THE LAPLACIAN SUBALGEBRA OF $\mathcal{L}(\mathbb{F}_N)^{\overline{\otimes} k}$ IS A STRONGLY SINGULAR MASA

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ABSTRACT. In this paper, we present a new class of strongly singular maximal abelian subalgebras living inside the k -folded tensor product of the free group factor $\mathcal{L}(\mathbb{F}_N)$ with itself ($N \geq 2$). The notions of strongly singular masas in type II_1 factors and that of asymptotic homomorphism were introduced by A. Sinclair and R. Smith. One of their first examples was the Laplacian subalgebra of the free group factor, generated by the sum of words of length 1 in $\mathbb{F}_N$. This subalgebra was known to be a singular masa. Using the results of A. Sinclair and R. Smith, we show that the unique trace-preserving conditional expectation onto the Laplacian subalgebra of $\mathcal{L}(\mathbb{F}_N)^{\overline{\otimes} k}$ is an asymptotic homomorphism, and hence the Laplacian subalgebra is a strongly singular masa for every $k \geq 1$.

1. INTRODUCTION

Consider a type II_1 factor M , with trace τ , and let $A \subset M$ be a maximal abelian von Neumann subalgebra (masa) of M . Following Dixmier [3], consider the group of normalizing unitaries of A ,

$$N_M(A) = \{u \in A \mid u \text{ unitary, } uAu^* = A\}.$$

According to the size of $B = N_M(A)''$ in M , two extreme situations can occur: $B = M$, that is, B has enough unitaries to generate M and in this case A is called *regular* or *Cartan*; $B = A$, in which case the only normalizing unitaries are those in A , and B is called *singular*. Regular masas appear in a natural way in cross-products: $L^\infty[0, 1]$ is such a regular masa in its crossed product with some countable group G . On the other hand, until recently, singular masas were hard to find. Dixmier [3], who started the investigation of masas, gave the example of the singular masa inside the free group factor $\mathcal{L}(\mathbb{F}_2)$. The masa is generated by one of the two generators of $\mathbb{F}_2$ inside the factor. Building on work of Pytlik [6] and Popa [4], [5], Rădulescu [1] showed, using the Pukansky invariant, that the Laplacian subalgebra of $\mathcal{L}(\mathbb{F}_N)$ is a singular masa. The Laplacian subalgebra is

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also a singly-generated algebra and the generator is the selfadjoint element

$$w = \sum_{i=1}^N (g_i + g_i^{-1}),$$

where we have identified the generators $g_1, \dots, g_N$ of $\mathbb{F}_N$ with the unitaries in $\mathcal{L}(\mathbb{F}_N)$ given by the left regular representation. Together with Boca [2], they gave more examples of singular masas in group von Neumann algebras of certain free products of groups. It was Popa who actually showed in [4] that every separable II_1 factor has a singular masa. On the other hand, it is worth mentioning that Voiculescu [10] proved the absence of Cartan subalgebras in free group factors.

The class of strongly singular masas, introduced by Sinclair and Smith in [9] turn out to be more tractable, and a plethora of examples can be found now in [8], [9], [7], mostly in the context of group von Neumann algebras or crossed products with groups. The good results are due to the notion of asymptotic homomorphism and to good criteria to detect such maps. Among the first examples of strongly singular masas was the Laplacian subalgebra of the free group factor.

The aim of this paper is to present new examples of strongly singular masas living inside tensor products of $\mathcal{L}(\mathbb{F}_N)$ with itself. The examples are natural generalizations of the Laplacian subalgebra of $\mathcal{L}(\mathbb{F}_N)$. In $\mathcal{L}(\mathbb{F}_N)^{\overline{\otimes}_k}$, we consider the abelian von Neumann subalgebra generated by

$$w^{(k)} = \sum_{i=1}^N (g_i^{\otimes k} + (g_i^{-1})^{\otimes k}),$$

and show that the unique trace-preserving conditional expectation onto this subalgebra is an asymptotic isomorphism, and hence the subalgebra is a strongly singular masa for each $k \geq 1$.

The paper is organized as follows: after some preliminaries in section 2, we prove in section 3 an essential lemma on free groups. In section 4, we review the notion of strongly singular masa and the tools needed to prove our result. The last section contains the main result.

