ON A PROBLEM OF D. H. LEHMER

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ABSTRACT. Let p be an odd prime number. For $n \in \{1, \dots, p-1\}$ we denote the inverse of n modulo p by n^* with $n^* \in \{1, \dots, p-1\}$. Given $\varepsilon > 0$, we prove that in any range $n \in \{N+1, \dots, N+L\} \subseteq \{1, \dots, p-1\}$ of length $L \geq p^{1/2+\varepsilon}$ the probability that n^* has the same parity as n tends to $1/2$ as $p \rightarrow +\infty$. This result was previously known only to hold true in the full range $n \in \{1, \dots, p-1\}$ of length $L = p-1$. We will also obtain quantitative results on the pseudorandomness of the sequence $(-1)^{n+n^*}$ for which we estimate the well-distribution W and correlation measures C_k as defined by Mauduit and Sárközy (1997).

1. INTRODUCTION

The aim of this paper is to use a construction proposed by D. H. Lehmer (See R. Guy [5, pp. 139-140]) to give new examples of sequences with good pseudorandom properties. Let p be an odd prime number. For $n \in \{1, \dots, p-1\}$ we denote the inverse of n modulo p by n^* with $n^* \in \{1, \dots, p-1\}$. By convention we fix $0^* = 0$. Let $E_p = (e_1, \dots, e_{p-1})$ defined by

$$(1) \quad e_n = (-1)^{n+n^*} = \begin{cases} +1 & \text{if } n^* \equiv n \pmod{2}, \\ -1 & \text{if } n^* \not\equiv n \pmod{2}. \end{cases}$$

Z. Wenpeng [11, Theorem 1] and [12, Corollary, p. 48] (see also [13, 14, 15, 16]) has proved that

$$(2) \quad \left| \sum_{1 \leq n \leq p-1} e_n \right| \ll p^{1/2} \log^2 p.$$

Roughly speaking, this means that half of the $p-1$ integers $n \in \{1, \dots, p-1\}$ satisfy $n^* \equiv n \pmod{2}$ (in which case $e_n = +1$) while the other half satisfy $n^* \not\equiv n \pmod{2}$ ($e_n = -1$). However, (2) does not prevent the e_n 's from all being equal to $+1$ (or to -1) for long strings of consecutive integers, say of length $(p-1)/2$. Our first result generalizes estimate (2) to short sums, thus proving that the e_n 's cannot all be equal to $+1$ (or to -1) for strings of consecutive integers of length $\gg p^{1/2} \log^2 p$.

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Theorem 1. For $1 \leq M < N < p$, we have

$$(3) \quad \left| \sum_{M \leq n \leq N} e_n \right| \leq \frac{8}{\pi^2} p^{1/2} \log^2(5p) + 2.$$

Not only is our bound (3) much more general than (2), but our proof is elementary and much simpler than Z. Wenpeng's proof of (2).

In a series of papers, partly with further co-authors, we studied finite pseudo-random binary sequences

$$E_N = \{e_1, \dots, e_N\} \in \{-1, +1\}^N.$$

Our goal was to develop a constructive theory of pseudorandomness, *i.e.*, to introduce and study measures of pseudorandomness which cover pseudorandom properties important in applications and which can be handled and estimated at least for large families of binary sequences. Indeed, in [6] Mauduit and Sárközy introduced the following measures of pseudorandomness: the *well-distribution measure* of E_N is defined by

$$W(E_N) = \max_{a,b,t} \left| \sum_{j=0}^{t-1} e_{a+jb} \right|,$$

where the maximum is taken over all $a, b, t \in \mathbf{N}$ such that $1 \leq a \leq a + (t-1)b \leq N$, and the *correlation measure of order k* of E_N is defined as

$$C_k(E_N) = \max_{M,D} \left| \sum_{n=1}^M e_{n+d_1} e_{n+d_2} \cdots e_{n+d_k} \right|$$

