WEIGHTED COMPOSITION OPERATOR ON THE FOCK SPACE

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(Communicated by Joseph A. Ball)

ABSTRACT. We characterize the boundedness and compactness of a weighted composition operator on the Fock space. Our results use a certain integral transform. We also estimate the essential norm of a weighted composition operator. The result could be extended to the higher-dimensional case.

1. INTRODUCTION

Throughout this paper, let dA denote the usual Lebesgue measure on $\mathbb{C}$. The Fock space $\mathcal{F}^2$ is the space of all entire functions f on $\mathbb{C}$ for which

$$\|f\|_{\mathcal{F}^2}^2 = \frac{1}{2\pi} \int_{\mathbb{C}} |f(z)|^2 e^{-\frac{|z|^2}{2}} dA(z) < \infty.$$

$\mathcal{F}^2$ is a Hilbert space with inner product

$$\langle f, g \rangle_{\mathcal{F}^2} = \frac{1}{2\pi} \int_{\mathbb{C}} f(z) \overline{g(z)} e^{-\frac{|z|^2}{2}} dA(z).$$

The reproducing kernel function for $\mathcal{F}^2$ is given by $K(z, w) = \exp \frac{\langle z, w \rangle}{2}$ (see [5]). We also use the normalized kernel function $k_w(z) = \exp \frac{\langle z, w \rangle}{2} - \frac{|w|^2}{4}$.

Many authors have studied various concrete operators on the Fock space. For these studies, refer to [1, 2, 4, 5, 8, 10, 11]. In this paper, our object is the weighted composition operator on $\mathcal{F}^2$. Let φ and u be entire functions on $\mathbb{C}$. The *weighted composition operator* uC_φ is defined by $uC_\varphi f = u \cdot (f \circ \varphi)$ for an entire function f . The purpose of this paper is to characterize the boundedness and the compactness of uC_φ . Our results will be expressed in terms of the integral transform

$$B_\varphi(|u|^2)(w) = \frac{1}{2\pi} \int_{\mathbb{C}} |u(z)|^2 |e^{\frac{\langle \varphi(z), w \rangle}{2}}|^2 e^{-\frac{|w|^2}{2}} e^{-\frac{|z|^2}{2}} dA(z).$$

The following two theorems are our main results.

Theorem 1. *Let φ be an entire function on $\mathbb{C}$ and $u \in \mathcal{F}^2$. Then uC_φ is a bounded operator on $\mathcal{F}^2$ if and only if $B_\varphi(|u|^2) \in L^\infty(\mathbb{C})$.*

Received by the editors October 25, 2005 and, in revised form, December 5, 2005.

2000 *Mathematics Subject Classification.* Primary 47B38; Secondary 30D15.

Key words and phrases. Weighted composition operators, Fock space, entire function.

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Theorem 2. Suppose that φ and u are entire functions on $\mathbb{C}$ such that uC_φ is bounded on $\mathcal{F}^2$. Then there is a positive constant C such that

$$\limsup_{|w| \rightarrow \infty} B_\varphi(|u|^2)(w) \leq \|uC_\varphi\|_e^2 \leq C \limsup_{|w| \rightarrow \infty} B_\varphi(|u|^2)(w).$$

In particular, uC_φ is a compact operator on $\mathcal{F}^2$ if and only if

$$\lim_{|w| \rightarrow \infty} B_\varphi(|u|^2)(w) = 0.$$

2. PROOFS OF THE MAIN THEOREMS

In order to prove our results, we need several lemmas. In this section, let φ be an entire function and $u \in \mathcal{F}^2$.

The following lemma appears in [6, 7]. However, we give another proof. Our proof is simpler than the proof in [7] and remains valid in the higher-dimensional case.

Lemma 1. There is a positive constant C such that for each entire function f and $z \in \mathbb{C}$,

$$|f(z)|^2 e^{-\frac{|z|^2}{2}} \leq C \int_{D(z,1)} |f(w)|^2 e^{-\frac{|w|^2}{2}} dA(w),$$

where $D(z, 1) = \{w \in \mathbb{C} : |w - z| < 1\}$.

