GROWTH OF RELATIVELY HYPERBOLIC GROUPS

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ABSTRACT. We show that a finitely generated group that is hyperbolic relative to a collection of proper subgroups either is virtually cyclic or has uniform exponential growth.

1. INTRODUCTION

Let G be a finitely generated group, and S a finite generating set. Denote by d_S the word metric on G with respect to S and by $\beta(G, S, k)$ the number of elements of G a d_S distance at most k from the identity. The exponential growth rate of G with respect to S is $\omega(G, S) = \lim_{k \rightarrow \infty} (\beta(G, S, k))^{1/k}$. Notice that the limit exists due to the submultiplicativity: $\beta(G, S, m+n) \leq \beta(G, S, m)\beta(G, S, n)$. We say G has *exponential growth* if $\omega(G, S) > 1$ for some (hence any) finite generating set S ; and we say G has *uniform exponential growth* if $\omega(G) = \inf_S \omega(G, S) > 1$, where S varies over all finite generating sets of G .

J. Wilson constructed groups that have exponential growth but not uniform exponential growth ([W]). On the other hand, among the following classes of groups, exponential growth implies uniform exponential growth: linear groups over fields with zero characteristic ([EMO]), hyperbolic groups ([K]), one-relator groups ([GD]), solvable groups ([O2]), and geometrically finite groups acting on pinched Hadamard manifolds ([AN]). In this paper we show that the same is true for relatively hyperbolic groups.

Theorem 1.1. *Let G be a finitely generated group that is hyperbolic relative to a collection of proper subgroups. Then G either has uniform exponential growth or has a finite index cyclic subgroup.*

Relatively hyperbolic groups are generalizations of Gromov hyperbolic groups. Typical examples of relatively hyperbolic groups include Gromov hyperbolic groups, fundamental groups of finite volume real hyperbolic manifolds, and groups acting properly and cocompactly on spaces with isolated flats ([HK]). There are five different definitions for relatively hyperbolic groups: one due to each of Gromov ([G]), B. Farb ([F]) and D. Osin ([O1]), and two due to B. Bowditch ([B]). These definitions are all equivalent for finitely generated groups. However, we remark that Osin's definition is also valid for infinitely generated groups. We shall use

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Bowditch's definition of relatively hyperbolic groups as geometrically finite groups. The main ingredient in our proof is Bowditch's theorem on the existence of an invariant collection of disjoint horoballs (see Proposition 2.2).

The usual way for proving uniform exponential growth is as follows: for any finite generating set S of G , find two elements $g_1, g_2 \in G$ with word length bounded independent of S , such that $\langle g_1, g_2 \rangle$ is free with basis $\{g_1, g_2\}$. We shall use the same strategy. Let G be a relatively hyperbolic group and S a finite generating set. For any positive integer n , let $S(n) = \{g \in G : d_S(g, id) \leq n\}$. Notice that $S(n)$ is also a finite generating set of G , and $S(n) \subset S(m)$ if $n \leq m$. In particular, $S \subset S(n)$ for all $n \geq 1$.

We prove uniform exponential growth in two steps:

Step 1: There exists some positive integer n_0 with the following property. For any finite generating set S of G , $S(n_0)$ contains a hyperbolic element (see Proposition 5.1).

Step 2: There exists some positive integer k_0 with the following property. If a finite generating set S of G contains a hyperbolic element, then there are $g_1, g_2 \in S(k_0)$ such that $\langle g_1, g_2 \rangle$ is free with basis $\{g_1, g_2\}$ (see Corollary 3.2 and Proposition 4.1).

It should be noted that uniform exponential growth for geometrically finite groups acting on pinched Hadamard manifolds has been established by R. Alperin and G. Noskov ([AN]). Some of our arguments are similar to theirs. Theorem 1.1 was conjectured by C. Drutu in [D].

It was pointed out by the referee that our proof actually establishes a stronger result: any non-elementary relatively hyperbolic group is uniformly non-amenable in the sense of Osin ([O3]).

Notation. For a metric space Y , any subset $A \subset Y$, and any $r \geq 0$, we denote by $N_r(A) = \{y \in Y : d(y, A) \leq r\}$ the closed r -neighborhood of A . For any $p, q \in Y$, pq denotes a geodesic segment between p and q , although it is not unique in general.

2. RELATIVELY HYPERBOLIC GROUPS AS GEOMETRICALLY FINITE GROUPS

Here we recall the notion of geometrically finite groups and Bowditch's result on the existence of an invariant system of disjoint horoballs (see Proposition 2.2). The reader is referred to [B] for more details.

Suppose that M is a compact metrizable topological space, and a group G acts by homeomorphisms on M . We say that G is a *convergence group* if the induced action on the space of distinct triples is properly discontinuous. Let G be a convergence group on M . A point $\xi \in M$ is a *conical limit point* if there exists a sequence in G , $\{g_n\}_{n=1}^\infty$, and two points $\zeta \neq \eta \in M$, such that $g_n(\xi) \rightarrow \zeta$ and $g_n(\xi') \rightarrow \eta$ for all $\xi' \neq \xi$. An element $g \in G$ is a *hyperbolic element* if it has infinite order and fixes exactly two points in M . A subgroup $H < G$ is *parabolic* if it is infinite, fixes a point $\xi \in M$, and contains no hyperbolic elements. In this case, the fixed point of H is unique and is referred to as a *parabolic point*. The nontrivial elements in a parabolic subgroup are called *parabolic elements*. A parabolic point ξ is a *bounded parabolic point* if its stabilizer $Stab(\xi) := \{g \in G : g(\xi) = \xi\}$ acts properly and cocompactly on $M \setminus \{\xi\}$. A convergence group G on M is a *geometrically finite group* if each point of M is either a conical limit point or a bounded parabolic point.

