BOUNDARY SPECTRA OF UNIFORM FROSTMAN BLASCHKE PRODUCTS

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ABSTRACT. A closed set F in the unit circle $\mathbb{T}$ is the boundary spectrum of a uniform Frostman Blaschke product if and only if F is nowhere dense in $\mathbb{T}$.

1. INTRODUCTION

A Blaschke product $B(z) = z^m \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n}z}$ is said to satisfy the Frostman condition at the point $\zeta \in \mathbb{T}$ if the quantity

$$\sigma_B(\zeta) = \sum_{n=1}^{\infty} \frac{1 - |a_n|}{|\zeta - a_n|}$$

is finite. Frostman [2, 3] showed that this condition is necessary and sufficient for the Blaschke product B and all of its subproducts to have nontangential limits of modulus one at ζ . The Blaschke product B is said to be uniform Frostman if the quantity

$$\sigma_B = \sup_{\zeta \in \mathbb{T}} \sigma_B(\zeta)$$

is finite. The function $\sigma_B(\zeta)$ is the *Frostman indicator* of B and the number σ_B is the *Frostman constant* of B . Evidently $\sigma_B > 1$ for any Blaschke product B with more than one zero. Moreover, any sequence (a_n) satisfying

$$\sup_{\zeta \in \mathbb{T}} \sum_{n=1}^{\infty} \frac{1 - |a_n|}{|\zeta - a_n|} < \infty$$

is a Blaschke sequence. It was shown by Hruščev and Vinogradov [5] that an inner function is a multiplier of the space of Cauchy transforms if and only if it is a uniform Frostman Blaschke product. An exposition of this result can also be found in [1].

For any inner function u , the boundary spectrum of u is the set

$$\text{spec}(u) = \left\{ \zeta \in \mathbb{T} \mid \liminf_{z \rightarrow \zeta} |u(z)| = 0 \right\}.$$

For a Blaschke product B , $\text{spec}(B)$ is just the accumulation set of the zeros of B . In general u admits an analytic continuation across $\mathbb{T} \setminus \text{spec}(u)$. Joseph A. Cima asked whether the boundary spectrum of an inner multiplier of the space of Cauchy

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transforms could be the whole unit circle. The theorem below provides a negative answer to this question.

Theorem 1. *If B is a uniform Frostman Blaschke product, then the set $\text{spec}(B)$ is nowhere dense in $\mathbb{T}$.*

The next theorem provides a converse to the above theorem.

Theorem 2. *If F is a closed nowhere dense subset of $\mathbb{T}$, then there is a uniform Frostman Blaschke product B with $\text{spec}(B) = F$. Moreover, for any $\epsilon > 0$, B can be chosen so that $\sigma_B < 1 + \epsilon$.*

Some definitions and known results will be presented in the next section. Theorem 1 will be proved in Section 3, except for a technical lemma, which will be proved in Section 4. Theorem 2 will be proved by an explicit construction in Section 5.

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2. DEFINITIONS AND KNOWN RESULTS

To each sequence $a = (a_n)$ in the unit disk there is associated a measure μ_a which puts the mass $1 - |a_n|^2$ at the point a_n for each n . The sequence a is a Blaschke sequence if μ_a is a finite measure. The (Blaschke) sequence a is a Carleson sequence if μ_a is a Carleson measure, that is, if there is a constant C such that

$$\mu_a(Q_I) \leq C|I|$$

for every arc I on $\mathbb{T}$, where $|I|$ denotes the normalized length of I , and Q_I is the Carleson square

$$Q_I = \{z \in \mathbb{D} \mid 1 - |z| < |I| \text{ and } z/|z| \in I\}.$$

The sequence a is separated if there is a positive number δ such that $\rho(a_m, a_n) > \delta$ whenever $m \neq n$. Here $\rho(a, b) = \left| \frac{a - b}{1 - \bar{a}b} \right|$ is the pseudohyperbolic distance between a and b . The sequence a is uniformly separated if there is a positive number δ such that

$$\inf_m \prod_{n \neq m} \left| \frac{a_n - a_m}{1 - \bar{a}_n a_m} \right| > \delta.$$

By Carleson's interpolation theorem, the uniformly separated sequences are precisely the interpolating sequences for H^∞ . Thus they are often called interpolating sequences, and that practice will be followed here. Clearly, interpolating sequences are separated. It is a result of McKenna [7] that every Carleson sequence can be partitioned into a finite number of interpolating sequences. Conversely, every interpolating sequence is Carleson. All of this and more can be found in Garnett's book [4].

