EQUIVARIANT CRYSTALLINE COHOMOLOGY AND BASE CHANGE

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ABSTRACT. Given a perfect field k of characteristic $p > 0$, a smooth proper k -scheme Y , a crystal E on Y relative to $W(k)$ and a finite group G acting on Y and E , we show that, viewed as a virtual $k[G]$ -module, the reduction modulo p of the crystalline cohomology of E is the de Rham cohomology of E modulo p . On the way we prove a base change theorem for the virtual G -representations associated with G -equivariant objects in the derived category of $W(k)$ -modules.

1. THE THEOREM

Let k be a perfect field of characteristic $p > 0$, let W denote its ring of Witt vectors, let $K = \text{Quot}(W)$. Let Y be a proper and smooth k -scheme and suppose that the finite group G acts (from the right) on Y . Let E be a locally free, finitely generated crystal of $\mathcal{O}_{Y/W}$ -modules and suppose that for each $g \in G$ we are given an isomorphism of crystals $\tau_g : E \rightarrow g^*E$ (where g^*E denotes the pull-back of E via $g : Y \rightarrow Y$) such that $g_2^*(\tau_{g_1}) \circ \tau_{g_2} = \tau_{g_2g_1}$ (equality as maps $E \rightarrow (g_2g_1)^*E = g_2^*g_1^*E$) for any two $g_1, g_2 \in G$. For $s \in \mathbb{Z}$ let $H_{crys}^s(Y/W, E)$ denote the s -th crystalline cohomology group (relative to $\text{Spf}(W)$) of the crystal E , a finitely generated W -module which is zero if $s \notin [0, 2\dim(Y)]$ (see [1]). On the other hand, the reduction modulo p of the crystal E is equivalent with a locally free $\mathcal{O}_Y$ -module E_k with connection $E_k \rightarrow \Omega_Y^1 \otimes_{\mathcal{O}_Y} E_k$; here Ω_Y^1 denotes the $\mathcal{O}_Y$ -module of differentials of Y/k . Let $\Omega_Y^\bullet \otimes E_k$ denote the corresponding de Rham complex. The cohomology group $H^s(Y, \Omega_Y^\bullet \otimes E_k)$ is a finite-dimensional k -vector space which is zero if $s \notin [0, 2\dim(Y)]$. The isomorphisms τ_g for $g \in G$ provide each $H_{crys}^s(Y/W, E)$, each $H^s(Y, \Omega_Y^\bullet \otimes E_k)$ and each $H^s(Y, \Omega_Y^t \otimes E_k)$ for $t \geq 0$ with an action of G (from the left). By definition, the reduction modulo p of the $K[G]$ -module $H_{crys}^s(Y/W, E) \otimes_W K$ is the $k[G]$ -module obtained by reducing modulo p the G -stable W -lattice $H_{crys}^s(Y/W, E)/(\text{torsion})$ in $H_{crys}^s(Y/W, E) \otimes_W K$.

Theorem 1.1. *For any j , the following three virtual $k[G]$ -modules are the same:*

- (i) *the reduction modulo p of the virtual $K[G]$ -module*

$$\sum_s (-1)^s H_{crys}^s(Y/W, E) \otimes_W K;$$

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- (ii) $\sum_s (-1)^s H^s(Y, \Omega_Y^\bullet \otimes E_k)$;
- (iii) $\sum_{s,t} (-1)^{s+t} H^s(Y, \Omega_Y^t \otimes E_k)$.

An obvious variant of Theorem 1.1 holds in logarithmic crystalline cohomology, for crystals E on the logarithmic crystalline site of Y/W with respect to a log structure defined by a normal crossings divisor on Y . Similarly, the proof which we give below also shows the analog of Theorem 1.1 for the ℓ -adic cohomology ($\ell \neq p$) of constructible ℓ -adic sheaves on Y , even if Y/k is not proper. Of course, the result in the ℓ -adic case (even for nonproper Y/k) is well known; it has been used for investigating the reduction modulo ℓ of the Deligne-Lusztig characters of groups $G = \mathbb{G}(\mathbb{F})$, where $\mathbb{G}$ is a reductive group over a finite field $\mathbb{F}$ of characteristic p . In [3] we use the variant of Theorem 1.1 in logarithmic crystalline cohomology to show that these Deligne-Lusztig characters, usually defined via ℓ -adic cohomology of certain $\mathbb{F}$ -varieties which are nonproper in general, can also be expressed through the log crystalline cohomology of suitable log crystals on suitable proper and smooth $\mathbb{F}$ -varieties with a normal crossings divisor. Unfortunately, the (more geometric) proof of the ℓ -adic analog of Theorem 1.1 (due to Deligne and Lusztig; see for example [2], Lemma 12.4 and A3.15) breaks down for crystalline cohomology. On the other hand, our proof of Theorem 1.1 contains a result (Theorem 2.1) on G -actions on strictly perfect complexes in the derived category which should be of independent interest.