2. PRELIMINARIES AND NOTATION

We denote by $\mathbb{F}_N$ the free group on a fixed set of $N \geq 2$ generators $g_1, g_2, \dots, g_N$. We will call $S_N = \{g_1, g_2, \dots, g_N, g_1^{-1}, g_2^{-1}, \dots, g_N^{-1}\}$ a generating set for $\mathbb{F}_N$. We denote by e the empty word. On $\mathbb{F}_N$, we will consider the canonical length function $|\cdot|$, defined by $|e| = 0$ and $|w| = |\alpha_1| + |\alpha_2| + \dots + |\alpha_p|$ if $w \in \mathbb{F}_N$ has the reduced form $g_{i_1}^{\alpha_1} g_{i_2}^{\alpha_2} \dots g_{i_p}^{\alpha_p}$ ($i_j \neq i_{j+1}$).

The von Neumann group algebra $\mathcal{L}(\mathbb{F}_N)$ is the norm closure of the left regular representation of $\mathbb{F}_N$ on $l^2(\mathbb{F}_N)$. It is well known that $\mathcal{L}(\mathbb{F}_N)$ is a type II_1 factor acting standardly on $l^2(\mathbb{F}_N)$ and that with this identification the $\|\cdot\|_\tau$ norm coincides with the usual norm $\|\cdot\|_2$ on $l^2(\mathbb{F}_N)$. We will use the same notation $\|\cdot\|_2$ for the norm $\|\cdot\|_\tau$ given by $\|x\|_\tau = (\tau(x^*x))^{1/2}$. We will identify elements in $\mathbb{F}_N$ with their left regular representations. We will also identify the elements of $\mathcal{L}(\mathbb{F}_N)$ with the corresponding vectors in $L^2(\mathcal{L}(\mathbb{F}_N), \tau) = l^2(\mathbb{F}_N)$.

If H is a Hilbert space, we denote by $H^{\otimes k}$ the k -folded tensor product of H with itself (with the standard convention $H^{\otimes 0} := \mathbb{C}$, $H^{\otimes 1} := H$). An element $v \in H$ embeds as $v \otimes \dots \otimes v =: v^{\otimes k} \in H^{\otimes k}$. For a von Neumann algebra M , $M^{\overline{\otimes}_k}$ denotes

the k -folded von Neumann algebra tensor product of M with itself ($M^{\overline{\otimes}_0} := \mathbb{C}$, $M^{\overline{\otimes}_1} := M$). For $x \in M$, $x \otimes \dots \otimes x =: x^{\otimes_k} \in M^{\overline{\otimes}_k}$.

For $n \geq 0$, we define $w_n^{(k)}$ to be the sum of all k -folded tensors of reduced words in $\mathbb{F}_N$ of length n :

$$\begin{aligned} w_0^{(k)} &= e^{\otimes_k}, \\ w_1^{(k)} &= \sum_{i=1}^N g_i^{\otimes_k} + \sum_{i=1}^N (g_i^{-1})^{\otimes_k}, \\ w_n^{(k)} &= \sum_{|v|=n} v^{\otimes_k}. \end{aligned}$$

The following relations are well known for $k = 1$ and hold for any $k \geq 1$:

$$(2.1) \quad w_1^{(k)} w_n^{(k)} = w_n^{(k)} w_1^{(k)} = w_{n+1}^{(k)} + (2N - 1)w_{n-1}^{(k)}.$$

These relations show that the von Neumann algebra $\mathcal{B}^{(k)}$ generated by $w_1^{(k)}$ contains each $w_n^{(k)}$, and hence is the weak closure of the span of these elements:

$$\mathcal{B}^{(k)} = \overline{\text{Sp}\{w_n^{(k)} | n \geq 0\}}^w \subset \mathcal{L}(\mathbb{F}_N)^{\overline{\otimes}_k}.$$

We call $\mathcal{B}^{(k)}$ the *Laplacian* or *radial subalgebra*, in analogy with the case $k = 1$. The selfadjoint elements $w_n^{(k)}$ are pairwise orthogonal with respect to the trace $\tau^{\otimes_k}$, and a simple counting argument shows that

$$(2.2) \quad \|w_n^{(k)}\|_2^2 = 2N(2N - 1)^{n-1}, \quad n \geq 1,$$

which represents the number of words of length n . By the previous remarks,

$$\left\{ \frac{w_n^{(k)}}{\|w_n^{(k)}\|_2} | n \geq 0 \right\}$$

is an orthonormal basis for $L^2(\mathcal{B}^{(k)}, \tau)$. Moreover, if $\mathbb{E}_k : \mathcal{L}(\mathbb{F}_N)^{\overline{\otimes}_k} \rightarrow \mathcal{B}^{(k)}$ is the unique trace-preserving conditional expectation, then

$$(2.3) \quad \mathbb{E}_k(x) = \sum_{n=0}^{\infty} \tau^{\otimes_k}(x w_n^{(k)}) \frac{w_n^{(k)}}{\|w_n^{(k)}\|_2^2}.$$