Proof. Since $|f|^2$ is subharmonic, we have

$$|f(0)|^2 \leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta,$$

for all $r > 0$. Multiplying both sides by $2\pi r e^{-\frac{r^2}{2}}$ and integrating with respect to r from 0 to 1 give

$$(1) \quad |f(0)|^2 \leq \frac{1}{2\pi(1 - e^{-\frac{1}{2}})} \int_{D(0,1)} |f(w)|^2 e^{-\frac{|w|^2}{2}} dA(w).$$

For fixed $c \in \mathbb{C}$, we set $I_c f(z) = k_c(z) f(z - c)$. By [3, Proposition 2], we see that I_c is a unitary operator on $\mathcal{F}^2$ and $I_c^{-1} = I_{-c}$. Since $|f(z)|^2 e^{-\frac{|z|^2}{2}} = |I_{-z} f(0)|^2$, it follows from (1) and a change of variable that

$$\begin{aligned} |f(z)|^2 e^{-\frac{|z|^2}{2}} &= |I_{-z} f(0)|^2 \\ &\leq \frac{1}{2\pi(1 - e^{-\frac{1}{2}})} \int_{D(0,1)} |I_{-z} f(w)|^2 e^{-\frac{|w|^2}{2}} dA(w) \\ &= \frac{1}{2\pi(1 - e^{-\frac{1}{2}})} \int_{D(0,1)} e^{-\frac{|z|^2}{2} - \operatorname{Re}\langle w, z \rangle} |f(w + z)|^2 e^{-\frac{|w|^2}{2}} dA(w) \\ &= \frac{1}{2\pi(1 - e^{-\frac{1}{2}})} \int_{D(0,1)} |f(w + z)|^2 e^{-\frac{|w+z|^2}{2}} dA(w) \\ &= \frac{1}{2\pi(1 - e^{-\frac{1}{2}})} \int_{D(z,1)} |f(w)|^2 e^{-\frac{|w|^2}{2}} dA(w). \end{aligned}$$

The proof is complete. $\square$

Lemma 2. Define the positive measure μ by

$$\mu(E) = \frac{1}{2\pi} \int_{\varphi^{-1}(E)} |u(z)|^2 e^{-\frac{|z|^2}{2}} dA(z),$$

where E is a Borel subset of $\mathbb{C}$. Then

$$\int_{D(w,1)} e^{\frac{|z|^2}{2}} d\mu(z) \leq e^{\frac{1}{2}} B_\varphi(|u|^2)(w),$$

for all $w \in \mathbb{C}$.

Proof. For each $z \in D(w, 1)$, we see that

$$|k_w(z)|^2 = e^{\operatorname{Re}\langle z, w \rangle - \frac{|w|^2}{2}} \geq e^{-\frac{1}{2} + \frac{|z|^2}{2}}.$$

Hence we get

$$e^{-\frac{1}{2}} \int_{D(w,1)} e^{\frac{|z|^2}{2}} d\mu(z) \leq \int_{D(w,1)} |k_w(z)|^2 d\mu(z) \leq \int_{\mathbb{C}} |k_w(z)|^2 d\mu(z).$$

By the definitions of the measure μ and the integral operator $B_\varphi(|u|^2)$, we see that

$$\int_{\mathbb{C}} |k_w(z)|^2 d\mu(z) = \frac{1}{2\pi} \int_{\mathbb{C}} |u(z)|^2 |k_w(\varphi(z))|^2 e^{-\frac{|z|^2}{2}} dA(z) = B_\varphi(|u|^2)(w).$$

So we obtain the desired inequality. $\square$

Proof of Theorem 1. First, we suppose that uC_φ is bounded on $\mathcal{F}^2$. Thus we have

$$\|uC_\varphi k_w\|_{\mathcal{F}^2}^2 \leq C \|k_w\|_{\mathcal{F}^2}^2 = C,$$

for some constant $C > 0$ and all $w \in \mathbb{C}$. The formula for the normalized kernel function k_w shows that $\|uC_\varphi k_w\|_{\mathcal{F}^2}^2 = B_\varphi(|u|^2)(w)$, and so $B_\varphi(|u|^2)(w) \leq C$ for all $w \in \mathbb{C}$. This implies that $B_\varphi(|u|^2) \in L^\infty(\mathbb{C})$.

Next, we prove the other direction. By the definition of the measure μ , we obtain

$$\|uC_\varphi f\|_{\mathcal{F}^2}^2 = \frac{1}{2\pi} \int_{\mathbb{C}} |u(z)|^2 |f(\varphi(z))|^2 e^{-\frac{|z|^2}{2}} dA(z) = \int_{\mathbb{C}} |f(z)|^2 d\mu(z).$$