where the maximum is taken over all $D = (d_1, \dots, d_k)$ and M such that $0 \leq d_1 < \cdots < d_k \leq N - M$. Then the sequence is considered as a “good” pseudorandom sequence if both these measures $W(E_N)$ and $C_k(E_N)$ (at least for small k) are “small” in terms of N (in particular both are $o(N)$ as $N \rightarrow \infty$). Notice that for almost all $E_N \in \{-1, +1\}^N$, both $W(E_N)$ and $C_k(E_N)$ are less than $N^{1/2}(\log N)^c$ (see [2] and [1]). One might like to know whether the sequence (e_n) defined by (1) is such a “good” pseudorandom sequence. We shall, in fact, answer this question for a family of sequences more general than (1). Let $f(x) \in \mathbf{F}_p[x]$. We identify the field $\mathbf{F}_p$ and the set $\{0, \dots, p-1\}$ and for $n \in \mathbf{Z}$ we assume that $f(n) \in \{0, \dots, p-1\}$. (These details are important in order to further reduce modulo 2.) Under these conditions we may generalize construction (1) as follows:

$$(4) \quad e_n = \begin{cases} +1 & \text{if } f(n)^* \equiv f(n) \pmod{2}, \\ -1 & \text{if } f(n)^* \not\equiv f(n) \pmod{2}. \end{cases}$$

Theorem 2. Let $f(x) \in \mathbf{F}_p[x]$ of degree d with $p > d \geq 1$, let s be its number of distinct roots in an algebraic closure $\overline{\mathbf{F}_p}$ of $\mathbf{F}_p$, and let $E_p = (e_1, \dots, e_{p-1})$ defined by (4). Then,

$$(5) \quad W(E_p) \ll (d+s)p^{1/2} \log^3 p,$$

$$(6) \quad C_2(E_p) \ll (d+s)p^{1/2} \log^5 p,$$

where the implied constants are absolute.

2. PROOF OF THEOREM 1

2.1. Exponential sums.

Lemma 3. *Let $A, B, m \in \mathbf{Z}_{\geq 0}$ such that m is odd and $0 \leq A < B \leq (m-1)/2$. Then, for $1 \leq n \leq m-1$, we have*

$$(7) \quad (-1)^n \mathbf{1}_{[2A+1, 2B]}(n) = \frac{1}{m} \sum_{1 \leq |h| \leq (m-1)/2} v_m(h; A, B) \exp(2\pi i h n / m),$$

where

$$\begin{aligned} v_m(h; A, B) &= (1 - \exp(2\pi i h / m)) \sum_{A+1 \leq k \leq B} \exp(-4\pi i h k / m) \\ &= -i \frac{\sin(2\pi h(B-A)/m)}{\cos(\pi h / m)} \exp(2\pi i h(A+B+1/2)/m) \end{aligned}$$

satisfies

$$(8) \quad \sum_{|h| \leq (m-1)/2} |v_m(h; A, B)| \leq \frac{2}{\pi} m \log(5m).$$

Proof. For odd $m \geq 3$ and $1 \leq n \leq (m-1)/2$, we have

$$\begin{aligned} (-1)^n \mathbf{1}_{[2A+1, 2B]}(n) &= \sum_{A+1 \leq k \leq B} (\mathbf{1}_{\{2k\}}(n) - \mathbf{1}_{\{2k-1\}}(n)) \\ &= \sum_{A+1 \leq k \leq B} \frac{1}{m} \sum_{|h| \leq (m-1)/2} \left(\exp\left(\frac{2\pi i h(n-2k)}{m}\right) - \exp\left(\frac{2\pi i h(n-(2k-1))}{m}\right) \right) \\ &= \frac{1}{m} \sum_{|h| \leq (m-1)/2} v_m(h; A, B) \exp(2\pi i h n / m). \end{aligned}$$

Observe that $v_m(0; A, B) = 0$, and this completes the proof of (7). To prove (8), we write

$$\sum_{|h| \leq (m-1)/2} |v_m(h; A, B)| \leq 2 \sum_{1 \leq h \leq (m-1)/2} \frac{1}{\cos(\pi h / m)} := 2S(m).$$

