A CONDITION UNDER WHICH $B = A = U^*BU$ FOLLOWS FROM $B \leq A \leq U^*BU$

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ABSTRACT. We will give some sufficient conditions for a p -hyponormal operator, $p > 0$, to be normal, and a sufficient condition for a triplet of operators A, B, U with A, B self-adjoint and U unitary such that $B \leq A \leq U^*BU$ necessarily satisfies $B = A = U^*BU$.

1. INTRODUCTION

Given A, B self-adjoint operators (by operators we mean bounded linear ones) on a Hilbert space $\mathcal{H}$, supposed to be separable herein, we denote $A \leq_u B$ if $A \leq U^*BU$ for some unitary U on $\mathcal{H}$ (see [5]). A noteworthy fact on the binary relation $\leq_u$ is that if A, B are self-adjoint and $A \leq_u B$, then we have $e^A \leq_u e^B$ (Kosaki [4]). Our interest in this paper is toward a deeper understanding of the relation $\leq_u$.

It is obvious that the relation $\leq_u$ satisfies the reflexive and transitive laws. However, the antisymmetric law, that is, the assertion

$$A \leq_u B \text{ and } B \leq_u A \implies A, B \text{ are unitarily equivalent,}$$

is true in some cases, but is not true in general. It would be noteworthy to point out that if A, B are operators of *trace class* and satisfy $A \leq_u B, B \leq_u A$, then they are unitarily equivalent, because, $U^*BU \leq A \leq V^*BV$ implies that the trace of $V^*BV - U^*BU \geq 0$ is zero, and that $U^*BU = A$. Further, it is known that if either A or B is a *positive compact* operator and that they satisfy $A \leq_u B, B \leq_u A$, then they are unitarily equivalent. This is shown by Takeaki Yamazaki, and the proof is as follows: $U^*BU \leq A \leq V^*BV$ together with $B \geq 0$ implies that $T = B^{1/2}VU^*$ is compact hyponormal, and is necessarily normal ([1]), so, we have $B = TT^* = T^*T = UV^*BVU^* = UAU^*$. But, on the other hand, we have examples of triplets of operators A, B, U with A, B self-adjoint, *not* unitarily equivalent, and U unitary for which $B \leq A \leq U^*BU$ holds (see Examples 2 and 3 below; the latter is due to Yamazaki).

We will give in Theorem 5 in Section 3 a sufficient condition for a triplet of operators A, B, U with A, B self-adjoint and U unitary, on $\mathcal{H}$, which satisfy that $B \leq A \leq U^*BU$ necessarily satisfy $B = A = U^*BU$, namely, that either $\sigma(A)$

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or $\sigma(B)$ is a null set with respect to Lebesgue measure on the real line $\mathbf{R}$ and $\sigma(U) \neq \mathbf{T}$, the unit circle on the complex plane $\mathbf{C}$.

In Section 2, we will investigate sufficient conditions for p -hyponormal operators, $p > 0$, to be normal; for one thing for interest in themselves, and for another for the discussion in Section 3. We will show in Theorem 3 that for a p -hyponormal operator T to be normal, it suffices that both $\sigma(|T|)$ and the set $\{r \geq 0 : C_r \subseteq \sigma(T)\}$ are null with respect to Lebesgue measure on $\mathbf{R}$, where $C_r = \{\lambda : |\lambda| = r\}$, and in Theorem 4 that $\sigma(|T|)$ is null with respect to Lebesgue measure on $\mathbf{R}$ and $\{\text{Arg} \lambda : \lambda \in \sigma(U) \setminus \{0\}\} \neq (-\pi, \pi]$ holds, where $T = U|T|$ is the polar decomposition of T .

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2. CONDITIONS FOR p -HYPONORMAL OPERATORS TO BE NORMAL

An operator T on $\mathcal{H}$ is called p -hyponormal, $p > 0$, if it satisfies the inequality

$$(T^*T)^p \geq (TT^*)^p.$$

An operator T is called *hyponormal* if T is 1-hyponormal. It is known that any compact hyponormal operator is necessarily normal (T. Andô [1]), and further, that any compact p -hyponormal operator, $p > 0$, is necessarily normal. This fact is due to the following

Theorem 1 (Chô and Itô [3]). *If an operator T on $\mathcal{H}$ is p -hyponormal, $1 \geq p > 0$, then we have*

$$\|(T^*T)^p - (TT^*)^p\| \leq \frac{p}{\pi} \int \int_{re^{i\theta} \in \sigma(T)} r^{2p-1} dr d\theta.$$

Theorem 1 was established for the case $p = 1$ by Putnam [6], and improved extensively by D. Xia [7], M. Chô and M. Itô. The inequality above is called the *Putnam-type inequality*.