2. THE PROOF

Proof of Theorem 1.1. (ii)=(iii) is clear. By [1] we know that the total crystalline cohomology $\mathbb{R}\Gamma_{crys}(Y/W, E)$, as an object in the derived category $D(W)$ of the category of W -modules, is represented by a complex of W -modules of finite tor-dimension and with finitely generated cohomology; by functoriality, G acts on $\mathbb{R}\Gamma_{crys}(Y/W, E)$. Also from [1] we know that the total crystalline cohomology commutes with base change, i.e. that $\mathbb{R}\Gamma_{crys}(Y/W, E) \otimes_W^{\mathbb{L}} k$ is the total crystalline cohomology of the reduction modulo p of E (as a crystal relative to $\text{Spec}(k)$). But the latter is known (see [1], Corollary 7.4) to be the de Rham cohomology of E_k ; i.e., its s -th cohomology group is $H^s(Y, \Omega_Y^\bullet \otimes E_k)$. Hence (i)=(ii) follows from Theorem 2.1 below. $\square$

Let A be a complete discrete valuation ring with perfect residue field k of characteristic $p > 0$ and fraction field K of characteristic 0. Let $L^\bullet$ be a complex of A -modules of finite tor-dimension and with finitely generated cohomology; by [1], Lemma 7.15, this is equivalent with saying that $L^\bullet$ is quasi-isomorphic to a strictly perfect complex, i.e. a bounded complex of finitely generated projective A -modules. Suppose the finite group G acts on $L^\bullet$ when $L^\bullet$ is viewed as an object in the derived category $D(A)$ of the category of A -modules. Then each cohomology group $H^i(L^\bullet \otimes_A K) = H^i(L^\bullet) \otimes_A K$ (resp. each cohomology group $H^i(L^\bullet \otimes_A^{\mathbb{L}} k)$) becomes a representation of G on a finite-dimensional K -vector space (resp. k -vector space).

Theorem 2.1. *The virtual $k[G]$ -module $\sum_i (-1)^i H^i(L^\bullet \otimes_A^{\mathbb{L}} k)$ is the reduction (modulo the maximal ideal of A) of the virtual $K[G]$ -module $\sum_i (-1)^i H^i(L^\bullet) \otimes_A K$. Equivalently, the restriction of the character of $\sum_i (-1)^i H^i(L^\bullet) \otimes_A K$ to the subset of p -regular elements of G is the Brauer character of $\sum_i (-1)^i H^i(L^\bullet \otimes_A^{\mathbb{L}} k)$.*

We say that the automorphism γ of the finitely generated A -module M is *prime* to p if and only if the following holds. For any finite extension $A' \supset A$ with a discrete

valuation ring A' and for any two $\gamma \otimes_A A'$ -stable submodules N, N' of $M \otimes_A A'$ with $N' \subset N$ and such that N/N' is a cyclic A' -module, the endomorphism which $\gamma \otimes_A A'$ induces on N/N' is of finite order prime to p .

