TRACE CLASS CRITERIA FOR BILINEAR HANKEL FORMS OF HIGHER WEIGHTS

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ABSTRACT. In this paper we give a complete characterization of higher weight Hankel forms, on the unit ball of $\mathbb{C}^d$, of Schatten-von Neumann class $\mathcal{S}_p$, $1 \leq p \leq \infty$. For this purpose we give an atomic decomposition for certain Besov-type spaces. The main result is then obtained by combining the decomposition and our earlier results.

1. INTRODUCTION

Hankel operators on the unit disc have been studied extensively; see [Pe1] for a systematic treatment. One of the main topics is to study Schatten-von Neumann properties of Hankel operators; see [Pe1] and [Pe2]. In [JP] Janson and Peetre introduced Hankel forms of higher weights on the unit disc. Their Schatten-von Neumann properties were studied in [Ro] and [Z].

In [P1] Peetre introduced Hankel forms of higher weights on the unit ball in $\mathbb{C}^d$. Their Schatten-von Neumann, $\mathcal{S}_p$, properties were studied in [Su] for $2 \leq p \leq \infty$. See also [FR] for a different approach.

The results for $2 \leq p \leq \infty$ in [Su] were proved by using interpolation between $\mathcal{S}_2$ and $\mathcal{S}_\infty$ (bounded operators) and boundedness of certain matrix-valued Bergman projections, but the case of $1 \leq p < 2$ was left open there.

In this paper we extend the results in [Su] to $1 \leq p \leq \infty$. For this purpose we study the atomic decomposition for some Besov spaces of vector-valued holomorphic functions, see Section 4, which then gives $\mathcal{S}_1$ properties. Our results follow by interpolation, and we get a full characterization for $1 \leq p \leq \infty$. Some of the proofs in this paper are based on techniques used in [Su] and will therefore be given briefly. The reader is referred to that article for more details.

The paper is organized as follows. In section 2 we recall briefly some notation and we prove Theorem 2.1, generalizing the result for $p = 2$ in [Su]. Section 3 is devoted to duality relations for the spaces of symbols. In Section 4 we give an atomic decomposition for a certain space of symbols, which will be used in Section 5 to prove the $\mathcal{S}_1$ criterion.

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2. PRELIMINARIES

2.1. The Banach space $\mathcal{H}_{\nu,s}^p$ for $1 \leq p \leq \infty$. Let dm denote the Lebesgue measure on the unit ball $\mathbb{B} \subset \mathbb{C}^d$ and let $dl(z)$ be the measure $(1 - |z|^2)^{-d-1} dm(z)$. For $d < \nu < \infty$ let $dl_\nu(z)$ be the measure $c_\nu(1 - |z|^2)^\nu dl(z)$, where c_ν is chosen such that

$$\int_{\mathbb{B}} dl_\nu(z) = 1.$$

The closed subspace of all holomorphic functions in $L^2(dl_\nu)$ is denoted by $L_a^2(dl_\nu)$ and is called a weighted Bergman space. Note that the space $L_a^2(dl_\nu)$ has a reproducing kernel $K_z(w) = (1 - \langle w, z \rangle)^{-\nu}$, that is,

$$(2.1) \quad f(z) = \langle f, K_z \rangle_\nu = \int_{\mathbb{B}} f(w) \overline{K_z(w)} dl_\nu(w), \quad f \in L_a^2(dl_\nu), \quad z \in \mathbb{B}.$$

Denote by $B(z, w)$ the Bergman operator on $V = \mathbb{C}^d$ as in [L], namely

$$(2.2) \quad B(z, w) = (1 - \langle z, w \rangle)(I - z \otimes w^*),$$

where $z \otimes w^*$ stands for the rank one operator given by $(z \otimes w^*)(v) = \langle v, w \rangle z$.

The Bergman metric at $z \in \mathbb{B}$, when we identify the tangent space with V , is $\langle B(z, z)^{-1}u, v \rangle$ for $u, v \in V$. We note that

$$(2.3) \quad B(z, w)^{-1} = (1 - \langle z, w \rangle)^{-2}((1 - \langle z, w \rangle)I + z \otimes w^*).$$

Let $B^t(z, w)$ denote the dual of $B(z, w)$ acting on the dual space V' of V . When acting on a vector $v' \in V'$ it is

$$(2.4) \quad B^t(z, w)v' = (1 - \langle z, w \rangle)v'(I - z\bar{w}^t).$$

For a nonnegative integer s , let $\otimes^s V'$ be the tensor product of s copies of V' and let $\otimes^0 V' = \mathbb{C}$. The space $\otimes^s V'$ is equipped with a natural Hermitian inner product induced by that of V' . Denote by $\odot^s V'$ the subspace of symmetric tensors of length s and denote by $\otimes^s B^t(z, z)$ the operator on $\otimes^s V'$ induced by the action of $B^t(z, z)$ on V' , where $\otimes^0 B^t(z, z) = I$. Recall, generally, that if A acts on V' , $\otimes^s A$ acts on $\otimes^s V'$ by

$$(\otimes^s A)(u_1 \otimes u_2 \otimes \cdots \otimes u_s) = (Au_1) \otimes (Au_2) \otimes \cdots \otimes (Au_s).$$

For example, in the case $s = 2$ the operator $\otimes^2 B^t(z, z)$ becomes

$$(1 - |z|^2)^2 (I \otimes I - I \otimes A_z - A_z \otimes I + A_z \otimes A_z),$$

where $A_z = \bar{z} \otimes z^*$. Let $L_{\nu,s}^p = L_\nu^p(\mathbb{B}, \odot^s V')$ be the space of functions $G : \mathbb{B} \rightarrow \odot^s V'$ such that

$$\|G\|_{\nu,s,p} = \left(\int_{\mathbb{B}} \langle (1 - |z|^2)^{2\nu} \otimes^s B^t(z, z)G(z), G(z) \rangle^{p/2} dl(z) \right)^{1/p} < \infty,$$

where $1 \leq p < \infty$, and let $L_{\nu,s}^\infty$ be the space of functions $G : \mathbb{B} \rightarrow \odot^s V'$ such that

$$\|G\|_{\nu,s,\infty} = \sup_{z \in \mathbb{B}} \langle (1 - |z|^2)^{2\nu} \otimes^s B^t(z, z)G(z), G(z) \rangle^{1/2} < \infty.$$

Let $\mathcal{H}_{\nu,s}^p$ be the closed subspace of all holomorphic functions in $L_{\nu,s}^p$, $1 \leq p \leq \infty$.