In particular, when v is a reduced word of length p :

$$(2.4) \quad \mathbb{E}_k(v^{\otimes_k}) = \frac{w_p^{(k)}}{\|w_p^{(k)}\|_2^2}.$$

Lemma 2.1. *For $k \geq 1$, let $x_1, \dots, x_k \in \mathbb{F}_N$. With the above notation,*

$$\mathbb{E}_k(x_1 \otimes \dots \otimes x_k) \neq 0 \Leftrightarrow x_1 = \dots = x_k.$$

Proof. Recall from (2.3) that

$$\mathbb{E}_k(x_1 \otimes \dots \otimes x_k) = \sum_{n=0}^{\infty} \tau^{\otimes_k}(x_1 \otimes \dots \otimes x_k w_n^{(k)}) \frac{w_n^{(k)}}{\|w_n^{(k)}\|_2^2}.$$

Therefore $\mathbb{E}_k(x_1 \otimes \dots \otimes x_k) \neq 0$ if and only if there is some $n \geq 0$ so that

$$\tau^{\otimes_k}(x_1 \otimes \dots \otimes x_k w_n^{(k)}) \neq 0.$$

Because of the definitions of $w_n^{(k)}$ and $\tau^{\otimes k}$, this can happen if and only if for some $n \geq 0$:

$$\sum_{|v|=n} \tau^{\otimes k}(x_1 v \otimes \dots \otimes x_k v) \neq 0,$$

or

$$\sum_{|v|=n} \tau(x_1 v) \dots \tau(x_k v) \neq 0.$$

But this is true exactly when for some v of length $n \geq 0$ we have

$$x_1 = \dots = x_k = v^{-1}.$$

□

3. COUNTING WORDS IN $\mathbb{F}_N$

In this section we prove a technical lemma about solutions of a conjugacy equation in $\mathbb{F}_N$. If $a, b \in \mathbb{F}_N$ are two words and no cancellations occur in their concatenation, we will denote this by $a \cdot b$. In general, the product (concatenation) will be denoted ab , meaning that cancellations may or may not take place.

Lemma 3.1. *Let $a, b \in \mathbb{F}_N$ be non-trivial words.*

(i) *The equation $x \cdot a = b \cdot x$ has in $\mathbb{F}_N$ at most one solution of fixed positive length $l \geq 1$.*

(ii) *The equation $xa = bx$ has in $\mathbb{F}_N$ at most one solution of fixed positive length $l > |a| + |b|$.*

Proof. (i) Fix a length $l \geq 1$. Suppose that a non-trivial word $x \in \mathbb{F}_N$ satisfies $x \cdot a = b \cdot x$ and has length $|x| = l$. This is possible only if $|a| = |b|$.

If x is shorter than a, b , then it is a suffix for a and a prefix for b . Furthermore, if $a = a_1 \cdot x$ with $a_1 \in \mathbb{F}_N$, then $b = x \cdot a_1$ and $x = a_1^{-1}a$ is the unique solution. As we can see, the solution of prescribed length exists only for particular a, b . It depends also on the desired length, so if it exists, it is unique by construction.

If the length l of x is greater than $|a| = |b|$, then x starts with b . Suppose $x = b^m \cdot x_1$, $m \geq 1$, maximal with this property. This forces $|x_1| < |b| = |a|$ and $x_1 \cdot a = b \cdot x_1$. As shown in the first paragraph, there is at most one solution x_1 , hence at most one solution x of prescribed length l .