For an angle α , $0 < \alpha < \pi/2$, and a point $\zeta \in \mathbb{T}$, the Stolz region $S_\alpha(\zeta)$ is the convex hull of the closed disk with center 0 and radius $\sin \alpha$ and the point ζ , with the point ζ removed. Near ζ , $S_\alpha(\zeta)$ is bounded by two straight lines each making angle α with the radius from 0 to ζ .

The next two lemmas are due to Hruščev and Vinogradov [5]. In each case the simple proof is given. If a is a Blaschke sequence, then $\sigma(a)$ is equal to σ_B , where B is the Blaschke product with zero sequence a .

Lemma 1. *Let a be a Frostman sequence. Then the number of elements of a in any Stolz region $S_\alpha(\zeta)$ does not exceed $2\sigma(a)/(1 - \sin \alpha)$.*

Proof. If z is any point in $\mathbb{D}$, the law of cosines gives

$$|z|^2 = 1 + |\zeta - z|^2 - 2|\zeta - z| \cos \theta,$$

where θ is the angle that the segment from z to ζ makes with the radius from 0 to ζ . Rearranging this yields

$$\frac{1 - |z|^2}{|\zeta - z|} = 2 \cos \theta - |\zeta - z|.$$

The chord of the unit circle containing z and ζ has length $2 \cos \theta$, and so the length of the segment of this chord on the opposite side of z from ζ has length $2 \cos \theta - |\zeta - z|$. In particular, if $z \in S_\alpha(\zeta)$, this length exceeds $1 - \sin \alpha$, the length of the segment of the diameter from $-\zeta$ to ζ lying outside $S_\alpha(\zeta)$. If N is the number of elements of the sequence a lying in $S_\alpha(\zeta)$, then

$$N \leq \frac{1}{1 - \sin \alpha} \sum_{a_n \in S_\alpha(\zeta)} \frac{1 - |a_n|^2}{|\zeta - a_n|} \leq \frac{2\sigma(a)}{1 - \sin \alpha}.$$

□

Lemma 2. *Let a be a Frostman sequence. Then μ_a is a Carleson measure.*

Proof. Let Q be a Carleson square. Then

$$\begin{aligned} \mu_a(Q) &= \sum_{a_n \in Q} 1 - |a_n|^2 \\ &\leq c|I| \sum_{a_n \in Q} \frac{1 - |a_n|^2}{|\zeta - a_n|} \\ &\leq 2c|I|\sigma(a), \end{aligned}$$

where c is an absolute constant, $|I|$ is the length of the arc I upon which Q rests, and ζ is the midpoint of I . □

Lemma 3. *Let B be a Blaschke product admitting a factorization $B = B_1 B_2 \dots B_n$. Then $\text{spec}(B) = \bigcup_{k=1}^n \text{spec}(B_k)$.*

Proof. This is obvious. □

3. PROOF OF THEOREM 1

The proof of Theorem 1 depends on the following lemma. This lemma will be proved in Section 4.

Lemma 4. *Let F be a countable subset of $\mathbb{D}$ such that for some fixed α , $0 < \alpha < \frac{\pi}{2}$, there is an integer N such that no Stolz region $S_\alpha(\zeta)$ contains more than N points from F . Then F admits a partition $F = \bigcup_{k=0}^{2N-1} F_k$, where $F_0 = F \cap \overline{D(0, \sin \alpha)}$ and each F_k , $k = 1, \dots, 2N - 1$, has at most one point in each Stolz region $S_\alpha(\zeta)$.*

If $F \subset \mathbb{T}$ is a finite union $F = \bigcup_{k=1}^n F_k$ of closed sets F_k nowhere dense in $\mathbb{T}$, then it is clear that F is also a closed set nowhere dense in $\mathbb{T}$. If B is a uniform Frostman Blaschke product which can be written as a finite product $B = \prod_{k=1}^n B_k$ of Blaschke products, then each factor B_k is uniform Frostman, and, by Lemma 3,

it will be enough to show that each factor B_k satisfies the conclusion of Theorem 1. Because the zero sequence of B is Carleson, B can be factored as a finite product so that each factor B_k is interpolating, and, in particular, has simple zeros. Then an application of Lemma 4 shows that B admits a factorization $B = \prod_{k=1}^n B_k$, where each B_k has simple zeros and has the property that no Stolz region $S_\alpha(\zeta)$ contains more than one zero of B_k .