Also, we need the group G of biholomorphic mappings of $\mathbb{B}$. Let P_z be the orthogonal projection of $\mathbb{C}^d$ onto $\mathbb{C}z$ and let $Q_z = I - P_z$. Put $s_z = (1 - |z|^2)^{1/2}$ and define a linear fractional mapping φ_z on $\mathbb{B}$ by (see [Ru])

$$(2.5) \quad \varphi_z(w) = \frac{z - P_z w - s_z Q_z w}{1 - \langle w, z \rangle}.$$

If $g \in G$ and $g(z) = 0$, then there is a unique unitary operator $U : \mathbb{C}^d \rightarrow \mathbb{C}^d$ such that

$$g = U\varphi_z.$$

Define the complex Jacobian J_g by $J_g(w) = \det(g'(w))$. Now, let $z_0 \in \mathbb{B}$. Then by arguments in Remark 3.1 in [Su] it follows that there is a constant c with $|c| = 1$ such that

$$(2.6) \quad J_{\varphi_{z_0}}(w)^{2\nu/(d+1)} = c \cdot \frac{(1 - |z_0|^2)^\nu}{(1 - \langle w, z_0 \rangle)^{2\nu}}.$$

The next theorem gives the reproducing properties for $\mathcal{H}_{\nu,s}^p$.

Theorem 2.1. *Let $1 \leq p \leq \infty$. There is a nonzero constant c such that, for any $G \in \mathcal{H}_{\nu,s}^p$ and any $v \in \odot^s V'$,*

$$\langle G(z), v \rangle = c \int_{\mathbb{B}} \langle \otimes^s B^t(w, w) G(w), K_{\nu,s}(w, z) v \rangle (1 - |w|^2)^{2\nu} d\mu(w),$$

where

$$K_{\nu,s}(w, z) = (1 - \langle w, z \rangle)^{-2\nu} \otimes^s B^t(w, z)^{-1}.$$

The proof of this theorem is given at the end of this subsection.

Remark 2.2. Consider $\mathcal{H}_{\nu,s}^2 \subset L_{\nu,s}^2$. According to Lemma 3.5 in [Su] the orthogonal projection operator $P_{\nu,s}$ of $L_{\nu,s}^2$ onto $\mathcal{H}_{\nu,s}^2$, is given by

$$(2.7) \quad P_{\nu,s} G(z) = c \int_{\mathbb{B}} (1 - |w|^2)^{2\nu} K_{\nu,s}(z, w) \otimes^s B^t(w, w) G(w) d\mu(w).$$

Namely, for any $G \in L_{\nu,s}^2$ and any $v \in \odot^s V'$ it follows that

$$\langle P_{\nu,s} G(z), v \rangle = c \int_{\mathbb{B}} \langle \otimes^s B^t(w, w) G(w), K_{\nu,s}(w, z) v \rangle (1 - |w|^2)^{2\nu} d\mu(w).$$

The orthogonal projection operator has the following boundedness property.

Proposition 2.3. *If $1 \leq p < \infty$, then $P_{\nu,s} : L_{\nu,s}^p \rightarrow \mathcal{H}_{\nu,s}^p$ is bounded.*

Proof. The case $1 < p < \infty$ is just Corollary 7.4 in [Su]. Now, consider the case $p = 1$. Let $F \in L_{\nu,s}^1$. Then it follows from Theorem 2.1 above and Lemma 7.1 in [Su] that

$$\left\| \otimes^s B^t(z, z)^{1/2} P_{\nu,s} F(z) \right\| \leq C_s \int_{\mathbb{B}} T(z, w) \left\| \otimes^s B^t(w, w)^{1/2} F(w) \right\| (1 - |w|^2)^{2\nu} d\mu(w),$$

where

$$T(z, w) = \frac{(1 - |z|^2)^{s/2} (1 - |w|^2)^{s/2}}{|1 - \langle z, w \rangle|^{2\nu+s}}.$$

Thus, by Fubini-Tonelli's theorem and Proposition 1.4.10 in [Ru] it follows that

$$\begin{aligned} \|P_{\nu,s}F\|_{\nu,s,1} &\leq C_s \int_{\mathbb{B}} \left\| \otimes^s B^t(w, w)^{1/2} F(w) \right\| (1 - |w|^2)^{2\nu} \\ &\quad \cdot \left(\int_{\mathbb{B}} T(z, w) (1 - |z|^2)^\nu d\mu(z) \right) d\mu(w) \\ &\leq C'_s \int_{\mathbb{B}} \left\| \otimes^s B^t(w, w)^{1/2} F(w) \right\| (1 - |w|^2)^\nu d\mu(w) = C'_s \|F\|_{\nu,s,1}. \end{aligned}$$

□

Note that it is proved in [Su], using the complex interpolation method of Banach spaces, that $\mathcal{H}_{\nu,s}^p = (\mathcal{H}_{\nu,s}^2, \mathcal{H}_{\nu,s}^\infty)_{[1-2/p]}$ if $2 < p < \infty$; see Theorem 8.2 in [Su]. However, Proposition 2.3 allows us to use the same proof as in [Su] to get the following result.

Corollary 2.4. *If $1 < p < \infty$, then*

$$\mathcal{H}_{\nu,s}^p = (\mathcal{H}_{\nu,s}^1, \mathcal{H}_{\nu,s}^\infty)_{[1-1/p]}.$$

Now we go back to Theorem 2.1. First we need a proposition.

