mathcal{R}_k(\alpha)$ be the subset of O such that $\|D^{-1}\| > \frac{|k|^\tau}{\alpha}$, $\tau > nL$. Then, there is a sufficiently small $\epsilon_0 > 0$ and a sufficiently large positive integer K only depending on M such that, if $\epsilon \leq \epsilon_0$, for $\alpha > 0$ and $|k| \geq K$, we have*

$$\text{mes}(\mathcal{R}_k(\alpha)) \leq \left(\frac{c\alpha}{|k|^{\tau-L}} \right)^{\frac{1}{L}},$$

where c is a constant depending on ϵ_0 and M .

Lemma 3.2. *Let D be a matrix depending on ξ and $\|D\|^L \leq M$. If D is invertible with $\|D^{-1}\| \leq N$, then $\|D^{-1}\|^L \leq cM^L N^{L+1}$, where c is a constant depending on L .*

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