Definition 2.1. A group G is *hyperbolic relative* to a family of finitely generated subgroups $\mathcal{G}$ if it acts properly discontinuously by isometries, on a proper geodesic hyperbolic space X , such that the induced action on ∂X is of convergence, geometrically finite, and such that the maximal parabolic subgroups are exactly the elements of $\mathcal{G}$.

By definition a relatively hyperbolic group is a geometrically finite group. A. Yaman ([Y]) proved that if G is a geometrically finite group on a perfect metrizable compact space M , and the maximal parabolic subgroups are finitely generated, then G is hyperbolic relative to the family of maximal parabolic subgroups.

Now let X be a δ -hyperbolic geodesic metric space for some $\delta \geq 0$. Recall that each geodesic triangle in X is δ -thin, that is, each edge is contained in the δ -neighborhood of the union of the other two edges. Below we shall denote by $c = c(\delta)$ a constant that depends only on δ . Let $\xi \in \partial X$. A (not necessarily continuous) function $h : X \rightarrow \mathbb{R}$ is a *horofunction* about ξ if there are constants $c_1 = c_1(\delta)$, $c_2 = c_2(\delta)$ such that if $x, a \in X$ and $d(a, x\xi) \leq c_1$ for some geodesic ray $x\xi$ from x to ξ , then $|h(a) - (h(x) + d(x, a))| \leq c_2$. A closed subset $B \subset X$ is a *horoball* about ξ if there is a horofunction h about ξ and a constant $c = c(\delta)$ such that $h(x) \geq -c$ for all $x \in B$, and $h(x) \leq c$ for all $x \in X \setminus B$. Note that ξ is uniquely determined by B , and we refer to it as the center of the horoball.

Proposition 2.2 (B. Bowditch, Proposition 6.13 in [B]). *Let G be a relatively hyperbolic group, and X a proper hyperbolic geodesic metric space that G acts upon as in Definition 2.1. Let Π be the set of all bounded parabolic points in ∂X . Then Π/G is finite. Moreover, for any $r \geq 0$, there is a collection of horoballs $\mathcal{B} = \{B_\xi : \xi \in \Pi\}$ indexed by Π with the following properties:*

- (1) $\mathcal{B}$ is r -separated, that is, $d(B_\xi, B_\eta) \geq r$ for all $\xi \neq \eta \in \Pi$;
- (2) $\mathcal{B}$ is G -invariant, that is, $g(B_\xi) = B_{g(\xi)}$ for all $g \in G$ and $\xi \in \Pi$;
- (3) $Y(\mathcal{B})/G$ is compact, where $Y(\mathcal{B}) = X \setminus \bigcup_{\xi \in \Pi} \text{int}(B_\xi)$.

3. AXES OF HYPERBOLIC ELEMENTS

In this section we study how a hyperbolic element “translates” its “axes”.

Let G and X be as in Definition 2.1. That is, X is a proper δ -hyperbolic geodesic space for some $\delta \geq 0$ and G acts properly discontinuously by isometries on X , such that the induced action on ∂X is of convergence and geometrically finite.

Recall that, in a δ -hyperbolic space X , any two complete geodesics that have the same endpoints in ∂X have Hausdorff distance at most 2δ . For a hyperbolic element $\gamma \in G$, let γ_+ and γ_- be the attracting and repelling fixed points of γ in ∂X respectively. We shall call any complete geodesic with γ_+, γ_- as endpoints an axis of γ , and denote by A_γ the union of all axes of γ . Note that γ may have many different axes and an axis of γ in general is not invariant under γ . However, for any axis c of γ , $\gamma(c)$ is also an axis of γ and hence the Hausdorff distance between c and $\gamma(c)$ is at most 2δ .

Proposition 2.2 implies that there is a 200δ -separated G -invariant collection of horoballs $\mathcal{B}$ centered at the parabolic points such that $Y(\mathcal{B})/G$ is compact.

Lemma 3.1. *There exists a positive integer k_1 with the following property: for any infinite order element $\gamma \in G$ and any $x \in Y(\mathcal{B})$, there is some k , $1 \leq k \leq k_1$, such that $d(\gamma^k(x), x) \geq 200\delta$.*

Proof. Notice that the action of G on $Y(\mathcal{B})$ is properly discontinuous and cocompact. It follows that there is some integer $k_1 \geq 1$, such that for any $x \in Y(\mathcal{B})$, the cardinality of $\{g \in G : d(x, g(x)) \leq 200\delta\}$ is less than k_1 . In particular, for any $x \in Y(\mathcal{B})$ and any infinite order element $\gamma \in G$, there is some k , $1 \leq k \leq k_1$, such that $d(\gamma^k(x), x) \geq 200\delta$. $\square$

Corollary 3.2. *If a finite generating set S of G contains a hyperbolic element, then there is a hyperbolic element $\gamma \in S(k_1)$ and some $x \in A_\gamma$ such that $d(\gamma(x), x) \geq 200\delta$, where k_1 is the constant in Lemma 3.1.*