Proposition 2.5. *Let s be a nonnegative integer and let $\nu > d$, $2\nu > \alpha > d$. Then there is a constant $C_s > 0$ such that*

$$(1 - |z|^2)^{2\nu-\alpha} \left\| K_{\nu,s}(\cdot, z) \otimes^s B^t(z, z)^{1/2} v \right\|_{\alpha,s,1} \leq C_s \|v\|$$

for all $z \in \mathbb{B}$ and all $v \in \odot^s V'$.

Proof. Let $v \in \odot^s V'$. It follows from Lemma 7.1 in [Su] and Proposition 1.4.10 in [Ru] that

$$\begin{aligned} &\|K_{\nu,s}(\cdot, z) \otimes^s B^t(z, z)^{1/2} v\|_{\alpha,s,1} \\ &= \int_{\mathbb{B}} \left\| \otimes^s B^t(w, w)^{1/2} \otimes^s B^t(w, z)^{-1} \otimes^s B^t(z, z)^{1/2} v \right\| \frac{(1 - |w|^2)^\alpha}{|1 - \langle w, z \rangle|^{2\nu}} d\mu(w) \\ &\leq C_s \|v\| \int_{\mathbb{B}} \frac{(1 - |z|^2)^{s/2} (1 - |w|^2)^{\alpha+s/2}}{|1 - \langle w, z \rangle|^{2\nu+s}} d\mu(w) \leq C'_s (1 - |z|^2)^{\alpha-2\nu} \|v\|. \end{aligned}$$

□

Lemma 2.6. *Let $z \in \mathbb{B}$. Then there is a constant $C_s > 0$ such that, for any $v \in \odot^s V'$ and any $1 \leq p \leq \infty$, it follows that*

$$\left\| (1 - |z|^2)^\nu K_{\nu,s}(\cdot, z) \otimes^s B^t(z, z)^{1/2} v \right\|_{\nu,s,p} \leq C_s \|v\|.$$

Proof. Let $T_z = (1 - |z|^2)^\nu K_{\nu,s}(\cdot, z) \otimes^s B^t(z, z)^{1/2}$. By Proposition 2.5 and by Lemma 7.1 in [Su] it follows that $\|T_z v\|_{\nu,s,1} \leq C_s \|v\|$ and $\|T_z v\|_{\nu,s,\infty} \leq C'_s \|v\|$ respectively, for all $v \in \odot^s V'$. Thus the result follows from Riesz-Thorin's interpolation theorem. □

Now we can prove Theorem 2.1.

Proof of Theorem 2.1. Let $G \in \mathcal{H}_{\nu,s}^p$, $1 \leq p \leq \infty$. Then it follows from Lemma 2.6 that, for all $v \in \odot^s V'$,

$$\int_{\mathbb{B}} |\langle \otimes^s B^t(w, w) G(w), K_{\nu,s}(w, z) v \rangle| d\mu_{2\nu}(w) \leq \|G\|_{\nu,s,p} \|K_{\nu,s}(\cdot, z) v\|_{\nu,s,q} < \infty.$$

In particular, if $z = 0$, then

$$\int_{\mathbb{B}} |\langle \otimes^s B^t(w, w) G(w), v \rangle| (1 - |w|^2)^{2\nu} d\iota(w) < \infty.$$

By the mean-value property for holomorphic functions and rotation invariance for integration,

$$\int_{\mathbb{B}} \langle (1 - |w|^2)^{2\nu} \otimes^s B^t(w, w) G(w), v \rangle d\iota(w) = c' \langle G(0), v \rangle,$$

where $c' \neq 0$ only depends on d, ν and s . Hence, there exists a nonzero constant c such that, for all $G \in \mathcal{H}_{\nu, s}^p$ and all $v \in \odot^s V'$,

$$(2.8) \quad \langle G(0), v \rangle = c \langle G, v \rangle_{\nu, s, 2},$$

where $\langle \cdot, \cdot \rangle_{\nu, s, 2}$ is the $\mathcal{H}_{\nu, s}^2$ -pairing. Now, define an isometry $\pi_{\nu, s}$ on $\mathcal{H}_{\nu, s}^2$ by

$$\pi_{\nu, s} : g \in G, S(z) \rightarrow \left(\otimes^s (dg^{-1}(z))^t \right) S(g^{-1}z) (J_{g^{-1}}(z))^{2\nu/(d+1)},$$

as in [Su]. Let $z_0 \in \mathbb{B}$. For notational convenience we prove the reproducing property only for $s = 1$; the case for general s is identically the same. On the one hand,

$$(2.9) \quad \langle (\pi_{\nu, 1}(\varphi_{z_0})G)(0), v \rangle = \left\langle G(z_0), \overline{J_{\varphi_{z_0}}(0)}^{2\nu/(d+1)} (\varphi'_{z_0}(0)^t)^* v \right\rangle.$$

By equation (2.6), $(1 - |z_0|^2)^{(d+1)/2} < |J_{\varphi_{z_0}}(w)| < (1 - |z_0|^2)^{-d-1}$ on $\mathbb{B}$, so $\pi_{\nu, 1}(\varphi_{z_0})G \in \mathcal{H}_{\nu, 1}^p$. However, using equation (2.8) above for $\pi_{\nu, 1}(\varphi_{z_0})G$ and the transformation properties

$$B(\varphi_{z_0}(w), \varphi_{z_0}(z)) = \varphi'_{z_0}(w) B(w, z) (\varphi'_{z_0}(z))^*$$

(see equation (9) in [Su]) and

$$\begin{aligned} K_{\nu, 1}(\varphi_{z_0}(w), \varphi_{z_0}(z)) &= J_{\varphi_{z_0}}(w)^{-2\nu/(d+1)} \cdot \overline{J_{\varphi_{z_0}}(z)}^{-2\nu/(d+1)} \\ &\quad \cdot (\varphi'_{z_0}(w)^t)^{-1} K_{\nu, 1}(w, z) \left(\overline{\varphi'_{z_0}(z)} \right)^{-1} \end{aligned}$$