By the convexity of the function $t \mapsto 1/\sin(\pi t/m)$, we get

$$\begin{aligned} S(m) &= \sum_{1 \leq k \leq (m-1)/2} \frac{1}{\sin(\frac{\pi}{m}(k - \frac{1}{2}))} \leq \frac{1}{\sin(\frac{\pi}{2m})} + \int_{3/2}^{m/2} \frac{dt}{\sin(\frac{\pi}{m}(t - \frac{1}{2}))} \\ &\leq \frac{1}{\sin(\frac{\pi}{2m})} + \frac{m}{\pi} \log \cot \frac{\pi}{2m} \leq \frac{m}{\pi} \log(5m), \end{aligned}$$

which completes the proof of (8). $\square$

2.2. Proof of Theorem 1. Let $1 \leq M < N \leq p-1$. Using (7) with $A = M' := [M/2]$ and $B = N' := [N/2]$, for $1 \leq n \leq p-1$ we have

$$(-1)^n \mathbf{1}_{[2M'+1, 2N']}(n) = \frac{1}{p} \sum_{1 \leq |h| \leq (p-1)/2} v_p(h; M', N') \exp(2\pi i h n / p).$$

Similarly, using (7) with $A = 0$, $B = (p-1)/2$, for $1 \leq n \leq p-1$ we have

$$(-1)^{n^*} = \frac{1}{p} \sum_{1 \leq |k| \leq (p-1)/2} v_p(k; 0, (p-1)/2) \exp(2\pi i k n^* / p).$$

Hence, since $2M' \leq M \leq 2M' + 1$ and $2N' \leq N \leq 2N' + 1$,

$$\begin{aligned} \sum_{M \leq n \leq N} e_n &= \sum_{1 \leq n \leq p-1} (-1)^n (-1)^{n^*} \mathbf{1}_{[2M'+1, 2N']} (n) + 2\theta(M, N) \\ &= \frac{1}{p^2} \sum_{1 \leq |h|, |k| \leq (p-1)/2} v_p(h; 0, N') v_p(k; 0, (p-1)/2) S(h, k) + 2\theta(M, N), \end{aligned}$$

where $|\theta(M, N)| \leq 1$ and $S(h, k)$ is a Kloosterman's sum:

$$S(h, k) = \sum_{1 \leq n \leq p-1} \exp(2\pi i(hn + kn^*)/p).$$

By Weil's theorem [10], we have $|S(h, k)| \leq 2\sqrt{p}$ whenever $(h, p) = 1$ or $(k, p) = 1$. This last condition is always satisfied. Therefore, applying (8) twice, we complete the proof of Theorem 1.

Remark 4. Further pseudorandom properties of Kloosterman's sums can be found in É. Fouvry, P. Michel, J. Rivat, A. Sárközy [4].

3. PROOF OF THEOREM 2

3.1. Exponential sums. We will use exponential sums which can be handled by Weil's theorem [10]. In the special case of interest for us, a convenient form was given by C.J. Moreno and O. Moreno [8, Theorem 2]. Using some arguments of J. Eichenauer-Herrmann and H. Niederreiter [3, p. 272], we proved the following.

Lemma 5. *Let p be a prime number and Q/R a rational function over $\mathbf{F}_p$, which is not constant. Let s be the number of distinct roots of the polynomial R in $\overline{\mathbf{F}_p}$. If ψ is a non-trivial additive character of $\mathbf{F}_p$, then*

$$\left| \sum_{\substack{n \in \mathbf{F}_p \\ R(n) \neq 0}} \psi \left(\frac{Q(n)}{R(n)} \right) \right| \leq (\max(\deg(Q), \deg(R)) + s - 1) \sqrt{p}.$$

Proof. This is Lemma 13 of [9]. □

Lemma 6. *Let $A, B, m \in \mathbf{Z}_{\geq 0}$ such that $m \geq 3$ is odd and $0 \leq A < B \leq m-1$. Then, for $n \in \mathbf{Z}$ with $1 \leq n \leq m-1$, we have*

$$(9) \quad \mathbf{1}_{[A+1, B]}(n) = \frac{1}{m} \sum_{|h| \leq (m-1)/2} u_m(h; A, B) \exp(2\pi i h n / m),$$

where $u_m(0, A, B) = B - A$ and $u_m(h; A, B)$ is defined for $0 < |h| \leq (m-1)/2$ by

$$\begin{aligned} u_m(h; A, B) &= \sum_{A+1 \leq k \leq B} \exp\left(\frac{-2\pi i h k}{m}\right) \\ &= \frac{\sin(\pi h(B-A)/m)}{\sin(\pi h/m)} \exp\left(\frac{-\pi i h}{m}(A+B+1)\right), \end{aligned}$$

and satisfies

$$(10) \quad \sum_{|h| \leq (m-1)/2} |u_m(h; A, B)| \leq B - A + \frac{2m}{\pi} \log \frac{4m}{\pi}.$$

