THE DEGREE OF THE BICANONICAL MAP OF A SURFACE WITH $p_g = 0$

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ABSTRACT. In this note it is shown that, given a smooth minimal complex surface of general type S with $p_g(S) = 0$, $K_S^2 = 3$, for which the bicanonical map φ_{2K} is a morphism, the degree of φ_{2K} is not 3. This completes our earlier results, showing that if S is a minimal surface of general type with $p_g = 0$, $K^2 \geq 3$ such that $|2K_S|$ is free, then the bicanonical map of S can have degree 1, 2 or 4.

1. INTRODUCTION

Complex surfaces of general type with $p_g = 0$ continue to be intriguing. Continuing our study of their bicanonical map, in this note we prove the following.

Theorem 1.1. *Let S be a smooth minimal complex surface of general type with $p_g(S) = 0$ and $K_S^2 = 3$ such that the bicanonical system $|2K_S|$ is base point free. Then the degree of the bicanonical map of S is different from 3.*

Theorem 1.1, together with previous results ([ML], [MP1], [MP2]) gives the following general statement on the degree of the bicanonical map of a surface of general type with $p_g = 0$ and base point free bicanonical system.

Theorem 1.2. *Let S be a smooth minimal complex surface of general type with $p_g(S) = 0$, and let $\varphi: S \rightarrow \mathbb{P}^{K^2}$ be the bicanonical map. Then:*

- (i) *if $K_S^2 = 9$, then φ is birational;*
- (ii) *if $K_S^2 = 7, 8$, then $\deg \varphi \leq 2$;*
- (iii) *if $3 \leq K_S^2 \leq 6$ and $|2K_S|$ is base point free, then $\deg \varphi$ is equal to either 1, 2 or 4.*

The proof of Theorem 1.1 is based on some properties of nonnormal rational quartic surfaces of $\mathbb{P}^3$, which are studied in the next section.

Notation and conventions. We work over the complex numbers; all varieties are assumed to be compact and algebraic.

We do not distinguish between line bundles and divisors on a smooth variety. Linear equivalence of divisors is denoted by $\equiv$ and numerical equivalence by $\sim$. The remaining notation is standard in algebraic geometry.

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2. NONNORMAL RATIONAL QUARTICS OF $\mathbb{P}^3$

In this section we show the following.

Theorem 2.1. *Let $\Sigma \subset \mathbb{P}^3$ be a rational quartic surface such that Σ is not normal, i.e. such that the singular locus of Σ has dimension 1. Let $\Upsilon \rightarrow \Sigma$ be the minimal desingularization and let $|H|$ be the pullback on Υ of the linear system of planes of $\mathbb{P}^3$. If the linear system $|H|$ is complete, then:*

- (i) *the genus $g(H)$ of H is equal to 2;*
- (ii) *$K_\Upsilon^2 = 0$ and the linear system $|K_\Upsilon + H|$ is a free pencil of rational curves.*

Proof. We start by noting that, since the linear system $|H|$ is the pullback on Υ of the linear system of planes of $\mathbb{P}^3$ and Σ has degree 4, $H^2 = 4$. The general curve H of $|H|$ is smooth and irreducible by Bertini's theorem, and it is mapped birationally onto a singular plane quartic, thus we have $g(H) = 1 + (K_\Upsilon + H)H/2 \leq 2$. On the other hand, by the regularity of Υ the restriction of $|H|$ to a general H is a complete system, hence Riemann–Roch on H gives $g(H) \geq 2$. This proves (i).

We divide the proof of assertion (ii) into steps. Consider the linear system $|D| := |K_\Upsilon + H|$. By (i), we have $K_\Upsilon H = -2$ and thus $DH = 2$, $D^2 = K_\Upsilon^2$. Using the adjunction sequence for a general H one sees that $h^0(D) = 2$, namely $|D|$ is a pencil.

Write $|D| = F + |M|$, where F is the fixed part of $|D|$ and $|M|$ is the moving part.