(see equation (9) in [Su] and Theorem 2.2.5 in [Ru]), the left-hand side in equation (2.9) above is

$$\langle G(z_0), u \rangle = c \langle G, K_{\nu, s}(\cdot, z_0)u \rangle_{\nu, s, 2},$$

where $u = \overline{J_{\varphi_{z_0}}(0)}^{2\nu/(d+1)} (\varphi'_{z_0}(0)^t)^* v$. Since v is arbitrary, then so is $u \in \odot^s V'$, which proves the theorem. $\square$

2.2. Hankel forms of higher weights. Let H_1 and H_2 be Hilbert spaces and let $T : H_1 \rightarrow H_2$ be a linear operator. Define the singular numbers $s_n(T) = \inf\{\|T - K\| : \text{rank}(K) \leq n\}$, $n \geq 0$. If T is compact, these singular numbers are equal to the eigenvalues of $|T| = (T^*T)^{1/2}$. We denote by $\mathcal{S}_p$ the ideal of operators for which $\{s_n(T)\}_{n \geq 0} \in l^p$, $0 < p \leq \infty$; see [S].

The transvectant $\mathcal{T}_s$ on $L_a^2(d\iota_\nu) \otimes L_a^2(d\iota_\nu)$ (introduced in [P1]; see also [P2], [PZ] and [Su]) is defined by

$$(2.10) \quad \mathcal{T}_s(f, g)(z) = \sum_{k=0}^s \binom{s}{k} (-1)^k \frac{\partial^{s-k} f(z) \odot \partial^k g(z)}{(\nu)_{s-k} (\nu)_k},$$

where

$$\partial^s f(z) = \sum_{j_1 \dots j_s=1}^d \partial_{j_1} \dots \partial_{j_s} f(z) dz_{j_1} \otimes \dots \otimes dz_{j_s} \in \odot^s V'$$

and $(\nu)_k = \nu(\nu+1) \dots (\nu+k-1)$, $(\nu)_0 = 1$, is the Pochhammer symbol.

Lemma 2.7. *There is a constant $C_s > 0$ such that*

$$\|\mathcal{T}_s(f, g)\|_{\nu, s, 1} \leq C_s \|f\|_\nu \|g\|_\nu$$

for all $f, g \in L_a^2(d\nu)$.

First we need a lemma, which actually is a consequence of Theorem 4.1 in [Su], but we give an independent and easier proof.

Lemma 2.8. *There is a constant $C_{\nu, s} > 0$ such that*

$$\int_{\mathbb{B}} \langle \otimes^s B^t(z, z) \partial^s f(z), \partial^s f(z) \rangle (1 - |z|^2)^\nu d\iota(z) \leq C_{\nu, s} \|f\|_\nu$$

for all $f \in L_a^2(d\nu)$.

Proof. First,

$$\partial^s f(z) = c_\nu(\nu)_s \int_{\mathbb{B}} \frac{f(w) \otimes^s \bar{w}}{(1 - \langle z, w \rangle)^{\nu+s}} \cdot (1 - |w|^2)^\nu d\iota(w),$$

so that

$$\left\| \otimes^s B^t(z, z)^{1/2} \partial^s f(z) \right\| \leq C_{\nu, s} \int_{\mathbb{B}} \frac{|f(w)| \cdot \|B^t(z, z)^{1/2} \bar{w}\|^s}{|1 - \langle z, w \rangle|^{\nu+s}} \cdot (1 - |w|^2)^\nu d\iota(w).$$

We can estimate

$$\begin{aligned} \|B^t(z, z)^{1/2} \bar{w}\| &= s_z (\|s_z P_{\bar{z}} \bar{w}\|^2 + \|Q_{\bar{z}} \bar{w}\|^2)^{1/2} \\ &= s_z (|w|^2 - |\langle z, w \rangle|^2)^{1/2} \\ &\leq \sqrt{2} \cdot s_z |1 - \langle z, w \rangle|^{1/2}. \end{aligned}$$

Hence,

$$\left\| \otimes^s B^t(z, z)^{1/2} \partial^s f(z) \right\| \leq C'_{\nu, s} \int_{\mathbb{B}} T(z, w) |f(w)| (1 - |w|^2)^\nu d\iota(w),$$

where

$$T(z, w) = \frac{(1 - |z|^2)^{s/2}}{|1 - \langle z, w \rangle|^{\nu+s/2}}.$$

Now, the result follows by exactly the same arguments as in the proof of Theorem 7.2 in [Su] (where we let $t = -(\nu - d)/4$). $\square$

Proof of Lemma 2.7. The transvectant is a linear combination of terms $\partial^k f(z) \otimes \partial^{s-k} g(z)$ so we need only to estimate $\|\partial^k f(z) \otimes \partial^{s-k} g(z)\|_{\nu, s, 1}$ for $0 \leq k \leq s$. First we observe that

$$\begin{aligned} &\left\| \otimes^s B^t(z, z)^{1/2} \partial^k f(z) \otimes \partial^{s-k} g(z) \right\| \\ &= \left\| \otimes^k B^t(z, z)^{1/2} \partial^k f(z) \right\| \cdot \left\| \otimes^{s-k} B^t(z, z)^{1/2} \partial^{s-k} g(z) \right\|. \end{aligned}$$

Thus by Hölder's inequality and Lemma 2.8 it follows that

$$\begin{aligned} \int_{\mathbb{B}} \left\| \otimes^s B^t(z, z)^{1/2} \partial^k f(z) \otimes \partial^{s-k} g(z) \right\| (1 - |z|^2)^\nu d\iota(z) \\ \leq C \|f\|_{\nu, k} \|g\|_{\nu, s-k} \leq C_s \|f\|_\nu \|g\|_\nu. \end{aligned}$$

□

The Hankel bilinear form H_F^s on $L_a^2(d\iota_\nu) \otimes L_a^2(d\iota_\nu)$ is defined by

$$(2.11) \quad H_F^s(f, g) = \int_{\mathbb{B}} \langle \otimes^s B^t(z, z) \mathcal{T}_s(f, g)(z), F(z) \rangle d\iota_{2\nu}(z)$$

where $F : \mathbb{B} \rightarrow \odot^s V'$ is holomorphic. We call F the symbol of the corresponding Hankel form. We remark that

$$H_F^0(f, g) = \int_{\mathbb{B}} f(z) g(z) \overline{F(z)} d\iota_{2\nu}(z).$$

This is the classical Hankel form studied in [JPR].