Proof. Let $g \in S$ be a hyperbolic element and c an axis of g . It follows from the definition of a horoball that c is not contained in any horoball. Since $\mathcal{B}$ is a disjoint collection of horoballs and c is connected, we have $c \cap Y(\mathcal{B}) \neq \emptyset$. Let $x \in c \cap Y(\mathcal{B})$. By Lemma 3.1, there is some k , $1 \leq k \leq k_1$, such that $d(g^k(x), x) \geq 200\delta$. Notice that $g^k \in S(k_1)$ is hyperbolic and c is also an axis of g^k . $\square$

For a complete geodesic c in X , we define a map $P_c : X \rightarrow c$ as follows: for any $x \in X$, let $P_c(x) \in c$ be a point with $d(x, P_c(x)) = d(x, c)$. Note that for any two points $x_1, x_2 \in c$ with $d(x, x_1) = d(x, x_2) = d(x, c)$, we have $d(x_1, x_2) \leq 4\delta$.

Lemma 3.3. *Let $c, c' : \mathbb{R} \rightarrow X$ be two geodesics with $c(+\infty) = c'(+\infty)$ and $c(-\infty) = c'(-\infty)$. Let $a, b \in \mathbb{R}$, with $b \geq a + 8\delta$, and $a', b' \in \mathbb{R}$ be determined by $c'(a') = P_{c'}(c(a))$ and $c'(b') = P_{c'}(c(b))$. Then $(b' - a') \geq (b - a) - 4\delta$.*

Proof. Recall that the Hausdorff distance between c and c' is at most 2δ . It follows from the triangle inequality that $|b' - a'| \geq (b - a) - 4\delta$. So either $(b' - a') \geq (b - a) - 4\delta$ or $(b' - a') \leq -(b - a) + 4\delta$. Suppose $(b' - a') \leq -(b - a) + 4\delta$. Let $x_i = c(b + i\delta)$, $i = 0, 1, \dots$, and $y_i = P_{c'}(x_i)$. Then $y_0 = c'(b')$, $y_1, \dots$ is a sequence of points on c' with $d(y_i, y_{i+1}) \leq 5\delta$. Since $(b' - a') \leq -(b - a) + 4\delta \leq -4\delta$ and $y_i \rightarrow c(+\infty)$, there is some $i \geq 1$ such that $d(y_i, c'(a')) \leq 2.5\delta$. The triangle inequality implies that $(b + i\delta) - a = d(x_i, c(a)) \leq d(x_i, y_i) + d(y_i, c'(a')) + d(c'(a'), c(a)) \leq 6.5\delta$, contradicting the fact that $b \geq a + 8\delta$. $\square$

Lemma 3.4. *Let $g \in G$ be a hyperbolic element, and c an axis of g . Suppose $x \in c$ is a point with $d(g(x), x) \geq 20\delta$. Then for any integers $i < j$, the point $P_c(g^j(x))$ lies between $P_c(g^i(x))$ and g_+ .*

Proof. Denote $x_i = P_c(g^i(x))$. Note that $x_i \rightarrow g_+$ as $i \rightarrow +\infty$. It suffices to show that for any i , x_i lies between x_{i-1} and x_{i+1} . Set $t = d(g(x), x)$. The triangle inequality implies that $t - 4\delta \leq d(x_i, x_{i+1}) \leq t + 4\delta$. Suppose x_i does not lie between x_{i-1} and x_{i+1} for some i . Then $d(x_{i-1}, x_{i+1}) \leq 8\delta$ and $d(g^{i-1}(x), g^{i+1}(x)) \leq 12\delta$. Since g is an isometry, $d(x, g^2(x)) \leq 12\delta$. We shall show that $d(x, g^2(x)) \geq 2t - 16\delta$, which is a contradiction.

Let $\sigma : \mathbb{R} \rightarrow X$ be the parameterization of c with $\sigma(0) = x$ and $\sigma(+\infty) = g_+$, $\sigma(-\infty) = g_-$. Then $g \circ \sigma$ is the parameterization of $g(c)$ with $g \circ \sigma(0) = g(x)$ and $g \circ \sigma(+\infty) = g_+$, $g \circ \sigma(-\infty) = g_-$. Since $t - 4\delta \leq d(x_1, x) \leq t + 4\delta$, we have $x_1 = \sigma(b)$ with either $-(t + 4\delta) \leq b \leq -(t - 4\delta)$ or $t - 4\delta \leq b \leq t + 4\delta$. We consider the case $-(t + 4\delta) \leq b \leq -(t - 4\delta)$, the other case can be handled similarly. Let $y_1 = P_c(g(x_1))$. Then $y_1 = \sigma(a)$ for some $a \in \mathbb{R}$. Now Lemma 3.3 applied to σ , $g \circ \sigma$ and the projections of $g(x)$, $g(x_1)$ onto c implies that $b - a \geq 0 - b - 4\delta \geq t - 8\delta$. It follows that $d(y_1, x) = 0 - a \geq -b + t - 8\delta \geq 2t - 12\delta$. By the triangle inequality $d(x, g^2(x)) \geq d(x, y_1) - d(y_1, g(x_1)) - d(g(x_1), g^2(x)) \geq 2t - 12\delta - 2\delta - 2\delta = 2t - 16\delta$. $\square$