(ii) Next we consider possible cancellations, that is, assume x of length $l > |a| + |b|$ satisfies $xa = bx$. This means that there are words a_1, a_2, b_1, b_2 , and x_1 such that $a = a_1 \cdot a_2 \neq 1$, $b = b_2 \cdot b_1 \neq 1$, and $x = b_1^{-1} \cdot x_1 \cdot a_1^{-1}$, $|x_1| \geq 1$, and at least one of a_1 or b_1 is a non-empty word. Looking at the lengths in the resulting equation

$$b_1^{-1} \cdot x_1 \cdot a_2 = b_2 \cdot x_1 \cdot a_1^{-1}$$

we see that

$$(3.1) \quad |b_1^{-1}| + |a_2| = |b_2| + |a_1^{-1}| \Leftrightarrow |a_1| - |a_2| = |b_1| - |b_2|.$$

These are necessary conditions on a, b for the existence of a solution. By hypothesis, we have $|a_1| > 0$ or $|b_1| > 0$. Note also that $|a_2 a_1| > 0, |b_1 b_2| > 0$. If $|a_1| < |a_2|$, then (3.1) implies $|b_1| < |b_2|$ and the equation can be written

$$x_1 \cdot (a_2 a_1) = (b_1 b_2) \cdot x_1.$$

The case $|a_1| > |a_2|$ leads to

$$(b_2^{-1} b_1^{-1}) \cdot x_1 = x_1 \cdot (a_1^{-1} a_2^{-1}).$$

As before, the above equations, and hence the original equation, have at most one solution of prescribed length. The case $|a_1| = |a_2|$ is not possible, since it leads to $a_1^{-1} = a_2$ and we have assumed $a = a_1 \cdot a_2$ is non-trivial. $\square$

4. STRONGLY SINGULAR MASA'S

A maximal abelian selfadjoint subalgebra $\mathcal{A}$ in a II_1 factor $\mathcal{M}$ is singular if any unitary $u \in \mathcal{M}$ which normalizes $\mathcal{A}$ (i.e., $u\mathcal{A}u^* = \mathcal{A}$) must lie in $\mathcal{A}$.

To define the notion of strongly singular masa, let $\phi : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ be a linear map between two type II_1 factors. There are several norms for ϕ , depending on the norms considered on the two algebras. When $\mathcal{M}_1$ has the operator norm and $\mathcal{M}_2$ has the $\|\cdot\|_2$ -norm given by the trace, we denote the resulting norm for ϕ by $\|\phi\|_{\infty,2}$, following [9].

Definition 4.1. Suppose $\mathcal{A}$ is a masa in a type II_1 factor $\mathcal{M}$.

- 1) $\mathcal{A}$ is called α -strongly singular (or simply strongly singular for $\alpha = 1$) if

$$\|\mathbb{E}_{u\mathcal{A}u^*} - \mathbb{E}_{\mathcal{A}}\|_{\infty,2} \geq \alpha \|u - \mathbb{E}_{\mathcal{A}}(u)\|_2,$$

for all unitaries $u \in \mathcal{M}$.

- 2) The conditional expectation $\mathbb{E}_{\mathcal{A}}$ is an asymptotic homomorphism if there is a unitary $u \in \mathcal{A}$ such that

$$\lim_{|k| \rightarrow \infty} \|\mathbb{E}_{\mathcal{A}}(xu^k y) - \mathbb{E}_{\mathcal{A}}(x)\mathbb{E}_{\mathcal{A}}(y)u^k\|_2 = 0$$

for all $x, y \in \mathcal{M}$.

The following provides the criteria we will use in the last section to prove the main result of this paper.

Theorem 4.2 (Sinclair and Smith). *Let $\mathcal{A}$ be an abelian von Neumann subalgebra of a type II_1 factor $(\mathcal{M}, \text{tr})$, and suppose that there is a $*$ -isomorphism $\pi : \mathcal{A} \rightarrow L^\infty[0, 1]$ which induces an isometry from $L^2(\mathcal{A}, \text{tr})$ onto $L^2[0, 1]$. Let $\{v_n \in \mathcal{A} | n \geq 0\}$ be an orthonormal basis for $L^2(\mathcal{A}, \text{tr})$, and let $Y \subseteq \mathcal{M}$ be a set whose linear span is norm dense in $L^2(\mathcal{M}, \text{tr})$. Let $\mathbb{E}_{\mathcal{A}} : \mathcal{M} \rightarrow \mathcal{A}$ be the unique conditional expectation satisfying $\text{tr} \circ \mathbb{E}_{\mathcal{A}} = \text{tr}$. If*

$$\sum_{n=0}^{\infty} \|\mathbb{E}_{\mathcal{A}}(xv_n y) - \mathbb{E}_{\mathcal{A}}(x)\mathbb{E}_{\mathcal{A}}(y)v_n\|_2^2 < \infty$$

for all $x, y \in Y$, then $\mathbb{E}_{\mathcal{A}}$ is an asymptotic homomorphism and $\mathcal{A}$ is a strongly singular masa.