The *normal approximate point spectrum* for an operator T , denoted by $\sigma_{na}(T)$, is the set of complex numbers λ for which there exists a sequence $\{\xi_n\}$ of unit vectors of $\mathcal{H}$ such that

$$\lim_{n \rightarrow \infty} \|(T - \lambda I)\xi_n\| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|(T^* - \bar{\lambda}I)\xi_n\| = 0.$$

The normal approximate point spectrum is characterized as follows.

Lemma 1 (cf. [7]). *Let T be an operator on $\mathcal{H}$, $T = U|T|$ the polar decomposition of T , and $\lambda = re^{i\theta}$, $r \neq 0$. Then, $\lambda \in \sigma_{na}(T)$ if and only if there exists a sequence $\{\xi_n\}$ of unit vectors of $\mathcal{H}$ such that*

$$\lim_{n \rightarrow \infty} \||T| - rI\|\xi_n\| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|(U - e^{i\theta}I)\xi_n\| = 0.$$

It is clear that for any hyponormal operator T we have the equality $\sigma_{na}(T) = \sigma_a(T)$, the approximate point spectrum of T . Xia proved that it holds also for any $1/2$ -hyponormal operator T , while Chô and Huruya [2] further proved the following

Theorem 2. *For any p -hyponormal T , $p > 0$, we have $\sigma_{na}(T) = \sigma_a(T)$.*

Combining Theorem 2 with Lemma 1, we have the following, where $|S|$ denotes the set $\{|\xi| : \xi \in S\}$ for a set S on $\mathbf{C}$.

Lemma 2. For any p -hyponormal operator T on $\mathcal{H}$, we have $|\sigma_a(T)| \subseteq \sigma(|T|)$.

Now we show the following theorems.

Theorem 3. Let T be a p -hyponormal operator on $\mathcal{H}$, $p > 0$. If both $\sigma(|T|)$ and the set $\{r \geq 0 : C_r \subseteq \sigma(T)\}$ are null sets with respect to Lebesgue measure on $\mathbf{R}$, where $C_r = \{\lambda : |\lambda| = r\}$, then T is normal.

Proof. Since the boundary $\partial\sigma(T)$ of $\sigma(T)$ is contained in $\sigma_a(T)$, we have $|\partial\sigma(T)| \subseteq |\sigma_a(T)| \subseteq \sigma(|T|)$. So, we have

$$|\sigma(T)| = |\partial\sigma(T)| \cup \{r \geq 0 : C_r \subseteq \sigma(T)\} \subseteq \sigma(|T|) \cup \{r \geq 0 : C_r \subseteq \sigma(T)\}$$

which shows that, if both $\sigma(|T|)$ and the set $\{r \geq 0 : C_r \subseteq \sigma(T)\}$ are null with respect to Lebesgue measure on $\mathbf{R}$, so is the set $|\sigma(T)|$. Therefore, $\sigma(T)$, which is contained in $\{re^{i\theta} : r \in |\sigma(T)|, \theta \in (-\pi, \pi]\}$, is null with respect to planar Lebesgue measure. Hence, by Theorem 1, it follows that T is normal. $\square$

Theorem 4. Let T be a p -hyponormal operator on $\mathcal{H}$, $p > 0$, and let $T = U|T|$ be the polar decomposition of T . If $\sigma(|T|)$ is a null set with respect to Lebesgue measure on $\mathbf{R}$, and $\{\text{Arg}\lambda : \lambda \in \sigma(U) \setminus \{0\}\} \neq (-\pi, \pi]$ holds, then T is normal.

Proof. Let $r > 0$ be such that $C_r \subseteq \sigma(T)$. Then, for any $\theta \in (-\pi, \pi]$, $re^{i\theta}$ is in $\sigma(T)$. Hence, there is an $r_\theta > 0$ such that $r_\theta e^{i\theta} \in \partial\sigma(T)$. But, $\partial\sigma(T)$ is contained in $\sigma_{na}(T) = \sigma_a(T)$ by Theorem 2, so, $e^{i\theta}$ is in $\sigma(U)$ by Lemma 1. It follows that no $r > 0$ satisfies that $C_r \subseteq \sigma(T)$. Therefore, by the preceding theorem we conclude that T is normal. $\square$

We give an example which shows that Theorem 4 fails to be true if the assumption $\{\text{Arg}\lambda : \lambda \in \sigma(U) \setminus \{0\}\} \neq (-\pi, \pi]$ fails to be fulfilled.

Example 1. Let T be a weighted shift defined by

$$T = \begin{pmatrix} 0 & & & \\ \lambda_1 & 0 & & \\ & \lambda_2 & \ddots & \\ & & \ddots & \ddots \end{pmatrix}.$$

Then, T is p -hyponormal if and only if $|\lambda_n| \geq |\lambda_{n+1}|$ ($n = 1, 2, \dots$). If $T = U|T|$ is the polar decomposition of T , then U is the unilateral shift

$$U = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & \ddots & \\ & & \ddots & \ddots \end{pmatrix},$$

whose spectrum is the closed unit disk, and $|T|$ the diagonal operator with the diagonal sequence $\{|\lambda_n|\}$. So, it suffices to choose $\{\lambda_n\}$ so that $\lambda_n \neq 0$ ($n = 1, 2, \dots$) and $\sigma(|T|)$ null with respect to Lebesgue measure on $\mathbf{R}$.