Now suppose that B is a Blaschke product with the property that every Stolz region $S_\alpha(\zeta)$ contains at most one zero of B . Let $F = \text{spec}(B)$, and let ζ be an interior point of F . There is a positive number ϵ such that F contains the arc $I_\zeta = \mathbb{T} \cap D(\zeta, \epsilon)$. For each $z \in \mathbb{D}$, let

$$J_z = \{ \xi \in \mathbb{T} : z \in S_\alpha(\xi) \}.$$

It is easy to see that there is a positive number $\eta < \epsilon$ which depends only on ϵ and α such that $J_z \subset I_\zeta$ for every $z \in \mathbb{D} \cap D(\zeta, \eta)$. Since $\zeta \in \text{spec}(B)$, B has a zero z which lies in the neighborhood $\mathbb{D} \cap D(\zeta, \eta)$ of ζ . Because each Stolz region contains at most one zero of B , the interior of the convex hull of z and J_z can contain no zeros of B , and so the interior of J_z is disjoint from $\text{spec}(B)$. This is a contradiction.

4. PROOF OF LEMMA 4

Suppose that F is a set satisfying the hypotheses of Lemma 4, that is, there is an α , $0 < \alpha < \frac{\pi}{2}$, and a positive integer N such that each Stolz region $S_\alpha(\zeta)$ contains at most N points from F . Let $F_0 = F \cap \overline{D}(0, \sin \alpha)$, and let $F' = F \setminus F_0$. Since F' is countable, its elements can be listed in a sequence (a_n) , arranged so that the moduli of the elements form an increasing sequence. For each positive integer n , let $I_n = J_{a_n}$ be the arc of $\mathbb{T}$ such that $a_n \in S_\alpha(\zeta)$ for each $\zeta \in I_n$. Clearly, $|I_1| \geq |I_2| \geq \dots$. The sequence of intervals (I_n) has the property that no point of the unit circle is contained in more than N of the I_n . This follows immediately from the observation that $\zeta \in I_n$ if and only if $a_n \in S_\alpha(\zeta)$.

Lemma 5. *The sequence $\mathbb{N}$ of natural numbers can be partitioned into at most $2N - 1$ subsequences M_k such that for each k , the sequence of intervals $\{I_n : n \in M_k\}$ is pairwise disjoint.*

Proof. The proof mimics the proof of Besicovitch's covering theorem [6, pp. 30–34]. In order to construct M_1 , let $n_1 = 1$, and let n_2 be the smallest positive integer such that I_{n_1} and I_{n_2} are disjoint. If $n_1 < n_2 < \dots < n_l$ have been selected so that the intervals I_{n_i} , $1 \leq i \leq l$, are pairwise disjoint, let n_{l+1} be the smallest integer greater than n_l such that $I_{n_{l+1}}$ is disjoint from the intervals I_{n_i} , $1 \leq i \leq l$. Continuing this process as long as possible creates M_1 .

Next, M_2 is constructed from $\mathbb{N} \setminus M_1$ in the same way. First, let n_1 be the smallest positive integer in $\mathbb{N} \setminus M_1$. Next let n_2 be the smallest positive integer in $\mathbb{N} \setminus M_1$ such that I_{n_2} is disjoint from I_{n_1} . The process continues as above.

As long as $\bigcup_{k=1}^l M_i \neq \mathbb{N}$ this process can be continued to create M_{l+1} . If $\bigcup_{k=1}^K M_k = \mathbb{N}$ for some $K \leq 2N - 1$, then the lemma is proved. Otherwise there is a smallest positive integer n not belonging to $\bigcup_{k=1}^{2N-1} M_k$. Then by construction, there is, for each k , $1 \leq k \leq 2N - 1$, an integer $n_i \in M_k$ such that $n_i < n$ and I_{n_i} intersects I_n , since otherwise, n would have been selected as a member of M_k . Since I_n is no longer than I_{n_k} for each k , I_{n_k} must contain at least one of the

endpoints of I_n . Consequently, at least one of the endpoints of I_n belongs to at least N intervals, contradicting the hypotheses. $\square$

Applying the lemma, there is a partition $\mathbb{N} = \bigcup_{k=1}^K M_k$ of $\mathbb{N}$ with $K \leq 2N - 1$, such that for each k , $1 \leq k \leq K$, the sequence $(I_n)_{n \in M_k}$ is pairwise disjoint. For each k , let $F_k = \{a_n : n \in M_k\}$. Evidently F_k has the property that it intersects any Stolz region in at most one point. This completes the proof of Lemma 4.