Proof. For odd $m \geq 3$ and $1 \leq n \leq m-1$, we have

$$\begin{aligned} \mathbf{1}_{[A+1, B]}(n) &= \sum_{A+1 \leq k \leq B} \mathbf{1}_{\{k\}}(n) = \sum_{A+1 \leq k \leq B} \frac{1}{m} \sum_{|h| \leq (m-1)/2} \exp\left(\frac{2\pi i h(n-k)}{m}\right) \\ &= \frac{1}{m} \sum_{|h| \leq (m-1)/2} u_m(h; A, B) \exp(2\pi i h n/m), \end{aligned}$$

which proves (9). To prove (10), it suffices to observe that

$$|u_m(h; A, B)| \leq 1/\sin(\pi|h|/m)$$

for $0 < |h| \leq (m-1)/2$. Therefore

$$\sum_{|h| \leq (m-1)/2} |u_m(h; A, B)| \leq B - A + 2 \sum_{1 \leq h \leq (m-1)/2} \frac{1}{\sin(\pi h/m)}$$

and by the convexity of $t \mapsto 1/\sin(\pi t/m)$, we have

$$\sum_{1 \leq h \leq (m-1)/2} \frac{1}{\sin(\pi h/m)} \leq \int_{1/2}^{m/2} \frac{dt}{\sin(\pi t/m)} = \frac{m}{\pi} \log \cot \frac{\pi}{4m} \leq \frac{m}{\pi} \log \frac{4m}{\pi},$$

which completes the proof of (10). $\square$

3.2. Proof of Theorem 2. Let $1 \leq t < p$ and $g(j) := f(a + jb)$. Using (7) with $A = 0$ and $B = (p-1)/2$, and writing $v_p(l) = v_p(l; 0, (p-1)/2)$, for j such that $g(j) \neq 0$ we have

$$\begin{aligned} (-1)^{g(j)} &= \frac{1}{p} \sum_{1 \leq |h| \leq (p-1)/2} v_p(h) \exp(2\pi i h g(j)/p), \\ (-1)^{g(j)^*} &= \frac{1}{p} \sum_{1 \leq |k| \leq (p-1)/2} v_p(k) \exp(2\pi i k g(j)^*/p). \end{aligned}$$

Furthermore, using (9) with $A = 0$ and $B = [(t-1)/2]$, and writing $u_p(l) = u_p(l; 0, [(t-1)/2])$, there exist $\theta_t \in \mathbf{Z}$ with $|\theta_t| \leq 1$ such that

$$\mathbf{1}_{[1, t-1]}(j) = \frac{1}{p} \sum_{|l| \leq (p-1)/2} u_p(l) \exp(2\pi i l j/p) + \theta_t.$$

Hence, we get

$$\sum_{\substack{0 \leq j \leq t-1 \\ g(j) \neq 0}} e_{a+jb} = \frac{1}{p^3} \sum_{|l| \leq (p-1)/2} \sum_{1 \leq |h|, |k| \leq (p-1)/2} u_p(l) v_p(h) v_p(k) S(h, k, l) + \theta'_t,$$

where $|\theta'_t| \leq 1$ and $S(h, k, l)$ is a sort of generalized Kloosterman's sum:

$$S(h, k, l) = \sum_{\substack{0 \leq j \leq p-1 \\ g(j) \neq 0}} \exp(2\pi i (lj + hg(j) + kg(j)^*)/p).$$

Since $0 \leq b < p$, the polynomial $g(x)$ is of degree $d = \deg f$, and has s distinct roots in $\overline{\mathbf{F}_p}$. Therefore, by Lemma 5, we have for $1 \leq |k| \leq (p-1)/2$,

$$|S(h, k)| \ll (d+s)\sqrt{p},$$

and by (10) and (8), we get

$$\left| \sum_{\substack{0 \leq j \leq t-1 \\ g(j) \neq 0}} e_{a+jb} \right| \ll (d+s)p^{1/2} \log^3 p.$$

Furthermore, there are at most s values of j such that $g(j) = f(a+jb) = 0$, and their contribution is negligible since $s \leq (d+s)p^{1/2} \log^3 p$. This completes the proof of (5).