Lemma 2.2. *Let γ be an automorphism of the finitely generated A -module M .*

(a) *If M is free, then γ is prime to p if and only if the roots of the characteristic polynomial of γ are roots of unity of order prime to p . In particular, $\gamma|_N : N \rightarrow N$ is prime to p for each submodule N of M with $\gamma(N) = N$.*

(b) *Let $M_1 \subset M$ be a submodule with $\gamma(M_1) = M_1$ and such that $M_2 = M/M_1$ is free. Let γ_1 , resp. γ_2 , be the induced automorphism of M_1 , resp. of M_2 . If γ_1 and γ_2 are prime to p , then γ is prime to p .*

Proof. Statement (a) is clear. (b) Let $N' \subset N \subset M \otimes_A A'$ be as in the definition. If $N \subset M_1 \otimes_A A'$ the hypothesis on γ_1 applies. Otherwise, since $M_2 \otimes_A A'$ is free over A' and N/N' is cyclic, N/N' maps injectively to $M_2 \otimes_A A'$ and the hypothesis on γ_2 applies. $\square$

Proof of Theorem 2.1. The problem is of course that the $H^i(L^\bullet)$ may have torsion, i.e., $H^i(L^\bullet) \otimes_A k \neq H^i(L^\bullet \otimes_A^{\mathbb{L}} k)$ in general. Similarly, the task would be easy if we knew that there is a strictly perfect complex $K^\bullet$ quasi-isomorphic to $L^\bullet$ such that the action of G on $L^\bullet$ in $D(A)$ is given by the action of G on $K^\bullet$ by true morphisms of complexes (not just by morphisms in $D(A)$). We introduce some notation. For an automorphism $\gamma : L^\bullet \rightarrow L^\bullet$ in $D(A)$ let $\epsilon_1^i, \dots, \epsilon_{n(i)}^i$ (with $n(i) = \dim_k H^i(L^\bullet \otimes_A^{\mathbb{L}} k)$) denote the roots of the characteristic polynomial of γ acting on $H^i(L^\bullet \otimes_A^{\mathbb{L}} k)$ and let $\tilde{\epsilon}_1^i, \dots, \tilde{\epsilon}_{n(i)}^i$ denote their Teichmüller liftings. On the other hand, let $\xi_1^i, \dots, \xi_{n'(i)}^i$ (with $n'(i) = \dim_K H^i(L^\bullet) \otimes_A K$) denote the roots of the characteristic polynomial of γ acting on $H^i(L^\bullet) \otimes_A K$. Then let

$$Br(\gamma, H^\heartsuit(L^\bullet \otimes_A^{\mathbb{L}} k)) = \sum_i (-1)^i \sum_{j=1}^{n(i)} \tilde{\epsilon}_j^i,$$

$$Tr(\gamma, H^\heartsuit(L^\bullet) \otimes_A K) = \sum_i (-1)^i \sum_{j=1}^{n'(i)} \xi_j^i.$$

What we must show is that for all p -regular elements $g \in G$ (those whose order in G is not divisible by p), if $\gamma : L^\bullet \rightarrow L^\bullet$ denotes the corresponding automorphism of $L^\bullet$ in $D(A)$, then

$$Br(\gamma, H^\heartsuit(L^\bullet \otimes_A^{\mathbb{L}} k)) = Tr(\gamma, H^\heartsuit(L^\bullet) \otimes_A K).$$

Clearly it is enough to show the following statement. For any strictly perfect complex $L^\bullet$ of A -modules (not necessarily endowed with a G -action in $D(A)$) and for any automorphism $\gamma : L^\bullet \rightarrow L^\bullet$ in $D(A)$ which on the cohomology modules induces automorphisms prime to p we have

$$Br(\gamma, H^\heartsuit(L^\bullet \otimes_A^{\mathbb{L}} k)) = Tr(\gamma, H^\heartsuit(L^\bullet) \otimes_A K).$$

We use induction on the minimal $m \in \mathbb{Z}_{\geq 0}$ with the following property: after a suitable degree shift we have $L^i = 0$ for all $i \notin [0, m]$. For $m = 0$ the statement is clear from Lemma 2.2(a). Now let $m \geq 1$; shifting degrees we may assume $L^i = 0$

for all $i \notin [0, m]$. Let $d^m : L^{m-1} \rightarrow L^m$ denote the differential. Choose a sub- k -vector space N_k^{m-1} of $L^{m-1} \otimes k$ which under $d^m \otimes k$ maps isomorphically to the kernel of

$$L^m \otimes k \rightarrow H^m(L^\bullet \otimes k) = H^m(L^\bullet) \otimes k.$$