5. A CONSTRUCTION

Suppose that F is a closed subset of $\mathbb{T}$ with no interior, and let (I_n) be the sequence of arcs contiguous to F , that is,

$$\mathbb{T} \setminus F = \bigcup_n I_n,$$

and the I_n are pairwise disjoint open arcs in $\mathbb{T}$. Let a_n, b_n denote the endpoints of I_n , with a_n coming before b_n as the arc I_n is traversed in the counterclockwise direction. Because F is nowhere dense in $\mathbb{T}$, it is the closure of the set of endpoints of the I_n .

The proof of Theorem 2 proceeds by constructing, for each I_n , a Blaschke product B_n whose zeros lie over I_n and accumulate at the two endpoints a_n and b_n , and so that the Frostman indicator $\sigma_{B_n}(\zeta)$ is small for $\zeta \notin I_n$, and not too big for $\zeta \in I_n$. Specifically, let $\epsilon > 0$, and let ϵ_n be such that $\sum_n \epsilon_n < \epsilon/2$. Suppose that for each n a Blaschke product B_n has been constructed such that the zeros of B_n lie over I_n and accumulate at both endpoints of I_n and such that

$$\sigma_{B_n}(\zeta) < \begin{cases} 1 + \epsilon/2 & \text{if } \zeta \in I_n, \\ \epsilon_n & \text{if } \zeta \notin I_n. \end{cases}$$

Now let $B = \prod_n B_n$. Since the Frostman indicator $\sigma_B(\zeta)$ is clearly additive over products, it follows that this product converges uniformly on compact sets to a Blaschke product B , and that $\sigma_B < 1 + \epsilon$. Obviously, $\text{spec}(B) = F$. Up to the construction of B_n , that completes the proof of Theorem 2.

In order to construct the B_n , it will be enough to consider a single arc I , and construct, for a given constant η , $0 < \eta < 1$, a Blaschke product B whose zeros lie over I and accumulate at one of the endpoints of I and such that

$$\sigma_B(\zeta) < \begin{cases} 1 + \eta & \text{if } \zeta \in I', \\ \eta & \text{otherwise,} \end{cases}$$

where I' is a subarc of I sharing the endpoint in question and having length less than or equal to one half of the length of I . If B_1 and B_2 are the Blaschke products constructed for each of the endpoints of I , then clearly the product $B = B_1 B_2$ satisfies

$$\sigma_B(\zeta) < \begin{cases} 1 + 2\eta & \text{if } \zeta \in I, \\ 2\eta & \text{otherwise.} \end{cases}$$

Now suppose that 1 is one of the endpoints of I and α , $0 < \alpha < \pi/4$, is such that I contains the shorter interval I'' with endpoints 1 and $e^{i\alpha}$, and finally that I'' has length less than or equal to one fourth of the length of I . Let I' be the arc containing I'' with endpoints 1 and $e^{2i\alpha}$. Now, for $n = 0, 1, \dots$, let $a_n = r_n e^{i\theta_n}$,

where $1 - r_n = \frac{\eta}{4} \frac{\alpha}{4^n}$ and $\theta_n = \frac{\alpha}{2^n}$. Because of the choice of α , for each n and any t ,

$$|e^{it} - r_n|^2 > (1 - r_n)^2 + t^2,$$

so that in any expression the denominator $|e^{i\theta} - a_n|$ can be replaced by $1 - r_n$ or by $|\theta - \theta_n|$ as appropriate. Let

$$\sigma(\theta) = \sum_{n=0}^{\infty} \frac{1 - r_n}{|e^{i\theta} - r_n e^{i\theta_n}|}.$$