The proof of (6) is similar. By the same arguments with $H = (p-1)/2$, we can write

$$\begin{aligned} & \sum_{\substack{1 \leq n \leq M \\ f(n+d_1)f(n+d_2) \neq 0}} e_{n+d_1} e_{n+d_2} \\ &= \frac{1}{p^5} \sum_{|l| \leq H} \sum_{\substack{1 \leq |h_1|, |h_2|, |k_1|, |k_2| \leq H}} u_p(l) v_p(h_1) v_p(h_2) v_p(k_1) v_p(k_2) T(h_1, h_2, k_1, k_2, l) \\ & \quad + O(1), \end{aligned}$$

where

$$T(h_1, h_2, k_1, k_2, l) = \sum_{\substack{0 \leq n \leq p-1 \\ f(n+d_1)f(n+d_2) \neq 0}} \exp(2\pi i P(n)/p) = \sum_{\substack{0 \leq n \leq p-1 \\ R(n) \neq 0}} \exp\left(2\pi i \frac{Q(n)}{R(n)}/p\right),$$

with

$$\begin{aligned} P(n) &= ln + h_1 f(n+d_1) + h_2 f(n+d_2) + k_1 f(n+d_1)^* + k_2 f(n+d_2)^*, \\ Q(n) &= (ln + h_1 f(n+d_1) + h_2 f(n+d_2)) f(n+d_1) f(n+d_2) \\ & \quad + k_1 f(n+d_2) + k_2 f(n+d_1), \\ R(n) &= f(n+d_1) f(n+d_2). \end{aligned}$$

In order to complete the proof of (6), using (10) and (8), it remains to prove, for all h_1, h_2, k_1, k_2, l with $1 \leq |h_1|, |h_2|, |k_1|, |k_2| \leq H$ and $|l| \leq H$, the estimate:

$$(11) \quad |T(h_1, h_2, k_1, k_2, l)| \ll (d+s)\sqrt{p}.$$

We may assume that $p \geq 11$ and $d = \deg f \leq \sqrt{p}$, for otherwise (6) is trivial. This ensures that $\max(\deg Q, \deg R) < p$. Then we distinguish two cases:

- If $n \mapsto ln + h_1 f(n+d_1) + h_2 f(n+d_2)$ is not constant, then $0 < \deg R < \deg Q < p$ and the rational function Q/R over $\mathbf{F}_p$ is not constant, so that by Lemma 5 we get (11).
- If $n \mapsto ln + h_1 f(n+d_1) + h_2 f(n+d_2)$ is constant, then we must estimate

$$\sum_{R(n) \neq 0} \exp\left(2\pi i \left(\frac{k_1 f(n+d_2) + k_2 f(n+d_1)}{f(n+d_1) f(n+d_2)}\right)/p\right).$$

If the numerator is a polynomial $\neq 0$, then the denominator is of higher degree than the numerator, and again by Lemma 5 we get (11). But if the numerator $= 0$, then considering the coefficients of higher degree we must have $k_2 = -k_1$, and then $d_1 \equiv d_2 \pmod{p}$, which is impossible since $0 \leq d_1 < d_2 < p$.

We remark that the correlation of order $k > 2$ can be expressed by a similar formula. Again, the estimate of the innermost sum leads to the estimate of an

exponential sum involving a rational function, i.e., of the type occurring in Lemma 5. However, in order to be able to use Lemma 5, we need the fact that this rational function is not constant. If $k > 2$, then to ensure this one needs further assumptions as in [7] on $f(n)$ and a lengthy discussion; here we do not go into the details of this.

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