Lemma 3.5. *Let $g \in G$ be a hyperbolic element. Suppose there is a point $x \in A_g$ with $d(x, g(x)) \geq 200\delta$. Then $|d(y, g(y)) - d(z, g(z))| \leq 40\delta$ for all $y, z \in A_g$.*

Proof. Let c be an axis of g that contains x . We first show $|d(y, g(y)) - d(x, g(x))| \leq 16\delta$ for all $y \in c$. Fix $y \in c$. Denote $t = d(x, g(x))$, $x_i = P_c(g^i(x))$, $x'_i = P_c(g(x_i))$ and $y' = P_c(g(y))$. Then $d(x_i, g^i(x)) \leq 2\delta$ and $d(x_i, x_{i-1}) \geq t - 4\delta$. By Lemma 3.4 there is some i such that y lies between x_i and x_{i+1} , and x_{i+1} lies between x_i and g_+ . The triangle inequality implies

$$|d(y', x'_{i-1}) - d(y, x_{i-1})| = |d(y', x'_{i-1}) - d(g(y), g(x_{i-1}))| \leq 4\delta$$

and

$$d(x_i, x'_{i-1}) \leq d(x_i, g^i(x)) + d(g^i(x), g(x_{i-1})) + d(g(x_{i-1}), x'_{i-1}) \leq 6\delta.$$

It follows that $|d(y', x_i) - d(y, x_{i-1})| \leq 10\delta$. In particular, $d(y', x_i) \geq d(y, x_{i-1}) - 10\delta = d(y, x_i) + d(x_i, x_{i-1}) - 10\delta \geq d(y, x_i) + t - 4\delta - 10\delta > d(y, x_i)$. On the other hand, Lemma 3.3 applied to $g(c)$, c and the projections of $g(y)$, $g(x_{i-1})$ onto c implies that y' lies between x_i and g_+ . It now follows from $d(y', x_i) > d(y, x_i)$ that y lies between x_i and y' . Now

$$\begin{aligned} |d(y, y') - d(x_{i-1}, x_i)| &= |[d(x_i, y) + d(y, y')] - [d(x_{i-1}, x_i) + d(x_i, y)]| \\ &= |d(x_i, y') - d(x_{i-1}, y)| \leq 10\delta. \end{aligned}$$

Finally

$$\begin{aligned} |d(y, g(y)) - d(x, g(x))| &= |d(y, g(y)) - d(g^{i-1}(x), g^i(x))| \\ &\leq |d(y, y') - d(x_{i-1}, x_i)| + d(g(y), y') \\ &\quad + d(g^{i-1}(x), x_{i-1}) + d(g^i(x), x_i) \\ &\leq 16\delta. \end{aligned}$$

Now fix any $y \in A_g$. Set $y' = P_c(y) \in c$. Then $|d(y, g(y)) - d(x, g(x))| \leq |d(y', g(y')) - d(x, g(x))| + d(y, y') + d(g(y'), g(y)) \leq 20\delta$. By the triangle inequality, we have $|d(y, g(y)) - d(z, g(z))| \leq 40\delta$ for all $y, z \in A_g$. $\square$

4. FREE SUBGROUPS

The goal in this section is to find two short hyperbolic elements that generate a free group.

Proposition 4.1. *Let G be a relatively hyperbolic group, and let X be a proper δ -hyperbolic geodesic metric space that G acts upon as in Definition 2.1. Then there exists a positive integer k_2 with the following property. If S is a finite generating set of G and $s \in S$ is a hyperbolic element such that $d(s(x), x) \geq 200\delta$ for some $x \in A_s$, then there are $g_1, g_2 \in S(k_2)$ such that $\langle g_1, g_2 \rangle$ is free with basis $\{g_1, g_2\}$.*

4.1. Ping-Pong lemma. The proof of Proposition 4.1 is based on the following Ping-Pong lemma.

Lemma 4.2. *Let G be a group acting on a set X , and g_1, g_2 two elements of G . If X_1, X_2 are disjoint subsets of X and for all $n \neq 0$, $i \neq j$, $g_i^n(X_j) \subset X_i$, then the subgroup $\langle g_1, g_2 \rangle$ is free with basis $\{g_1, g_2\}$.*

We will apply the Ping-Pong lemma in the following setting.

Lemma 4.3. *Let X be a metric space and g_1, g_2 isometries of X . Let $B_i \subset A_i \subset X$ ($i = 1, 2$) be subsets of X , and $p_i : X \rightarrow A_i$ a map. Denote $X_i = p_i^{-1}(A_i - B_i)$. Then the assumptions in the Ping-Pong lemma are satisfied if the following conditions hold:*

- (1) $X_1 \cap X_2 = \emptyset$;
- (2) $g_i^n(p_i^{-1}(B_i)) \subset X_i$ for all $n \neq 0$, $i = 1, 2$. □

In our case, g_i is a hyperbolic element, A_i is an axis of g_i , $B_i \subset A_i$ is a segment, and $p_i = P_{A_i}$.

4.2. Two hyperbolic elements. In this section we prove Proposition 4.1. For this, we shall find two short hyperbolic elements and segments in their axes such that the two conditions in Lemma 4.3 are satisfied.