Remark 4.3. As abelian von Neumann algebras, $\mathcal{B}^{(k)} \subset B(L^2(\mathcal{B}^{(k)}, \tau^{\otimes k}))$ and $\mathcal{B}^{(1)} \subset B(L^2(\mathcal{B}^{(1)}, \tau))$ are $*$ -isomorphic and the isomorphism induces an isometry between the L^2 -spaces, sending $w_n^{(k)}$ to $w_n^{(1)}$, $n \geq 0$. In [8] it is shown that $\mathcal{B}^{(1)}$ and $\mathbb{E}_1 : \mathcal{L}(\mathbb{F}_N) \rightarrow \mathcal{B}^{(1)}$ satisfy the hypothesis of the above theorem (so $\mathbb{E}_1$ is an asymptotic homomorphism and $\mathcal{B}^{(1)}$ is a strongly singular masa). Via the $*$ -isomorphism between $\mathcal{B}^{(k)}$ and $\mathcal{B}^{(1)}$ it follows that $\mathcal{B}^{(k)}$ also satisfies the hypothesis of the theorem, for all $k \geq 2$.

We further exploit the $*$ -isomorphism between $\mathcal{B}^{(k)}$ and $\mathcal{B}^{(1)}$ in the following.

Lemma 4.4. *Let $x, y \in \mathbb{F}_N$ with $|x| = l \geq 2, |y| = m \geq 2$. The following equality holds for every $k \geq 1, n \geq l + m$:*

$$(4.1) \quad \|\mathbb{E}_k(x^{\otimes k} w_n^{(k)} y^{\otimes k}) - \mathbb{E}_k(x^{\otimes k}) \mathbb{E}_k(y^{\otimes k}) w_n^{(k)}\|_2^2 = \|\mathbb{E}_1(x w_n^{(1)} y) - \mathbb{E}_1(x) \mathbb{E}_1(y) w_n^{(1)}\|_2^2.$$

Proof. To simplify the notation, let $w_n^{(1)} := w_n$. Note that if M is a von Neumann algebra with trace tr and $a, b \in M$, then $\|a - b\|_2^2 = \|a\|_2^2 + \|b\|_2^2 - 2\text{Re}(\text{tr}(a^*b))$. Hence to prove (4.1) means to show

$$(4.2) \quad \|\mathbb{E}_k(x^{\otimes k} w_n^{(k)} y^{\otimes k})\|_2^2 = \|\mathbb{E}_1(x w_n y)\|_2^2,$$

$$(4.3) \quad \|\mathbb{E}_k(x^{\otimes k}) \mathbb{E}_k(y^{\otimes k}) w_n^{(k)}\|_2^2 = \|\mathbb{E}_1(x) \mathbb{E}_1(y) w_n\|_2^2 \text{ and}$$

$$(4.4) \quad \tau^{\otimes k}(\mathbb{E}_k(x^{\otimes k} w_n^{(k)} y^{\otimes k})^* \mathbb{E}_k(x) \mathbb{E}_k(y) w_n^{(k)}) = \tau(\mathbb{E}_1(x w_n y)^* \mathbb{E}_1(x) \mathbb{E}_1(y) w_n).$$

Let $\mu(r, s, n; x, y)$ be the number of reduced words in the product $x w_n y$ which result from r cancellations on the left and s cancellations on the right. Remark that we have assumed $n \geq l + m$. We have

$$\mathbb{E}_k(x^{\otimes k} w_n^{(k)} y^{\otimes k}) = \sum_{r=0}^l \sum_{s=0}^m \mu(r, s, n; x, y) w_{m+l+n-2(r+s)}^{(k)} \|w_{m+l+n-2(r+s)}^{(k)}\|_2^{-2}.$$