Then $N_k^{m-1} = N^{m-1} \otimes k$ for a direct summand N^{m-1} of L^{m-1} . By construction, N^{m-1} maps isomorphically to its image N^m in L^m . Thus, setting $N^i = 0$ if $i \notin \{m-1, m\}$, the subcomplex $N^\bullet$ of $L^\bullet$ is acyclic. Dividing it out we may therefore assume $L^m \otimes k = H^m(L^\bullet \otimes k)$. Since the functor $K^-(\text{proj-}A) \rightarrow D(A)$ from the homotopy category of complexes of projective A -modules bounded above to $D(A)$ is fully faithful, the action of γ on $L^\bullet$ in $D(A)$ is in fact represented by a true morphism of complexes $\gamma^\bullet : L^\bullet \rightarrow L^\bullet$. Base changing to a finite extension of A by a discrete valuation ring (this does not affect the numbers Br and Tr) we may suppose that the characteristic polynomial of $\gamma^m : L^m \rightarrow L^m$ splits in A (we remark that γ^m is bijective: this follows from $L^m \otimes k = H^m(L^\bullet \otimes k)$ and the fact that γ acts bijectively on $H^m(L^\bullet \otimes k)$). We therefore find a γ^m -stable filtration

$$(1) \quad (0) = F^0 \subset F^1 \subset \dots \subset F^s = L^m \quad (s = \text{rk}(L^m))$$

such that $G^e = F^e/F^{e-1}$ is free of rank one, for any $1 \leq e \leq s$. The cyclic A -module

$$\frac{F^e}{(F^e \cap \text{im}(d^m)) + F^{e-1}}$$

is a γ^m -stable subquotient of $H^m(L^\bullet)$ (it is nonzero because of $L^m \otimes k = H^m(L^\bullet \otimes k)$); hence γ^m acts on it by multiplication with a root of unity of order prime to p . Let $\xi_e \in A^\times$ denote its Teichmüller lifting. Choose $\ell_e \in F^e$ which represents a basis element of G^e ; then $\ell_1, \dots, \ell_s$ is a basis of L^m . Modulo F^{e-1} the class of $\xi_e \ell_e - \gamma^m(\ell_e) \in F^e$ lies in $\text{im}(d^m)$. Choose a $t_e \in L^{m-1}$ with

$$d^m(t_e) = \xi_e \ell_e - \gamma^m(\ell_e) \text{ modulo } F^{e-1}.$$

Let $t : L^m \rightarrow L^{m-1}$ denote the A -linear map which sends ℓ_e to t_e , for each $1 \leq e \leq s$. Using t we see that we may modify $\gamma^\bullet$ within its homotopy class to achieve that the filtration (1) is still γ^m -stable and such that γ^m acts on each G^e by multiplication with a root of unity of prime-to- p order in $A^\times$. Therefore we may assume that $\gamma^m : L^m \rightarrow L^m$ is prime to p . Let $L_1^m = L^m$ and $L_1^i = 0$ for $i \neq m$. Then $L_1^\bullet$ is a $\gamma^\bullet$ -stable subcomplex of $L^\bullet$ and since $Br(\gamma)$ and $Tr(\gamma)$ are additive in exact $\gamma^\bullet$ -equivariant sequences of complexes it suffices to show $Br(\gamma) = Tr(\gamma)$ for the complexes $L_1^\bullet$ and $L^\bullet/L_1^\bullet$. Since these complexes are shorter than $L^\bullet$ this follows from the induction hypothesis. Indeed, the prime-to- p hypothesis is clearly satisfied for $L_1^\bullet$, so it remains to show that $\gamma^\bullet$ induces automorphisms prime to p on the cohomology modules of $L^\bullet/L_1^\bullet$. In degrees smaller than $m-1$ this is clear from the corresponding hypothesis on $L^\bullet$; only $H^{m-1}(L^\bullet/L_1^\bullet)$ is critical. But $H^{m-1}(L^\bullet)$ is a submodule of $H^{m-1}(L^\bullet/L_1^\bullet)$ and the quotient

$$Q = H^{m-1}(L^\bullet/L_1^\bullet)/H^{m-1}(L^\bullet)$$

maps isomorphically to a submodule of $L_1^m = L^m$. By Lemma 2.2(b) it suffices to show that $\gamma^\bullet$ induces automorphisms prime to p on $H^{m-1}(L^\bullet)$ and on Q . For $H^{m-1}(L^\bullet)$ this holds by hypothesis; for Q this follows from Lemma 2.2(a). $\square$

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