Let $\mathcal{B}$ be a fixed 200δ -separated G -invariant collection of horoballs centered at the parabolic points. Recall (see the proof of Lemma 3.1) that there is an integer $k_1 \geq 1$ such that for any $x \in Y(\mathcal{B})$, the cardinality of $\{g \in G : d(x, g(x)) \leq 200\delta\}$ is less than k_1 .

Lemma 4.4. *Let $g \in G$ be a hyperbolic element such that there is some $x \in A_g$ with $d(x, g(x)) \geq 200\delta$, and $\gamma \in G$ an element such that $\{\gamma(g_+), \gamma(g_-)\} \cap \{g_+, g_-\} = \emptyset$. Let c_1 and c_2 be axes of g and $h := \gamma g \gamma^{-1}$ respectively. Let $a = \min\{t \in \mathbb{R} : d(c_1(t), c_2) \leq 2\delta\}$ and $b = \max\{t \in \mathbb{R} : d(c_1(t), c_2) \leq 2\delta\}$, and denote $y = c_1(a)$, $z = c_1(b)$. Then $d(y, z) \leq 3k_1 d(x, g(x))$.*

Proof. Denote $t = d(x, g(x))$. Suppose $d(y, z) > 3k_1 t$. Let $y' = c_2(a') \in c_2$, $z' = c_2(b') \in c_2$ with $d(y, y'), d(z, z') \leq 2\delta$. It is easy to see from the δ -thin condition that the Hausdorff distance between yz and $y'z'$ is at most 4δ . After replacing h by its inverse if necessary, we may assume g and h translate in the “same direction”, that is, if both c_1 and c_2 are parameterized from the repelling fixed point toward the attracting fixed point, then $b' > a'$ (we have $b > a$ by definition).

Since h is a conjugate of g or its inverse, Lemma 3.5 implies that the inequality $|d(p, g^i(p)) - d(q, h^i(q))| \leq 40\delta$ holds for all $i \geq 1$, and all $p \in A_g$, $q \in A_h$. In particular, $|d(y, g^i(y)) - d(y', h^i(y'))| \leq 40\delta$ for all $i \geq 1$.

For $i \geq 0$, let $y_i = P_{c_1}(g^i(y))$ and $y'_i = P_{c_2}(h^i(y'))$. Lemma 3.4 implies that y_{i+1} lies between y_i and $c_1(+\infty)$. Since $d(y_i, y_{i+1}) \leq d(y, g(y)) + 4\delta \leq d(x, g(x)) + 40\delta + 4\delta \leq 2t$, we have $y_i \in yz$ for all $1 \leq i \leq k_1$. Similarly $y'_i \in y'z'$ for all $1 \leq i \leq k_1$. Hence for $1 \leq i \leq k_1$, there exists $z_i \in yz$ with $d(z_i, y'_i) \leq 4\delta$.

We first assume $y \in Y(\mathcal{B})$. The triangle inequality implies $|d(y, y_i) - d(y', y'_i)| \leq |d(y, g^i(y)) - d(y', h^i(y'))| + d(y'_i, h^i(y')) + d(y_i, g^i(y)) \leq 40\delta + 2\delta + 2\delta = 44\delta$ and $|d(y', y'_i) - d(y, z_i)| \leq d(y, y') + d(z_i, y'_i) \leq 6\delta$. It follows that

$$d(y_i, z_i) = |d(y, y_i) - d(y, z_i)| \leq 50\delta$$

and

$$\begin{aligned} d(g^i(y), h^i(y')) &\leq d(g^i(y), y_i) + d(y_i, z_i) + d(z_i, y'_i) + d(y'_i, h^i(y')) \\ &\leq 2\delta + 50\delta + 4\delta + 2\delta = 58\delta. \end{aligned}$$

Now for any i , $1 \leq i \leq k_1$, we have

$$\begin{aligned} d(y, h^{-i} \circ g^i(y)) &= d(h^i(y), g^i(y)) \leq d(h^i(y), h^i(y')) + d(h^i(y'), g^i(y)) \\ &= d(y, y') + d(h^i(y'), g^i(y)) \leq 60\delta. \end{aligned}$$

Since $y \in Y(\mathcal{B})$, it follows from the definition of k_1 that there are $i \neq j$, $1 \leq i, j \leq k_1$, with $h^{-i} \circ g^i = h^{-j} \circ g^j$. Consequently $h^{j-i} = g^{j-i}$, contradicting the fact that g and h do not share any fixed points in ∂X .

Now suppose $y \notin Y(\mathcal{B})$. Since $Y(\mathcal{B})$ is G -invariant, $g(y) \notin Y(\mathcal{B})$. There are horoballs $B_1, B_2 \in \mathcal{B}$ with $y \in B_1$ and $g(y) \in B_2$. Note $B_1 \neq B_2$, otherwise $g(B_1) = B_1$ and so g fixes the center of B_1 , contradicting the fact that g is a hyperbolic element. Since $\mathcal{B}$ is 200δ -separated and $d(y_1, g(y)) \leq 2\delta$, there is some $p \in c_1$ between y and y_1 such that $p \in Y(\mathcal{B})$. Now we run the argument from the pervious paragraph using p instead of y . $\square$

Remark 4.1. The proof of Lemma 4.4 actually shows that for any $\gamma \in G$, either $\{\gamma(g_+), \gamma(g_-)\} = \{g_+, g_-\}$ or $\{\gamma(g_+), \gamma(g_-)\} \cap \{g_+, g_-\} = \emptyset$ holds.