3. A CONDITION UNDER WHICH $B = A = U^*BU$
FOLLOWS FROM $B \leq A \leq U^*BU$

Now we are about to show the following

Theorem 5. *Let A, B be self-adjoint operators and let U be a unitary operator. If either $\sigma(A)$ or $\sigma(B)$ is a null set with respect to Lebesgue measure on $\mathbf{R}$, $\sigma(U) \neq \mathbf{T}$, and $B \leq A \leq U^*BU$, then we have $B = A = U^*BU$.*

Proof. Without loss of generality, we may assume that $\sigma(B)$ is a null set on $\mathbf{R}$. Choose $\lambda > 0$ such that $B + \lambda I$ becomes positive and invertible. Put $T = (B + \lambda I)^{1/2}U$; then by assumption, $TT^* \leq A + \lambda I \leq T^*T$. We also have that $|T| = U^*(B + \lambda I)^{1/2}U = U^*T$, and that $\sigma(|T|)$ is a null set on $\mathbf{R}$ because $\sigma((B + \lambda I)^{1/2})$ is also a null set. Let $T = V|T|$ be the polar decomposition of T . Then $T = VU^*T$, and so, $I = VU^*$, that is, $U = V$. Now we have found that T satisfies the conditions in Theorem 4. Hence T turns out to be normal, which implies that $B = A = U^*BU$. $\square$

The following is immediate.

Corollary. *Let A, B be self-adjoint operators, and let U, V be unitaries. If either $\sigma(A)$ or $\sigma(B)$ is a null set with respect to Lebesgue measure on $\mathbf{R}$, $\sigma(UV) \neq \mathbf{T}$, and $A \leq U^*BU$, $B \leq V^*AV$, then we have $A = U^*BU$, $B = V^*AV$.*

We conclude by giving two examples which show that for Theorem 5 to be true the condition $\sigma(U) \neq \mathbf{T}$ cannot be removed.

Example 2. Let A, B be bilateral diagonal operators with diagonal sequences $\{\alpha_n\}_{-\infty}^{\infty}$, $\{\beta_n\}_{-\infty}^{\infty}$ of real numbers, respectively, and let U be the bilateral shift. Then, U^*BU is the diagonal operator with diagonal sequence $\{\beta_{n+1}\}_{-\infty}^{\infty}$. So, for our purpose, it suffices to take sequences $\{\alpha_n\}, \{\beta_n\}$ so that $\beta_n \leq \alpha_n \leq \beta_{n+1}$ and that $\{\alpha_n\} \neq \{\beta_n\}$ set-theoretically. Indeed, the former requirement is equivalent to $B \leq A \leq U^*BU$, and the latter implies that A and B are not unitarily equivalent.

Example 3. Let U be the unilateral shift, V the diagonal operator with the diagonal sequence $\{1/2, 1, 1, \dots\}$, and put

$$A = \begin{pmatrix} V & O \\ O & O \end{pmatrix}, \quad B = \begin{pmatrix} UU^* & O \\ O & O \end{pmatrix}, \quad W = \begin{pmatrix} U & I - UU^* \\ O & U^* \end{pmatrix}.$$

Then A, B are positive operators and W a unitary operator. It is seen that $B \leq A \leq W^*BW$ because $O \leq UU^* \leq V \leq I = U^*U$. But $\sigma(A) \neq \sigma(B)$, which shows that A and B are not unitarily equivalent.

REFERENCES

- [1] T. Andô, *On hyponormal operators*, Proc. Amer. Math. Soc. **14** (1963), 290–291. MR0145353 (26:2884)
- [2] M. Chô and T. Huruya, *p-hyponormal operators* ($0 < p < 1/2$), Comment. Math. **33** (1993), 23–29. MR1269396 (95b:47021)
- [3] M. Chô and M. Itô, *Putnum's inequality for p-hyponormal operators*, Proc. Amer. Math. Soc. **123** (1995), no. 8, 2435–2440. MR1246519 (95j:47027)
- [4] H. Kosaki, *On some trace inequalities*, Proc. of the Centre for Math. Anal., Australian National Univ. vol. **29** (1992), 129–134. MR1188890 (93j:47027)
- [5] T. Okayasu and Y. Ueta, *Some order-like relations concerned with unitary equivalence, to appear.*

- [6] C. R. Putnam, *An inequality for the area of hyponormal spectra*, Math. Z. **116** (1970), 323–330. MR0270193 (42:5085)
- [7] D. Xia, *Spectral theory of hyponormal operators*, Birkhäuser Verlag, Basel, 1983. MR0806959 (87j:47036)

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