With the form H_F^s one can associate the operator A_F^s defined by

$$H_F^s(f, g) = \langle f, A_F^s g \rangle_\nu$$

as in [JPR]. Notice that A_F^s is an anti-linear operator on $L_a^2(d\iota_\nu)$. To get a linear operator one combines A_F^s with a conjugation, i.e., one instead considers the operator $\overline{A_F^s} : g \rightarrow \overline{A_F^s g}$. We say that H_F^s is of Schatten-von Neumann class $\mathcal{S}_p$, for $0 < p < \infty$, if and only if $\overline{A_F^s} : L_a^2(d\iota_\nu) \rightarrow \overline{L^2(d\iota_\nu)}$ is of class $\mathcal{S}_p$.

3. DUALITY OF $\mathcal{H}_{\nu, s}^p$

In this section we determine the dual space $(\mathcal{H}_{\nu, s}^p)^*$ of $\mathcal{H}_{\nu, s}^p$, $1 \leq p < \infty$.

Lemma 3.1. *Let $1 \leq p < \infty$. If $\Phi \in (\mathcal{H}_{\nu, s}^p)^*$, then there is a function $G \in L_{\nu, s}^q$ such that*

$$\Phi(F) = \int_{\mathbb{B}} \langle \otimes^s B^t(z, z) F(z), G(z) \rangle (1 - |z|^2)^{2\nu} d\iota(z)$$

and $\|\Phi\| = \|G\|_{\nu, s, q}$ where $1/q + 1/p = 1$.

Proof. Define $A(z) = (1 - |z|^2)^\nu \otimes^s B^t(z, z)^{1/2}$ and $(M_A F)(z) = A(z) F(z)$. Then M_A is an isometry from $L_{\nu, s}^p$ onto L^p , where $L^p = \{F : \mathbb{B} \rightarrow V : \|F\|_p < \infty\}$ and

$$\|F\|_p = \left(\int_{\mathbb{B}} \|F(z)\|^p d\iota(z) \right)^{1/p}.$$

Consider $\Theta = \Phi M_A^{-1}$. Then Θ is a bounded linear functional on L^p and $\Theta(AF) = \Phi(F)$. Then we can find a function $H \in L^q$ such that

$$\Phi(F) = \int_{\mathbb{B}} \langle (AF)(z), H(z) \rangle d\iota(z)$$

with $\|\Theta\| = \|H\|_q$. Let $G = M_A^{-1} H$. Then $G \in L_{\nu, s}^q$ and

$$\Phi(F) = \int_{\mathbb{B}} \langle \otimes^s B^t(z, z) F(z), G(z) \rangle (1 - |z|^2)^{2\nu} d\iota(z).$$

Also $\|\Phi\| = \|G\|_{\nu, s, q}$.

□

Theorem 3.2. For $1 \leq p < \infty$ we have $(\mathcal{H}_{\nu,s}^p)^* = \mathcal{H}_{\nu,s}^q$, under the integral pairing

$$\langle F, G \rangle_{\nu,s,2} = \int_{\mathbb{B}} \langle \otimes^s B^t(z, z) F(z), G(z) \rangle (1 - |z|^2)^{2\nu} d\mu(z), \quad F \in \mathcal{H}_{\nu,s}^p, \quad G \in \mathcal{H}_{\nu,s}^q,$$

where $1/p + 1/q = 1$. Namely, for any bounded linear functional $\Phi : \mathcal{H}_{\nu,s}^p \rightarrow \mathbb{C}$ there is a function $G \in \mathcal{H}_{\nu,s}^q$ such that $\Phi(F) = \langle F, G \rangle_{\nu,s,2}$ for all $F \in \mathcal{H}_{\nu,s}^p$ with

$$C\|G\|_{\nu,s,q} \leq \|\Phi\| \leq \|G\|_{\nu,s,q}.$$

Proof. By Hölder's inequality, every function $G \in \mathcal{H}_{\nu,s}^q$ defines a bounded linear functional Φ on $\mathcal{H}_{\nu,s}^p$ under the above integral pairing with $\|\Phi\| \leq \|G\|_{\nu,s,q}$.

Conversely, let $\Phi \in (\mathcal{H}_{\nu,s}^p)^*$. By the Hahn-Banach theorem we can extend Φ to a bounded linear functional $\tilde{\Phi}$ on $L_{\nu,s}^p$ such that $\Phi(F) = \tilde{\Phi}(F)$ for all $F \in \mathcal{H}_{\nu,s}^p$ with $\|\Phi\| = \|\tilde{\Phi}\|$. By Lemma 3.1 there is a function $H \in L_{\nu,s}^q$ such that

$$(3.1) \quad \tilde{\Phi}(F) = \int_{\mathbb{B}} \langle \otimes^s B^t(z, z) F(z), H(z) \rangle (1 - |z|^2)^{2\nu} d\mu(z)$$

for all $F \in L_{\nu,s}^p$, with $\|\tilde{\Phi}\| = \|H\|_{\nu,s,q}$. However, Theorem 2.1 implies that, for any $F \in \mathcal{H}_{\nu,s}^p$,

$$F(z) = (P_{\nu,s}F)(z) = c \int_{\mathbb{B}} (1 - |w|^2)^{2\nu} K_{\nu,s}(w, z)^* \otimes^s B^t(w, w) F(w) d\mu(w).$$

Substituting this into formula (3.1) and using Fubini-Tonelli's theorem we get that

$$\Phi(F) = \tilde{\Phi}(F) = \int_{\mathbb{B}} \langle \otimes^s B^t(w, w) F(w), (P_{\nu,s}H)(w) \rangle (1 - |w|^2)^{2\nu} d\mu(w).$$