Now let S be a finite generating set of G and suppose $s \in S$ is a hyperbolic element such that $d(s(x), x) \geq 200\delta$ for some $x \in A_s$. If $\{s'(s_+), s'(s_-)\} = \{s_+, s_-\}$ for all $s' \in S$, then A_s is invariant under G . In this case, G is 2-ended and hence is virtually infinite cyclic.

We now assume there is some $\gamma \in S$ with $\{\gamma(s_+), \gamma(s_-)\} \cap \{s_+, s_-\} = \emptyset$. Let c be an axis of s . Set $g_1 = s^{10k_1}$ and $g_2 = \gamma g_1 \gamma^{-1}$, $A_1 = c$, $A_2 = \gamma(c)$, $p_1 = P_c$ and $p_2 = P_{\gamma(c)}$. If $d(A_1, A_2) > 2\delta$, let $p \in A_1$, $q \in A_2$ with $d(p, q) = d(A_1, A_2)$; in this case, let $B_1 \subset A_1$ be the closed segment with midpoint p and length 40δ and similarly for $B_2 \subset A_2$. If $d(A_1, A_2) \leq 2\delta$, let $a = \min\{t \in \mathbb{R} : d(c(t), A_2) \leq 2\delta\}$ and $b = \max\{t \in \mathbb{R} : d(c(t), A_2) \leq 2\delta\}$, and denote $y = c(a)$, $z = c(b)$. Also let $y', z' \in A_2$ with $d(y, y') \leq 2\delta$ and $d(z, z') \leq 2\delta$; in this case, let $B_1 \subset A_1$ be the closed 20δ -neighborhood of yz in A_1 and let $B_2 \subset A_2$ be the closed 20δ -neighborhood of $y'z'$ in A_2 . We shall show that the two conditions in Lemma 4.3 are satisfied, and hence Proposition 4.1 holds.

Lemma 4.5. *Condition (2) of Lemma 4.3 is satisfied.*

Proof. We only write down the proof in the case $d(A_1, A_2) \leq 2\delta$, $i = 1$ and $n > 0$; the other cases are similar or simpler.

Let $y \in p_1^{-1}(B_1)$. We need to show $p_1(g_1^n(y)) \notin B_1$. Denote $z = p_1(y)$, $y' = g_1^n(y)$, $z' = g_1^n(z)$ and $y'' = p_1(y')$. Lemma 3.5 implies $d(z, s(z)) \geq 160\delta$. Set $t = d(z, s(z))$. By Lemma 3.4 and the fact that $d(p_1(s^i(z)), p_1(s^{i+1}(z))) \geq t - 4\delta$, we obtain $d(p_1(z'), z) \geq 10k_1n(t - 4\delta)$.

Let $z'' = p_1(z')$. We claim $d(y'', z'') \leq 13\delta$. We first finish the proof assuming the claim. The claim implies $d(z, y'') \geq 10k_1(t - 4\delta)n - 13\delta$. Note Lemma 3.5 implies $t - 4\delta \geq 2d(x, s(x))/5$. It follows that $d(z, y'') \geq 4nk_1d(x, s(x)) - 13\delta \geq 4k_1d(x, s(x)) - 13\delta > 3k_1d(x, s(x)) + 40\delta \geq \text{length}(B_1)$. The last inequality follows from the definition of B_1 and Lemma 4.4. Since $z = p_1(y) \in B_1$, we have $p_1(g_1^n(y)) = y'' \notin B_1$.

We now prove the claim. We may assume $d(y'', z'') > 3\delta$. Fix geodesic segments $z'z''$, $y'y''$ and $y'y'$. Let $p \in y'y''$ with $d(p, y'') = 3\delta$. We use the δ -thin condition. Consider the geodesic triangle $(y'y''z'')$. There is some $q \in y'y'' \cup y'y'$ with $d(p, q) \leq \delta$. If $q \in y'y''$, then $d(q, y'') \geq d(p, y'') - d(p, q) \geq 2\delta$ and hence $d(q, p) < d(q, y'')$, contradicting the fact that $y'' = p_1(y')$. So $q \in y'y'$. Now consider the geodesic triangle $(y'z'z'')$. There is some $r \in z'z'' \cup z'y'$ with $d(q, r) \leq \delta$. If $r \in z'z''$, then $d(r, z'') \leq d(z', z'') \leq 2\delta$ and hence $d(y'', z'') = d(y'', p) + d(p, z'') \leq 3\delta + d(p, q) + d(q, r) + d(r, z'') \leq 7\delta$. Suppose $r \in y'y'$. Then $d(r, z') = d(r, g_1^n(A_1)) \leq$

$d(r, p) + d(p, g_1^n(A_1)) \leq d(r, q) + d(q, p) + 2\delta \leq 4\delta$. Therefore $d(z'', p) \leq d(z'', z') + d(z', r) + d(r, p) \leq 8\delta$ and $d(z'', y'') \leq 11\delta$. $\square$

The following lemma can be easily seen using approximating trees ([GhD]). For completeness, here we include a proof using only the δ -thin condition.

Lemma 4.6. *Condition (1) of Lemma 4.3 is satisfied.*

Proof. We give the proof for the case $d(A_1, A_2) > 2\delta$; the proof when $d(A_1, A_2) \leq 2\delta$ is similar.