It follows from Lemma 1 that

$$\begin{aligned} \|uC_\varphi f\|_{\mathcal{F}^2}^2 &= \int_{\mathbb{C}} |f(z)|^2 d\mu(z) \\ &\leq C \int_{\mathbb{C}} e^{\frac{|z|^2}{2}} d\mu(z) \int_{\mathbb{C}} \chi_{D(z,1)}(w) |f(w)|^2 e^{-\frac{|w|^2}{2}} dA(w), \end{aligned}$$

where $\chi_{D(z,1)}(w)$ is the characteristic function of the set $D(z, 1)$. Since $\chi_{D(z,1)}(w) = \chi_{D(w,1)}(z)$, Lemma 2 and Fubini's theorem show that

$$\begin{aligned} \|uC_\varphi f\|_{\mathcal{F}^2}^2 &\leq C \int_{\mathbb{C}} |f(w)|^2 e^{-\frac{|w|^2}{2}} dA(w) \int_{\mathbb{C}} \chi_{D(w,1)}(z) e^{\frac{|z|^2}{2}} d\mu(z) \\ &\leq C e^{\frac{1}{2}} \int_{\mathbb{C}} |f(w)|^2 e^{-\frac{|w|^2}{2}} B_\varphi(|u|^2)(w) dA(w) \\ &\leq 2\pi C e^{\frac{1}{2}} \sup_{w \in \mathbb{C}} B_\varphi(|u|^2)(w) \|f\|_{\mathcal{F}^2}^2. \end{aligned}$$

This implies that uC_φ is bounded on $\mathcal{F}^2$. $\square$

For an entire function $f(z) = \sum_{k=0}^{\infty} a_k z^k$, define $R_n f(z) = \sum_{k=n}^{\infty} a_k z^k$, acting on $\mathcal{F}^2$. Note that $\|R_n\| = 1$ on $\mathcal{F}^2$. By [9, 5.1.Proposition], if uC_{φ} is bounded on $\mathcal{F}^2$, then

$$(2) \quad \|uC_{\varphi}\|_e = \lim_{n \rightarrow \infty} \|uC_{\varphi} R_n\|.$$

Moreover, the following lemma holds.

Lemma 3. *For each entire function f and $w \in \mathbb{C}$,*

$$|R_n f(w)| \leq \|f\|_{\mathcal{F}^2} \left\{ \sum_{k=n}^{\infty} \frac{|w|^{2k}}{2^k k!} \right\}^{\frac{1}{2}}.$$

Proof. The orthogonality of w^k shows that $\|R_n K_w\|_{\mathcal{F}^2}^2 = \sum_{k=n}^{\infty} \frac{|w|^{2k}}{2^k k!}$. Since R_n is self-adjoint, we have

$$\begin{aligned} |R_n f(w)| &= |\langle R_n f, K_w \rangle_{\mathcal{F}^2}| = |\langle f, R_n K_w \rangle_{\mathcal{F}^2}| \\ &\leq \|f\|_{\mathcal{F}^2} \|R_n K_w\|_{\mathcal{F}^2} = \|f\|_{\mathcal{F}^2} \left\{ \sum_{k=n}^{\infty} \frac{|w|^{2k}}{2^k k!} \right\}^{\frac{1}{2}}. \end{aligned}$$

□

Proof of Theorem 2. First we prove the lower estimate. Take a compact operator $\mathcal{K}$ on $\mathcal{F}^2$. Since $k_w \rightarrow 0$ weakly as $|w| \rightarrow \infty$, we see $\|\mathcal{K}k_w\|_{\mathcal{F}^2} \rightarrow 0$ as $|w| \rightarrow \infty$. Therefore,

$$\begin{aligned} \|uC_{\varphi} - \mathcal{K}\| &\geq \limsup_{|w| \rightarrow \infty} \|(uC_{\varphi} - \mathcal{K})k_w\|_{\mathcal{F}^2} \\ &\geq \limsup_{|w| \rightarrow \infty} (\|uC_{\varphi} k_w\|_{\mathcal{F}^2} - \|\mathcal{K}k_w\|_{\mathcal{F}^2}) \\ &= \limsup_{|w| \rightarrow \infty} \|uC_{\varphi} k_w\|_{\mathcal{F}^2}. \end{aligned}$$

Thus

$$\|uC_{\varphi}\|_e^2 \geq \|uC_{\varphi} - \mathcal{K}\|^2 \geq \limsup_{|w| \rightarrow \infty} \|uC_{\varphi} k_w\|_{\mathcal{F}^2}^2 = \limsup_{|w| \rightarrow \infty} B_{\varphi}(|u|^2)(w).$$