Because the selfadjoint elements $w_j^{(k)}, j \geq 0$, are pairwise orthogonal with respect to the trace, we get

$$\|\mathbb{E}_k(x^{\otimes k} w_n^{(k)} y^{\otimes k})\|_2^2 = \sum_{r=0}^l \sum_{s=0}^m \left(\frac{\mu(r, s, n; x, y)}{\|w_{m+l+n-2(r+s)}^{(k)}\|_2} \right)^2$$

and (4.2) follows using (2.2). Next

$$\begin{aligned} \|\mathbb{E}_k(x^{\otimes k}) \mathbb{E}_k(y^{\otimes k}) w_n^{(k)}\|_2^2 &= \frac{1}{\|w_l^{(k)}\|_2^4} \cdot \frac{1}{\|w_m^{(k)}\|_2^4} \cdot \|w_l^{(k)} w_m^{(k)} w_n^{(k)}\|_2^2 \\ &= \frac{1}{\|w_l\|_2^4 \cdot \|w_m\|_2^4} \cdot \sum_{\substack{|v_1| = |t_1| = n \\ |v_2| = |t_2| = l \\ |v_3| = |t_3| = m}} (\tau(v_1 v_2 v_3 t_3 t_2 t_1))^k = \frac{1}{\|w_l\|_2^4 \cdot \|w_m\|_2^4} \cdot \|w_l w_m w_n\|_2^2 \\ &= \|\mathbb{E}_1(x) \mathbb{E}_1(y) w_n\|_2^2, \end{aligned}$$

and this proves (4.3). Finally,

$$\begin{aligned} &\tau^{\otimes k}((\mathbb{E}_k(x^{\otimes k} w_n^{(k)} y^{\otimes k}))^* \mathbb{E}_k(x) \mathbb{E}_k(y) w_n^{(k)}) \\ &= \tau^{\otimes k}((\mathbb{E}_k(x^{\otimes k} w_n^{(k)} y^{\otimes k}))^* \frac{w_l^{(k)}}{\|w_l^{(k)}\|_2^2} \frac{w_m^{(k)}}{\|w_m^{(k)}\|_2^2} w_n^{(k)}) \\ &= \sum_{r=0}^l \sum_{s=0}^m \mu(r, s, n; x^{-1}, y^{-1}) \tau^{\otimes k} \left(\frac{w_{m+l+n-2(r+s)}^{(k)}}{\|w_{m+l+n-2(r+s)}^{(k)}\|_2^2} \frac{w_l^{(k)}}{\|w_l^{(k)}\|_2^2} \frac{w_m^{(k)}}{\|w_m^{(k)}\|_2^2} w_n^{(k)} \right). \end{aligned}$$

The coefficients depend only on x, y . The norms in the denominator also coincide for all k . It only remains to compute the trace. As before, this reduces to linear combinations of powers of the initial trace (the canonical trace of the free group factor). Since on group elements the trace is 0 or 1, we see that (4.4) follows. Hence we have proved (4.1). $\square$

5. THE CASE OF TENSORS OF THE FREE GROUP FACTORS

In what follows, we will identify elements $g \in \mathbb{F}_N$ with their left regular representations $\lambda(g) \in \mathcal{L}(\mathbb{F}_N)$ and we will use the notation from section 1.

The next technical lemma contains the calculations needed to prove the main result.

Lemma 5.1. *Let $k \geq 1$ and $w_n^{(k)} = \sum_{|v|=n} v^{\otimes k}$. Let $x_1, \dots, x_k, y_1, \dots, y_k$ be words in $\mathbb{F}_N$. Then*

$$(5.1) \quad \sum_{n=0}^{\infty} \frac{1}{\|w_n^{(k)}\|_2^2} \|\mathbb{E}_k(x_1 \otimes \dots \otimes x_k w_n^{(k)} y_1 \otimes \dots \otimes y_k) - \mathbb{E}_k(x_1 \otimes \dots \otimes x_k) \mathbb{E}_k(y_1 \otimes \dots \otimes y_k) w_n^{(k)}\|_2^2 < \infty.$$

Proof. We look first at the case $\mathbb{E}_k(x_1 \otimes \dots \otimes x_k) \mathbb{E}_k(y_1 \otimes \dots \otimes y_k) \neq 0$. By Lemma 2.1, it follows that $x_1 = \dots = x_k =: x$ and $y_1 = \dots = y_k =: y$. Let $l = |x|, m = |y|$. If $l = 1$ or $m = 1$ the inequality (5.1) is trivial. Indeed, suppose $l = 1$ (the other case is even easier). Then we actually have