We prove by contradiction. Assume $d(A_1, A_2) > 2\delta$ and $z \in X_1 \cap X_2$, where $X_i = p_i^{-1}(A_i - B_i)$. Without loss of generality, we may assume $d(z, A_2) \leq d(z, A_1)$. We claim that there is some $q' \in A_2$ with $d(p_1(z), q') \leq 8.2\delta$.

We first finish the proof of the lemma assuming the above claim. Fix geodesic segments $q'p_1(z)$, pq' and pq . Let $m \in A_1$ be the midpoint of $p_1(z)p \subset A_1$. Since $p_1(z) \notin B_1$, we have $d(p_1(z), p) > 20\delta$ and $d(m, p) = d(m, p_1(z)) > 10\delta$. The δ -thin condition applied to the triangle $(p, q', p_1(z))$ shows that there is some $n \in pq' \cup q'p_1(z)$ with $d(m, n) \leq \delta$. Since $d(p_1(z), q') \leq 8.2\delta$ and $d(m, p_1(z)) > 10\delta$, the point n must lie on pq' . Now the δ -thin condition applied to the triangle (p, q, q') shows that there is some $x \in pq \cup qq'$ with $d(n, x) \leq \delta$. If $x \in qq' \subset A_2$, then $d(m, x) \leq d(m, n) + d(n, x) \leq 2\delta$, contradicting the assumption that $d(A_1, A_2) > 2\delta$. Hence $x \in pq$. Since $p \in A_1$ and $q \in A_2$ realize the distance between A_1 and A_2 , we have $d(x, p) = d(x, A_1)$; it follows that $d(x, p) \leq d(x, n) + d(n, m) \leq 2\delta$. Consequently, $d(p, m) \leq d(p, x) + d(x, n) + d(n, m) \leq 4\delta$, contradicting $d(m, p) > 10\delta$.

Now we prove the claim. Recall that $d(z, A_2) \leq d(z, A_1)$. Since $d(A_1, A_2) > 2\delta$, we have $d(z, A_1) > \delta$. Pick $y \in zp_1(z)$ with $\delta < d(y, p_1(z)) \leq 1.1\delta$. Fix geodesic segments $zp_1(z)$, zp , zq and $zp_2(z)$. The δ -thin condition applied to the triangle $(z, p, p_1(z))$ shows that there is some $w_1 \in p_1(z)p \cup pz$ with $d(y, w_1) \leq \delta$. Since $d(z, p_1(z)) = d(z, A_1)$ and $y \in zp_1(z)$ with $d(y, p_1(z)) > \delta$, we must have $w_1 \in pz$. Similarly there is some $w_2 \in qz \cup pq$ with $d(w_1, w_2) \leq \delta$. The argument in the preceding paragraph shows $w_2 \in qz$. The δ -thin condition then shows that there is some $w_3 \in qp_2(z) \cup p_2(z)z$ with $d(w_2, w_3) \leq \delta$. If $w_3 \in qp_2(z)$, then $d(w_3, p_1(z)) \leq d(w_3, w_2) + d(w_2, w_1) + d(w_1, y) + d(y, p_1(z)) \leq 4.1\delta$, and the claim holds. Assume $w_3 \in zp_2(z)$. Then $d(w_3, z) \geq d(z, y) - d(y, w_1) - d(w_1, w_2) - d(w_2, w_3) \geq d(z, y) - 3\delta = d(z, p_1(z)) - d(y, p_1(z)) - 3\delta \geq d(z, p_1(z)) - 4.1\delta \geq d(z, p_2(z)) - 4.1\delta$. It follows that $d(w_3, p_2(z)) \leq 4.1\delta$. Consequently, $d(p_2(z), p_1(z)) \leq d(p_2(z), w_3) + d(w_3, p_1(z)) \leq 8.2\delta$. The proof of the claim is complete. $\square$

5. EXISTENCE OF HYPERBOLIC ELEMENTS

In this section we establish the following result that guarantees the existence of hyperbolic elements with short word length.

Proposition 5.1. *Let G be an infinite relatively hyperbolic group and X a δ -hyperbolic geodesic metric space that G acts upon as in Definition 2.1. Then there exists a positive integer n_0 with the following property: for any finite generating set S of G , $S(n_0)$ contains a hyperbolic element.*

For a finite set of isometries F of a metric space X and $x \in X$, we let $\lambda(x, F) = \max\{d(f(x), x) : f \in F\}$. We use the following result of M. Koubi ([K]).

Proposition 5.2. *Let X be a δ -hyperbolic geodesic metric space, and G a group of isometries of X with finite generating set S . If $\lambda(x, S) > 100\delta$ for all $x \in X$, then G contains a hyperbolic element g such that $d_S(id, g) = 1$ or 2 .*

Proposition 5.1 follows from Proposition 5.2 and the following result.