Next, we prove the upper estimate. Take $f \in \mathcal{F}^2$ with $\|f\|_{\mathcal{F}^2} \leq 1$. Using the same argument as in the proof of Theorem 1, we get

$$\begin{aligned} \|uC_{\varphi} R_n f\|_{\mathcal{F}^2}^2 &= \frac{1}{2\pi} \int_{\mathbb{C}} |u(z)|^2 |(R_n f)(\varphi(z))|^2 e^{-\frac{|z|^2}{2}} dA(z) = \int_{\mathbb{C}} |R_n f(z)|^2 d\mu(z) \\ &\leq C \int_{\mathbb{C}} e^{-\frac{|z|^2}{2}} d\mu(z) \int_{\mathbb{C}} \chi_{D(z,1)}(w) |R_n f(w)|^2 e^{-\frac{|w|^2}{2}} dA(w) \\ &= C \int_{\mathbb{C}} |R_n f(w)|^2 e^{-\frac{|w|^2}{2}} dA(w) \int_{\mathbb{C}} \chi_{D(w,1)}(z) e^{-\frac{|z|^2}{2}} d\mu(z) \\ &\leq e^{\frac{1}{2}} C \int_{\mathbb{C}} |R_n f(w)|^2 e^{-\frac{|w|^2}{2}} B_{\varphi}(|u|^2)(w) dA(w) \\ (3) \quad &= e^{\frac{1}{2}} C \left(\int_{\{|w| \geq r\}} + \int_{\{|w| < r\}} \right) |R_n f(w)|^2 e^{-\frac{|w|^2}{2}} B_{\varphi}(|u|^2)(w) dA(w), \end{aligned}$$

for fixed $r > 0$. Since $\|R_n f\|_{\mathcal{F}^2} \leq 1$, we see that

$$\int_{\{|w| \geq r\}} |R_n f(w)|^2 e^{-\frac{|w|^2}{2}} B_\varphi(|u|^2)(w) dA(w) \leq 2\pi \sup_{|w| \geq r} B_\varphi(|u|^2)(w).$$

By Lemma 3, we have

$$\begin{aligned} & \int_{\{|w| < r\}} |R_n f(w)|^2 e^{-\frac{|w|^2}{2}} B_\varphi(|u|^2)(w) dA(w) \\ & \leq \sup_{w \in \mathbb{C}} B_\varphi(|u|^2)(w) \int_{\mathbb{C}} e^{-\frac{|w|^2}{2}} dA(w) \sum_{k=n}^{\infty} \frac{r^{2k}}{2^k k!}. \end{aligned}$$

Since $\sum_{k=n}^{\infty} \frac{r^{2k}}{2^k k!} \rightarrow 0$ as $n \rightarrow \infty$ and, by Theorem 1, $B_\varphi(|u|^2)$ is bounded, so

$$\sup_{w \in \mathbb{C}} B_\varphi(|u|^2)(w) \int_{\mathbb{C}} e^{-\frac{|w|^2}{2}} dA(w) \sum_{k=n}^{\infty} \frac{r^{2k}}{2^k k!} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Hence, taking the supremum over entire functions f with $\|f\|_{\mathcal{F}^2} \leq 1$, and letting $n \rightarrow \infty$ in (3), we obtain

$$\lim_{n \rightarrow \infty} \sup_{\|f\|_{\mathcal{F}^2} \leq 1} \|uC_\varphi R_n f\|_{\mathcal{F}^2}^2 \leq 2\pi e^{\frac{1}{2}} C \sup_{|w| \geq r} B_\varphi(|u|^2)(w).$$

Combining this with (2), we get $\|uC_\varphi\|_e^2 \leq 2\pi e^{\frac{1}{2}} C \sup_{|w| \geq r} B_\varphi(|u|^2)(w)$. Letting $r \rightarrow \infty$, we have

$$\|uC_\varphi\|_e^2 \leq 2\pi e^{\frac{1}{2}} C \lim_{r \rightarrow \infty} \sup_{|w| \geq r} B_\varphi(|u|^2)(w) = 2\pi e^{\frac{1}{2}} C \limsup_{|w| \rightarrow \infty} B_\varphi(|u|^2)(w).$$

This completes the proof. $\square$

Remark. The N -dimensional Fock space $\mathcal{F}^2(\mathbb{C}^N)$ is defined by

$$\mathcal{F}^2(\mathbb{C}^N) = \left\{ f \text{ is entire} : \frac{1}{(2\pi)^N} \int_{\mathbb{C}^N} |f(z)|^2 e^{-\frac{|z|^2}{2}} dm_{2N}(z) < \infty \right\},$$

where m_{2N} denotes the Lebesgue measure on $\mathbb{C}^N$. All arguments used in this paper can apply to the N -dimensional case. Thus, Theorem 1 and Theorem 2 are still valid in $\mathcal{F}^2(\mathbb{C}^N)$.

ACKNOWLEDGEMENT

The author would like to thank the referee for pointing out mistakes.

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