Step 1. *D is nef. In particular, we have $D^2 \geq 0$.* Since $q(\Upsilon) = 0$, the restriction of $|D|$ to the general curve of $|H|$ is the complete canonical system. So for any irreducible component θ of F we have $\theta H = 0$ and, by the index theorem, $\theta^2 < 0$.

Let θ be an irreducible curve such that $\theta D < 0$. Since D is effective, θ is a component of F . Hence $\theta H = 0$, $\theta K_\Upsilon < 0$, $\theta^2 < 0$, namely θ is a -1 -curve contracted by $|H|$, against the assumption that $\Upsilon \rightarrow \Sigma$ is the minimal desingularization.

Step 2. *One has $D^2 \leq 1$. If $D^2 = 1$, then $H = -2K_\Upsilon$.* The inequality $D^2 \leq 1$ follows by the index theorem applied to D and H . If $D^2 = 1$, the index theorem gives also $H \sim 2D$, i.e. $H \sim -2K_\Upsilon$. Since Υ is rational, we actually have $H \equiv -2K_\Upsilon$.

Step 3. *The case $D^2 = 1$ does not occur.* By Step 2, if $D^2 = 1$, then $-K_\Upsilon$ is nef and $K_\Upsilon^2 = 1$. Riemann–Roch gives $h^0(-K_\Upsilon) = 2$. Write $|-K_\Upsilon| = \Delta + |N|$, where Δ is the fixed part and $|N|$ is the moving part. We have

$$1 = K_\Upsilon^2 = -K_\Upsilon \Delta - K_\Upsilon N \geq -K_\Upsilon N = N^2 + \Delta N \geq 0.$$

Assume that $\Delta \neq 0$. Then, since a nef and big divisor is 1-connected, we have $\Delta N > 0$. Hence in this case all the previous inequalities are equalities, and we have

$$\Delta N = 1, \quad N^2 = 0, \quad K_\Upsilon \Delta = 0.$$

It follows that $\Delta^2 = -K_\Upsilon \Delta - N \Delta = -1$, contradicting the fact that $\Delta^2 + K_\Upsilon \Delta$ is even by the adjunction formula. So we conclude that $\Delta = 0$. Since $K_\Upsilon^2 = 1$, it follows that the general curve of $|-K_\Upsilon|$ is smooth and irreducible of genus 1. Since $(-K_\Upsilon)H = 2$, the restriction of $|H|$ to the general curve of $|-K_\Upsilon|$ is not birational, contradicting the assumption that $|H|$ is the pullback of a very ample system via a birational morphism. Hence the case $D^2 = 1$ does not occur.

Step 4. $D^2 = 0$ and $|D|$ is a pencil of rational curves. We have $D^2 = K_Y^2 = 0$ by the previous steps. Since D and M are nef, we have

$$0 = D^2 \geq DM = M^2 + MF \geq MF \geq 0.$$

It follows that $M^2 = MF = F^2 = 0$. Hence, by Zariski's lemma (see [BPV], Lemma (8.2), p. 90) there is a rational number $\alpha \geq 0$ such that $F \equiv \alpha M$ as $\mathbb{Q}$ -divisors. One has: $2 = -K_Y D = -K_Y M(1 + \alpha)$. Since $-K_Y M = \frac{2}{1+\alpha} > 0$, by the adjunction formula we must have $-K_Y M = 2$, $\alpha = 0$. Hence $F = 0$, and $|D|$ is a free pencil of rational curves. $\square$

3. THE PROOFS OF THEOREM 1.2 AND THEOREM 1.1

Proof of Theorem 1.1. Since $K_S^2 = 3$, $h^0(S, 2K_S) = K_S^2 + \chi(\mathcal{O}_S) = 4$. Let $\varphi: S \rightarrow \mathbb{P}^3$ be the bicanonical map and denote by $\Sigma \subset \mathbb{P}^3$ the image of φ , which is a surface by [Xi].