Let $G = P_{\nu,s}H$. By Proposition 2.3, $\|P_{\nu,s}H\|_{\nu,s,q} \leq C'\|H\|_{\nu,s,q}$. Then $G \in \mathcal{H}_{\nu,s}^q$, $\Phi(F) = \langle F, G \rangle_{\nu,s,2}$ for all $F \in \mathcal{H}_{\nu,s}^p$ and $C\|G\|_{\nu,s,q} \leq \|\Phi\|$. $\square$

4. ATOMIC DECOMPOSITION OF $\mathcal{H}_{\nu,s}^1$

Following [JPR], we denote by $l^1(\mathbb{B}, \odot^s V')$ the space of all functions $a : \mathbb{B} \rightarrow \odot^s V'$, with support in $\{z_j\}_{j=1}^\infty \subset \mathbb{B}$, such that

$$\|a\|_{l^1} = \sum_{j=1}^\infty \|a(z_j)\| < \infty.$$

Also, denote by $l^\infty(\mathbb{B}, \odot^s V')$ the space of all functions $a : \mathbb{B} \rightarrow \odot^s V'$ such that

$$\|a\|_{l^\infty} = \sup_{z \in \mathbb{B}} \|a(z)\| < \infty.$$

Then it is elementary that

$$(4.1) \quad l^\infty(\mathbb{B}, \odot^s V') = (l^1(\mathbb{B}, \odot^s V'))^*,$$

under the pairing

$$\langle a, b \rangle' = \sum_{j=1}^\infty \langle a(z_j), b(z_j) \rangle$$

where $a \in l^1(\mathbb{B}, \odot^s V')$ with support $\{z_j\}_{j=1}^\infty \subset \mathbb{B}$ and $b \in l^\infty(\mathbb{B}, \odot^s V')$. Namely, for any bounded linear functional $\Phi : l^1(\mathbb{B}, \odot^s V') \rightarrow \mathbb{C}$ there is a function b in $l^\infty(\mathbb{B}, \odot^s V')$ such that $\Phi(a) = \langle a, b \rangle'$ for all $a \in l^1(\mathbb{B}, \odot^s V')$ with $\|\Phi\| = \|b\|_{l^\infty}$.

Theorem 4.1. *It follows that $F \in \mathcal{H}_{\nu,s}^1$ if and only if there is a sequence $\{z_j\}_{j=1}^\infty \subset \mathbb{B}$ and a sequence $\{a_j\}_{j=1}^\infty \in l^1(\mathbb{B}, \odot^s V')$ such that*

$$F(w) = \sum_{j=1}^{\infty} (1 - |z_j|^2)^\nu K_{\nu,s}(w, z_j) \otimes^s B^t(z_j, z_j)^{1/2} a_j.$$

Proof. By Proposition 2.5, for any $v \in \odot^s V'$ and any $z \in \mathbb{B}$,

$$\left\| K_{\nu,s}(\cdot, z) \otimes^s B^t(z, z)^{1/2} v \right\|_{\nu,s,1} \leq C_s (1 - |z|^2)^{-\nu} \|v\|.$$

Thus, the operator $T : l^1(\mathbb{B}, \odot^s V') \rightarrow \mathcal{H}_{\nu,s}^1$ defined by

$$(Ta)(w) = \sum_{j=1}^{\infty} (1 - |z_j|^2)^\nu K_{\nu,s}(w, z_j) \otimes^s B^t(z_j, z_j)^{1/2} a_j$$

is bounded, where $a_j = a(z_j)$ and the support of a is $\{z_j\}_{j=1}^\infty$. We need to prove that T is onto. Consider $T^* : (\mathcal{H}_{\nu,s}^1)^* \rightarrow (l^1(\mathbb{B}, \odot^s V'))^*$, $T^*(\Phi)(a) = \Phi(Ta)$, which is bounded, where $\Phi \in (\mathcal{H}_{\nu,s}^1)^*$ and $a \in l^1(\mathbb{B}, \odot^s V')$. By Theorem 3.2, for any $\Phi \in (\mathcal{H}_{\nu,s}^1)^*$ there is a $G \in \mathcal{H}_{\nu,s}^\infty$ such that $\Phi(F) = \langle F, G \rangle_{\nu,s,2}$ for all $F \in \mathcal{H}_{\nu,s}^1$ with $C\|G\|_{\nu,s,\infty} \leq \|\Phi\| \leq \|G\|_{\nu,s,\infty}$. Now, let $a \in l^1(\mathbb{B}, \odot^s V')$ with support $\{z_j\}_{j=1}^\infty \subset \mathbb{B}$. By the reproducing property in Theorem 2.1 it follows that

$$T^*(\Phi)(a) = \Phi(Ta) = \langle Ta, G \rangle_{\nu,s,2} = c \sum_{j=1}^{\infty} \langle a_j, (1 - |z_j|^2)^\nu \otimes B^t(z_j, z_j) G(z_j) \rangle.$$

Hence, by (4.1) and Theorem 3.2 it follows that

$$(4.2) \quad \frac{1}{c} \cdot \|T^* \Phi\|_{(l^1)^*} = \sup_{z \in \mathbb{B}} \left\| (1 - |z|^2)^\nu \otimes^s B^t(z, z) G(z) \right\| = \|G\|_{\nu,s,\infty} \geq \|\Phi\|.$$

On the one hand, (4.2) yields that $\ker T^* = \{0\}$ and consequently the range of T is dense in $\mathcal{H}_{\nu,s}^1$. On the other hand, (4.2) yields that the range of T^* is closed and so is the range of T by the Closed Range Theorem. $\square$

5. TRACE CLASS $\mathcal{S}_1$

We consider now the trace class property of H_F^s in (2.11).

Theorem 5.1. *The Hankel form H_F^s is of trace class $\mathcal{S}_1$ if and only if $F \in \mathcal{H}_{\nu,s}^1$.*

Combining the results in [Su] we have now a complete characterization of the Schatten-von Neumann class Hankel forms.