Lemma 5.3. *There exists a positive integer n_0 with the following property. For any finite generating set S of G , the inequality $\lambda(x, S(n_0)) > 100\delta$ holds for all $x \in X$.*

Proof. Let $\mathcal{B}$ be a 200δ -separated invariant system of horoballs centered at the parabolic points. Recall that the action of G on $Y(\mathcal{B})$ is proper and cocompact. Let $K \subset Y(\mathcal{B})$ be a compact set such that $\bigcup_{g \in G} g(K) = Y(\mathcal{B})$. Set $a = \text{diam}(K)$ and fix a point $p \in K$. Let $A = \{g \in G : d(g(p), p) \leq 2a + 100\delta\}$. Notice that A is a finite set. Since G is infinite, there is some $g_0 \in G$ with $d(g_0(p), p) > 100\delta + 2a$. Notice that if A generates G , then

$$n_0 := \max\{d_S(id, g_0) : S \subset A \text{ and } S \text{ generates } G\}$$

is finite.

Now let S be a finite generating set of G . Suppose there is some $x \in X$ with $\lambda(x, S) \leq 100\delta$. By the definition of $\lambda(x, S)$, we have $d(s(x), x) \leq 100\delta$ for all $s \in S$. Recall that $\mathcal{B}$ is 200δ -separated. If $x \in B$ for some $B \in \mathcal{B}$, then $s(B) = B$ for all $s \in S$ and consequently the center of B is fixed by the entire group G , a contradiction. Hence $x \in Y(\mathcal{B})$. There is some $\gamma \in G$ with $\gamma(x) \in K$. For $s \in S$, we have $d(p, \gamma s \gamma^{-1}(p)) \leq d(p, \gamma(x)) + d(\gamma(x), \gamma s(x)) + d(\gamma s(x), \gamma s \gamma^{-1}(p)) = d(p, \gamma(x)) + d(x, s(x)) + d(\gamma(x), p) \leq 2a + 100\delta$. It follows that $\gamma S \gamma^{-1} := \{\gamma s \gamma^{-1} : s \in S\} \subset A$. Clearly $\gamma S \gamma^{-1}$ generates G . There is some integer k , $1 \leq k \leq n_0$, and $s_i \in S \cup S^{-1}$ ($1 \leq i \leq k$) such that $g_0 = (\gamma s_1 \gamma^{-1}) \cdots (\gamma s_k \gamma^{-1}) = \gamma(s_1 \cdots s_k) \gamma^{-1}$. Now

$$\begin{aligned} d(\gamma^{-1} g_0 \gamma(x), x) &= d(g_0 \gamma(x), \gamma(x)) \geq d(g_0(p), p) - d(g_0(p), g_0 \gamma(x)) - d(\gamma(x), p) \\ &= d(g_0(p), p) - d(p, \gamma(x)) - d(\gamma(x), p) > 100\delta + 2a - a - a = 100\delta. \end{aligned}$$

Notice $\gamma^{-1} g_0 \gamma = s_1 \cdots s_k \in S(n_0)$. It follows that $\lambda(x, S(n_0)) > 100\delta$. $\square$

REFERENCES

- [AN] R. Alperin, G. Noskov, *Nonvanishing of algebraic entropy for geometrically finite groups of isometries of Hadamard manifolds*, preprint(2004).
- [B] B. Bowditch, *Relatively hyperbolic groups*, preprint(1999).
- [D] C. Drutu, *Quasi-isometric rigidity of groups*, preprint(2004).
- [EMO] A. Eskin, S. Mozes, H. Oh, *Uniform exponential growth for linear groups*, Int. Math. Res. Not. 2002, no. **31**, 1675–1683. MR1916428 (2003g:20074)
- [F] B. Farb, *Relatively hyperbolic groups*, Geom. Funct. Anal. **8** (1998), no. 5, 810–840. MR1650094 (99j:20043)
- [G] M. Gromov, *Hyperbolic Groups*, in “Essays in Group Theory” (ed. S. Gersten) M.S.R.I. Publications No.8, Springer-Verlag(1987) 75–263. MR0919829 (89e:20070)
- [GD] R. Grigorchuk, P. de la Harpe, *One-relator groups of exponential growth have uniformly exponential growth*, translation in Math. Notes 69 (2001), no. 3-4, 575–577. MR1846003 (2002b:20041)
- [GhD] E. Ghys and P. de la Harpe, *Sur les groupes hyperboliques d’après Mikhael Gromov*, Progress in Mathematics **83**.
- [HK] G. Hruska, B. Kleiner, *Hadamard spaces with isolated flats*, Geom. Topol. **9** (2005), 1501–1538. MR2175151
- [K] M. Koubi, *Croissance uniforme dans les groupes hyperboliques*, Ann. Inst. Fourier (Grenoble) **48** (1998), no. 5, 1441–1453. MR1662255 (99m:20080)

- [O1] D. Osin, *Relatively hyperbolic groups: Intrinsic geometry, algebraic properties, and algorithmic problems*, Mem. Amer. Math. Soc. **179** (2006), no. 843, vi+100pp. MR2182268
- [O2] D. Osin, *The entropy of solvable groups*, Ergodic Theory Dynam. Systems **23** (2003), no. 3, 907–918. MR1992670 (2004f:20065)
- [O3] D. Osin, *Weakly amenable groups*, Contemp. Math., **298** (2002), 105–113.
- [W] J. Wilson, *On exponential growth and uniformly exponential growth for groups*, Invent. Math. **155** (2004), no. 2, 287–303. MR2031429 (2004k:20085)
- [Y] A. Yaman, *A topological characterisation of relatively hyperbolic groups*, J. Reine Angew. Math. **566** (2004), 41–89. MR2039323 (2005e:20064)

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