We argue by contradiction. So from now on we assume that $\deg \varphi = 3$. Since $|2K_S|$ is free, we have

$$4K_S^2 = 12 = \deg \varphi \deg \Sigma = 3 \deg \Sigma,$$

namely Σ is a quartic surface.

Let $Y \rightarrow \Sigma$ be the minimal desingularization. In principle, Y is either a K3 surface or a ruled surface. Since Y is birationally dominated by S and $q(S) = 0$ by the inequality $\chi(\mathcal{O}_S) > 0$ for surfaces of general type, we have $p_g(Y) = q(Y) = 0$, and so Y is rational. We note also that, since Σ is the image of S by a complete linear system, the linear system $|H|$, as in the assumptions of Theorem 2.1, is complete.

We wish to show that Σ is not normal. Assume by contradiction that Σ is normal. Being a quartic hypersurface, Σ is a Gorenstein surface with $K_\Sigma = 0$ and $p_g(\Sigma) = 1$, namely there exists a nonzero 2-form w on Σ which is regular outside the singular locus of Σ , which is a finite set. The pullback w' of w to S is a nonzero 2-form which is regular on the complement of the union of the -2 -curves of S and of a finite set of points. It is well known (cf. [St], Lemmas 1.8 and 1.11) that such a form w' is actually regular on S , but this contradicts the assumption $p_g(S) = 0$. So Σ is not normal, and by Theorem 2.1 we have $g(H) = 2$ and $K_Y H = -2$.

Consider the linear system $|K_Y + H|$. By Theorem 2.1, this system defines a fibration $f: Y \rightarrow \mathbb{P}^1$ whose general fibre is a rational curve. Since $K_Y^2 = 0$ by Theorem 2.1, the fibration f is not relatively minimal. Let θ be an irreducible -1 -curve contained in a fibre of f . We have $\theta(K_Y + H) = 0$, namely $\theta H = 1$. So the image of θ in Σ is a line L . If L' is the pullback of L on S , then we have $3 = L' \psi^* H = L'(2K_S)$, a contradiction. $\square$

Proof of Theorem 1.2. Statement (i) and (ii) are the main result of [MP1].

So we consider the case $3 \leq K_S^2 \leq 6$. We have $\deg \varphi \leq 4$, by [ML]. Since $|2K_S|$ is free by assumption, we have

$$4K_S^2 = \deg \varphi \deg \Sigma.$$

Hence we have to exclude the cases $K_S^2 = 6$, $\deg \varphi = 3$ and $K_S^2 = \deg \varphi = 3$. The former does not occur by [MP2, Thm. 1.1], and the latter is excluded by Theorem 1.1. $\square$

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REFERENCES

- [BPV] W. Barth, C. Peters, A. Van de Ven, *Compact complex surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, **3**. Folge, Band **4**, Springer 1984. MR0749574 (86c:32026)
- [ML] M. Mendes Lopes, *The degree of the generators of the canonical ring of surfaces of general type with $p_g = 0$* , Arch. Math., **69** (1997), 435–440. MR1473098 (98j:14050)
- [MP1] M. Mendes Lopes, R. Pardini, *The bicanonical map of surfaces with $p_g = 0$ and $K^2 \geq 7$* , Bull. London Math. Soc., **33** (2001), 265–274. MR1817764 (2002a:14042)
- [MP2] M. Mendes Lopes, R. Pardini, *The classification of surfaces with $p_g = 0$, $K^2 = 6$ and non birational bicanonical map*, Math. Annalen, vol. 329, Number 3 (July 2004), 535–552. MR2127989 (2005m:14067)
- [St] J. H. M. Steenbrink, *Mixed Hodge structure on the vanishing cohomology*, in “Real and Complex Singularities, Oslo 1976” (P. Holm ed.), Sijthoff & Noordhoff, The Netherlands (1977), 525–564. MR0485870 (58:5670)
- [Xi] G. Xiao, *Finitude de l’application bicanonique des surfaces de type général*, Bull. Soc. Math. France, **113** (1985), 23–51. MR0807825 (87a:14035)

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