Theorem 5.2. *The Hankel form H_F^s is of Schatten-von Neumann class $\mathcal{S}_p$ if and only if $F \in \mathcal{H}_{\nu,s}^p$, $1 \leq p \leq \infty$.*

Proof of Theorem 5.2. It follows from Lemma 5.5 below and Theorem 1.1(a) in [Su] that the operator $\Gamma : F \rightarrow H_F^s$ is bounded from $\mathcal{H}_{\nu,s}^1$ into $\mathcal{S}_1$ and from $\mathcal{H}_{\nu,s}^\infty$ into $\mathcal{S}_\infty$, respectively. Since $\mathcal{S}_p = (\mathcal{S}_1, \mathcal{S}_\infty)_{[1-1/p]}$ if $1 < p < \infty$, then it follows by Riesz-Thorin's interpolation theorem and Corollary 2.4 that Γ is bounded from $\mathcal{H}_{\nu,s}^p$ into $\mathcal{S}_p$ if $1 < p < \infty$.

On the other hand, it follows from Lemma 5.6 below and Theorem 1.1(a) in [Su] that $\tilde{T}_s$, defined in (5.2), is bounded from $\mathcal{S}_1$ into $\mathcal{H}_{\nu,s}^1$ and from $\mathcal{S}_\infty$ into $\mathcal{H}_{\nu,s}^\infty$, respectively. Again, by interpolation $\tilde{T}_s$ is bounded from $\mathcal{S}_p$ into $\mathcal{H}_{\nu,s}^p$ if $1 < p < \infty$.

Also, if $H_F^s \in \mathcal{S}_p$ for $1 \leq p < \infty$, then $\tilde{\mathcal{T}}_s(H_F^s) = F$, which follows by the same arguments as in the proof of Lemma 8.6 in [Su]. $\square$

The proof of Theorem 5.1 will be divided into a few lemmas. We will first show in Lemma 5.3 that every H_F^s is of trace class $\mathcal{S}_1$ if F is in $\mathcal{H}_{\nu,s}^1$ and then in Lemma 5.4 that $\mathcal{H}_{\nu,s}^1$ can be continuously embedded into $\mathcal{H}_{\nu,s}^\infty$. Using these results we prove, in Lemma 5.5, that $F \rightarrow H_F^s$ is bounded from $\mathcal{H}_{\nu,s}^1$ into $\mathcal{S}_1$. Finally, in Lemma 5.6 we find a bounded mapping $\tilde{\mathcal{T}}_s$ from the trace class $\mathcal{S}_1$ into $\mathcal{H}_{\nu,s}^1$ such that $\tilde{\mathcal{T}}_s(H_F^s) = F$.

Lemma 5.3. *If $F \in \mathcal{H}_{\nu,s}^1$, then $H_F^s \in \mathcal{S}_1$.*

Proof. Let $F \in \mathcal{H}_{\nu,s}^1$. By Theorem 4.1, $F = \sum_{j=1}^\infty F_j$ where

$$F_j(w) = (1 - |z_j|^2)^\nu K_{\nu,s}(w, z_j) \otimes^s B^t(z_j, z_j)^{1/2} a_j$$

for some $\{z_j\}_{j=1}^\infty \subset \mathbb{B}$ and some $\{a_j\}_{j=1}^\infty \in l^1(\mathbb{B}, \odot^s V')$. We claim that

$$(5.1) \quad \text{rank } H_{F_j}^s \leq M_s \quad \text{for all } j = 1, 2, 3, \dots,$$

where M_s depends only on s and d . Accepting temporarily the claim and using Theorem 1.1(a) in [Su] we get that

$$\|H_F^s\|_{\mathcal{S}_1} \leq \sum_{j=1}^\infty \|H_{F_j}^s\|_{\mathcal{S}_1} \leq M_s \sum_{j=1}^\infty \|H_{F_j}^s\|_{\mathcal{S}_\infty} \leq M'_s \sum_{j=1}^\infty \|F_j\|_{\nu,s,\infty} \leq M''_s \sum_{j=1}^\infty \|a_j\|.$$

Now we go back to claim (5.1). By Lemma 2.7, $\mathcal{T}_s(f, g) \in \mathcal{H}_{\nu,s}^1$ for all $f, g \in L_a^2(d\nu)$. Thus, by the reproducing property in Theorem 2.1,

$$H_{F_j}^s(f, g) = c \left\langle \mathcal{T}_s(f, g)(z_j), (1 - |z_j|^2)^\nu \otimes^s B^t(z_j, z_j)^{1/2} a_j \right\rangle.$$

Fix $z_0 \in \mathbb{B}$. Then $\mathcal{T}_s(f, g)(z_0)$ is a sum of finitely many rank one forms where the number M_s of summands depends only on s and d . To see this, we consider $f(z_0) = \langle f, K_{z_0} \rangle_\nu$. Since

$$\partial^{s-k} f(z_0) \otimes \partial^k g(z_0) = \left\langle f, \overline{\partial^{s-k}} K_{z_0} \right\rangle_\nu \otimes \left\langle g, \overline{\partial^k} K_{z_0} \right\rangle_\nu,$$

then $(f, g) \rightarrow \partial^{s-k} f(z_0) \otimes \partial^k g(z_0)$ is a rank one form. Thus, the bilinear form $(f, g) \rightarrow \mathcal{T}_s(f, g)(z_0)$ has rank at most M_s and so has $H_{F_j}^s$. $\square$

Lemma 5.4. *The operator $\mathcal{I} : \mathcal{H}_{\nu,s}^1 \rightarrow \mathcal{H}_{\nu,s}^\infty$, $\mathcal{I}(F) = F$, is bounded.*

Proof. First, let $F \in \mathcal{H}_{\nu,s}^1$. Then $H_F^s \in \mathcal{S}_1$ by Lemma 5.3. Hence $H_F^s \in \mathcal{S}_\infty$, so by Theorem 1.1(a) in [Su] it follows that $F \in \mathcal{H}_{\nu,s}^\infty$. Thus $\mathcal{I}$ is well-defined.