First suppose that $2\theta_0 \leq \theta \leq \pi$. Then for each n , $|\theta - \theta_n| \geq |\theta_0| = \alpha$, so that

$$\sigma(\theta) \leq \frac{1}{\alpha} \sum_{n=0}^{\infty} (1 - r_n) \leq \frac{\eta}{4} \sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{4}{3} \frac{\eta}{4} = \frac{\eta}{3} < \eta.$$

Next, if $\theta_0 \leq \theta \leq 2\theta_0$, then for $n \geq 1$, $|\theta - \theta_n| \geq |\theta_0 - \theta_1| = \frac{\alpha}{2}$, so that

$$\begin{aligned} \sigma(\theta) &\leq \frac{1 - r_0}{|e^{i\theta} - a_n|} + \sum_{n=1}^{\infty} \frac{1 - r_n}{|e^{i\theta} - a_n|} \\ &\leq 1 + \frac{2}{\alpha} \sum_{n=1}^{\infty} \frac{\eta}{4} \frac{\alpha}{4^n} \\ &= 1 + \frac{2}{3} \frac{\eta}{4} = 1 + \frac{\eta}{6} < 1 + \eta. \end{aligned}$$

Now suppose that $\theta_n \leq \theta \leq \theta_{n-1}$. Suppose that θ is closer to θ_n than to θ_{n-1} , i.e., $\theta_n \leq \theta \leq \frac{\theta_n + \theta_{n-1}}{2}$. If $k < n - 1$, then $\theta_k - \theta \geq \theta_k - \theta_{k+1} = \frac{\alpha}{2^k} - \frac{\alpha}{2^{k+1}} = \frac{\alpha}{2^{k+1}}$. Similarly, if $k > n$, $\theta - \theta_k \geq \theta_{k-1} - \theta_k = \frac{\alpha}{2^k} \geq \frac{\alpha}{2^{k+1}}$. If $k = n - 1$, then $\theta_{n-1} - \theta \geq \theta_{n-1} - \frac{\theta_n + \theta_{n-1}}{2} = \frac{\theta_{n-1} - \theta_n}{2} = \frac{\alpha}{2^{n+1}}$. Thus

$$\begin{aligned} \sigma(\theta) &= \sum_{k=0}^{n-1} \frac{1 - r_k}{|e^{i\theta} - a_k|} + \frac{1 - r_n}{|e^{i\theta} - a_n|} + \sum_{k=n+1}^{\infty} \frac{1 - r_k}{|e^{i\theta} - a_k|} \\ &\leq \frac{\eta}{4} \sum_{k=0}^{n-1} \frac{2^{n+1}}{4^n} + 1 + \frac{\eta}{4} \sum_{k=n+1}^{\infty} \frac{2^{n+1}}{4^n} \\ &< 1 + \frac{\eta}{2} \sum_{n=0}^{\infty} \frac{1}{2^n} \\ &= 1 + \eta. \end{aligned}$$

A similar argument works if θ is closer to θ_{n-1} than it is to θ_n . Finally, if $-\pi \leq \theta \leq 0$, then $\theta_n - \theta \geq \theta_n$, and so

$$\begin{aligned} \sigma(\theta) &\leq \sum_{n=0}^{\infty} \frac{1 - r_n}{\theta_n} \\ &= \frac{\eta}{4} \sum_{n=0}^{\infty} \frac{2^n}{4^n} \\ &= \frac{\eta}{2} < \eta. \end{aligned}$$

This completes the construction of B and the proof of Theorem 2.

6. CONCLUSION

Finally, the referee has suggested an alternate, less quantitative, proof of Theorem 1 which uses the Baire category theorem. This proof proceeds as follows. Let $S_\alpha^\circ(\zeta)$ be the interior of $S_\alpha(\zeta)$. As observed by Frostman, each $S_\alpha^\circ(\zeta)$ contains only finitely many zeros of B , else B would not have a radial limit of modulus one at ζ . Now suppose I is a closed arc in the boundary spectrum F of B . For each positive integer n , let

$$E_n = \left\{ \zeta \in I : S_\alpha^\circ(\zeta) \text{ contains no zero } z \text{ of } B \text{ with } |z| \geq 1 - \frac{1}{n} \right\}.$$

Clearly $\bigcup_{n=1}^\infty E_n = I$. Since each $S_\alpha^\circ(\zeta)$ is open, each E_n is closed. By Baire's theorem some E_n contains an open arc J . Evidently J is disjoint from F , and that is a contradiction.

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