$$\begin{aligned} & \|\mathbb{E}_k(x^{\otimes k} w_n^{(k)} y^{\otimes k}) - \mathbb{E}_k(x^{\otimes k}) \mathbb{E}_k(y^{\otimes k}) w_n^{(k)}\|_2^2 \\ &= \|\mathbb{E}_k(w_n^{(k)} y^{\otimes k}) - \mathbb{E}_k(y^{\otimes k}) w_n^{(k)}\|_2^2 \\ &= \|w_n^{(k)} \mathbb{E}_k(y^{\otimes k}) - \mathbb{E}_k(y^{\otimes k}) w_n^{(k)}\|_2^2 = 0 \end{aligned}$$

because $\mathcal{B}^{(k)}$ is abelian. We are left with the case $l, m \geq 2$. Using Lemma 4.4, the finiteness of the sum in (5.1) follows in this case from the corresponding result in [8].

For the remainder of the proof, assume

$$\mathbb{E}_k(x_1 \otimes \dots \otimes x_k) \mathbb{E}_k(y_1 \otimes \dots \otimes y_k) = 0.$$

By Lemma 2.1, there must be at least two distinct x_{i_1}, x_{i_2} and at least two distinct y_{j_1}, y_{j_2} . The non-zero terms in the series (5.1),

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{1}{\|w_n^{(k)}\|_2^2} \|\mathbb{E}_k(x_1 \otimes \dots \otimes x_k w_n^{(k)} y_1 \otimes \dots \otimes y_k)\|_2^2 \\ &= \sum_{n=0}^{\infty} \frac{1}{\|w_n^{(k)}\|_2^2} \left\| \sum_{|v|=n} \mathbb{E}_k(x_1 v y_1 \otimes \dots \otimes x_k v y_k) \right\|_2^2, \end{aligned}$$

are given by the solutions v of length $n \geq 0$ of the equations

$$x_1 v y_1 = x_2 v y_2 = \dots = x_k v y_k.$$

A solution exists only if $x_i = x_j \Leftrightarrow y_i = y_j$ for $1 \leq i, j \leq k+1$. Without loss of generality, assume $x_1 \neq x_2 (\Leftrightarrow y_1 \neq y_2)$. Any solution of length n for the above

system of equations is a solution of $v(y_1 y_2^{-1}) = (x_1^{-1} x_2) v$. Clearly $x_1^{-1} x_2 \neq e \neq y_1 y_2^{-1}$. By Lemma 3.1, for each fixed $n > \max\{|x_i| : 1 \leq i \leq k+1\} + \max\{|y_j| : 1 \leq j \leq k+1\} =: n_0$ there is at most one solution to this equation. So

$$\begin{aligned} \sum_{n=n_0}^{\infty} \frac{1}{\|w_n^{(k)}\|_2^2} \left\| \sum_{|v|=n} \mathbb{E}_k(x_1 v y_1 \otimes \dots \otimes x_k v y_k) \right\|_2^2 \\ \leq \sum_{n=n_0}^{\infty} \frac{1}{\|w_n^{(k)}\|_2^2} \left\| \frac{w_n^{(k)}}{\|w_n^{(k)}\|_2^2} \right\|_2^2 \\ = \sum_{n=n_0}^{\infty} \frac{1}{\|w_n^{(k)}\|_2^4} \stackrel{(2.2)}{<} \infty. \end{aligned}$$

□

Theorem 5.2. *For each $k \geq 1$, the conditional expectation $\mathbb{E}_k$ onto the Laplacian subalgebra $\mathcal{B}^{(k)}$ of $\mathcal{L}(\mathbb{F}_N)^{\overline{\otimes}_k}$, generated by*

$$w_1^{(k)} = \sum_{v \in \mathbb{F}_N, |v|=1} v^{\otimes k},$$

is an asymptotic homomorphism and $\mathcal{B}^{(k)}$ is a strongly singular masa.

Proof. For $k = 1$, $\mathcal{L}(\mathbb{F}_N)^{\overline{\otimes}_1} := \mathcal{L}(\mathbb{F}_N)$ and the result was proved in [8]. For $k \geq 2$, apply Theorem 4.2: use Remark 4.3 and the proof for $k = 1$ to verify the statement about the existence of the $*$ -isometry; use $Y = \{x_1 \otimes \dots \otimes x_k | x_i \in \mathbb{F}_N, 1 \leq i \leq k\}$ and $v_n := w_n^{(k)} / \|w_n^{(k)}\|_2, n \geq 0$. The theorem is a direct consequence of the above technical lemma and Theorem 4.2. □

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