Now, assume that $F_n \rightarrow F$ in $\mathcal{H}_{\nu,s}^1$ and that $\mathcal{I}(F_n) \rightarrow G$ in $\mathcal{H}_{\nu,s}^\infty$. We shall prove that $\mathcal{I}(F) = G$. On the one hand, since $F_n \rightarrow F$ in $\mathcal{H}_{\nu,s}^1$, then there is a subsequence $\mathcal{I}(F_{n_j})$ converging pointwise to $\mathcal{I}(F)$. On the other hand, since $\mathcal{I}(F_n) \rightarrow G$ in $\mathcal{H}_{\nu,s}^\infty$, then $\mathcal{I}(F_{n_j}) \rightarrow G$ pointwise. Thus $\mathcal{I}(F) = G$ and the operator $\mathcal{I}$ is bounded by the Closed Graph Theorem. $\square$

Lemma 5.5. *The operator $\Gamma : \mathcal{H}_{\nu,s}^1 \rightarrow \mathcal{S}_1$, $\Gamma(F) = H_F^s$, is bounded.*

Proof. The operator Γ is well defined by Lemma 5.3. We use the Closed Graph Theorem. Assume that $F_n \rightarrow F$ in $\mathcal{H}_{\nu,s}^1$ and that $\Gamma(F_n) \rightarrow B$ in $\mathcal{S}_1$. We shall prove that $H_F^s = B$. On the one hand, by Theorem 1.1(a) in [Su] and Lemma 5.4 it follows that

$$\|H_{F_n-F}^s\|_{\mathcal{S}_\infty} \leq C \|F_n - F\|_{\nu,s,\infty} \leq C' \|F_n - F\|_{\nu,s,1}$$

so that $H_{F_n}^s \rightarrow H_F^s$ in $\mathcal{S}_\infty$. On the other hand,

$$\|\Gamma(F_n) - B\|_{\mathcal{S}_\infty} \leq \|\Gamma(F_n) - B\|_{\mathcal{S}_1}$$

so that $H_{F_n}^s \rightarrow B$ in $\mathcal{S}_\infty$. Thus $H_F^s = B$ so that Γ has the closed graph property. Hence, Γ is bounded. $\square$

We recall the transvectant $\tilde{\mathcal{T}}_s : \mathcal{S}_\infty(L_a^2(d\iota_\nu), L_a^2(d\iota_\nu)) \rightarrow \mathcal{A}_s(\mathbb{B} \times \mathbb{B})$ defined in [Su] (see also [FR] and [PZ]), where $\mathcal{A}_s(\mathbb{B} \times \mathbb{B})$ consists of all holomorphic functions $G : \mathbb{B} \times \mathbb{B} \rightarrow \odot^s V'$. We recall further that the transvectant $\mathcal{T}_s$ in (2.10) can be defined for any holomorphic function $G(z, w)$ on $\mathbb{B} \times \mathbb{B}$, namely

$$(\mathcal{T}_s G)(z, w) = \sum_{k=0}^s \binom{s}{k} (-1)^k \frac{\partial_z^k \odot \partial_w^{s-k} G(z, w)}{(\nu)_k (\nu)_{s-k}}.$$

For bounded bilinear forms A on $L_a^2(d\iota_\nu)$, we define

$$(5.2) \quad \tilde{\mathcal{T}}_s(A)(z) = (\mathcal{T}_s G)(z, z),$$

where $G(z, w) = \overline{A(K_z, K_w)}$.

Lemma 5.6. *The operator $\tilde{\mathcal{T}}_s : \mathcal{S}_1 \rightarrow \mathcal{H}_{\nu, s}^1$ defined in (5.2) is bounded. Also, $\tilde{\mathcal{T}}_s(H_F^s) = F$ if $H_F^s \in \mathcal{S}_1$.*

Proof. First, let $B \in \mathcal{S}_1$ be of rank one. Then there exists $\phi, \varphi \in L_a^2(d\iota_\nu)$ such that

$$B(f, g) = \langle f, \phi \rangle_\nu \langle g, \varphi \rangle_\nu$$

for all $f, g \in L_a^2(d\iota_\nu)$. Then $\|B\|_{\mathcal{S}_1} = \|\phi\|_\nu \|\varphi\|_\nu$ and $\tilde{\mathcal{T}}_s(B)(z) = \mathcal{T}_s(\phi, \varphi)(z)$, so by Lemma 2.7 it follows that

$$(5.3) \quad \|\tilde{\mathcal{T}}_s(B)\|_{\nu, s, 1} \leq C_s \|\phi\|_\nu \|\varphi\|_\nu \leq C_s \|B\|_{\mathcal{S}_1}.$$

In general, if $B \in \mathcal{S}_1$ we can write $B = \sum_{n=1}^\infty B_n$, $\text{rank } B_n = 1$ such that

$$\|B^N\|_{\mathcal{S}_1} = \sum_{n=1}^N \|B_n\|_{\mathcal{S}_1} \rightarrow \|B\|_{\mathcal{S}_1}, \text{ as } N \rightarrow \infty,$$

where $B^N = \sum_{n=1}^N B_n$. By (5.3) the sequence $\{\tilde{\mathcal{T}}_s(B^N)\}_{n=1}^\infty$ is Cauchy and hence converges to some G in $\mathcal{H}_{\nu, s}^1$. Now, since $B^N \rightarrow B$ in $\mathcal{S}_\infty$ it follows by Lemma 8.4 in [Su] that $\tilde{\mathcal{T}}_s(B^N) \rightarrow \tilde{\mathcal{T}}_s(B)$ in $\mathcal{H}_{\nu, s}^\infty$. Hence $\tilde{\mathcal{T}}_s(B) = G$ so that (5.3) holds for any $B \in \mathcal{S}_1$.

Also, if $H_F^s \in \mathcal{S}_1$, then $\tilde{\mathcal{T}}_s(H_F^s) = F$. (As in the proof of Theorem 5.2 we refer to the proof of Lemma 8.6 in [Su].) $\square$

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