VARIETIES WITH A REDUCIBLE HYPERPLANE SECTION WHOSE TWO COMPONENTS ARE HYPERSURFACES

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ABSTRACT. We classify smooth complex projective varieties $X \subset \mathbb{P}^N$ of dimension $n \geq 2$ admitting a divisor of the form $A + B$ among their hyperplane sections, both A and B of codimension ≤ 1 in their respective linear spans. In this setting, one of the following holds: 1) X is either the Veronese surface in $\mathbb{P}^5$ or its general projection to $\mathbb{P}^4$, 2) $n \leq 3$ and $X \subset \mathbb{P}^{n+2}$ is contained in a quadric cone of rank 3 or 4, 3) $n = 2$ and $X \subset \mathbb{P}^3$.

INTRODUCTION

Let $X \subset \mathbb{P}^N$ be an irreducible, nondegenerate, smooth projective variety over $\mathbb{C}$ of dimension $n \geq 2$, embedded by a very ample divisor H on X .

In [3], Chandler, Howard and Sommese started to study pairs (X, H) as above for $n \geq 4$ when $|H|$ contains an element of type $A + B$, with A, B smooth irreducible divisors meeting transversally and of sectional genus zero. They obtained the following possibilities: 1) X is a rational scroll of $(n - 1)$ -planes embedded by its tautological line bundle, 2) X is the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$, 3) $X \cong \mathbb{P}(\mathcal{E})$, where $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)$ and H is the tautological line bundle or 4) X is a Mukai 4-fold. This result was slightly improved in [5] excluding case 4).

Moreover, for $n = 3$, the list of all possible numerical invariants concerning A, B and $A \cap B$, when both A and B are rational scrolls meeting along a smooth curve of positive genus, is provided in [2]. A detailed classification result when H is only an ample and spanned line bundle and $A \cap B$ is also a rational curve is given in [6].

The technics used in the quoted papers are typical of the adjunction theory, but the use of the adjunction mapping seems not to be a powerful tool in low dimensions. On the other hand, one of the advantages of the adjunction theory is that sometimes we can consider H only as an ample and spanned line bundle (see [3] and [6]).

In this paper our approach is different. Let H be a very ample divisor on X and consider the embedding $X \subset \mathbb{P}(V) = \mathbb{P}^N$ given by a very ample linear system $V \subseteq H^0(X, \mathcal{O}_X(H))$ (that is, we are not assuming $X \subset \mathbb{P}^N$ to be linearly normal). Suppose that $|V|$ contains a divisor of the form $A + B$, with A, B effective. We will

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study the case in which both A and B (embedded by V) have low codimension in their linear spans. Let us note that by this geometric assumption we can work in a more general setting than in [2], [3], [5] and [6], since we consider $n \geq 2$ and we do not impose any restriction on the divisors A, B . In particular, we do not require a priori that both components are either smooth or irreducible.

To be precise, we will concentrate on the cases in which one of the divisors, say A , is either a linear space (Section 2) or a hypersurface (Section 3). We get the main result of the paper if also B is either a linear space or a hypersurface: 1) X is either the Veronese surface in $\mathbb{P}^5$ or its general projection to $\mathbb{P}^4$, 2) $n \leq 3$ and $X \subset \mathbb{P}^{n+2}$ is contained in a quadric cone of rank 3 or 4, 3) $n = 2$ and $X \subset \mathbb{P}^3$ (see Corollary 3.5).

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1. NOTATION AND BACKGROUND MATERIAL

Let $X \subset \mathbb{P}(V) = \mathbb{P}^N$ be an irreducible, nondegenerate, smooth projective variety over $\mathbb{C}$ of dimension $n \geq 2$ (from now on n -fold) embedded by a very ample linear system $V \subseteq H^0(X, \mathcal{O}_X(H))$, where H is a very ample divisor on X .

Note that the general element of $|V|$ is a smooth divisor by the following well-known theorem.

Theorem 1.1 (Bertini). *The general element of a linear system $|V|$ on X is smooth away from the base locus of $|V|$.*

Remark 1.2. A divisor $D \in |V|$ on X is singular at $x \in X$ if and only if the corresponding hyperplane $h_D \in \mathbb{P}(V)^*$ contains the embedded tangent n -plane $\mathbb{T}_x X$ to X at x .

We suppose that X has a reducible hyperplane section, that is, an element $A + B \in |V|$ with A, B effective divisors. Let us recall the definition of m -connectedness for $n = 2$.

Definition 1.3. *An effective divisor D on a surface is called (numerically) m -connected if for every splitting $D = D_1 + D_2$ with D_1, D_2 effective, the inequality $D_1 \cdot D_2 \geq m$ holds.*

Remark 1.4. A very ample divisor is always 1-connected.

Theorem 1.5 (Van de Ven [8]). *Let X be a smooth surface and H a very ample divisor on X . If H is not 2-connected, then X is either the Veronese surface or a scroll of lines.*

We recall some topological results about the Picard group $\text{Pic } X$ of a smooth projective variety $X \subset \mathbb{P}^N$.

Theorem 1.6. *Let $X \subset \mathbb{P}^N$ be an n -fold and consider the restriction homomorphism $i_k : H^k(\mathbb{P}^N, \mathbb{C}) \rightarrow H^k(X, \mathbb{C})$. Then i_k is an isomorphism in the following situations:*

- a) (Lefschetz) X is a complete intersection and $k \leq n - 1$;
- b) (Barth-Larsen [1], [7]) $k \leq 2n - N$.

The same is valid if we replace $\mathbb{C}$ by $\mathbb{Z}$.

The above theorem shows two cases in which $\text{Pic } X$ is generated by the hyperplane section.

Corollary 1.7. *Let $X \subset \mathbb{P}^N$ be an n -fold. Then $\text{Pic } X \cong \mathbb{Z}$ (generated by the hyperplane section) in the following cases:*

- a) X is a complete intersection and $n \geq 3$;
- b) $n \geq \frac{N+2}{2}$.

Proof. Both a) and b) respectively follow by applying Theorem 1.6 to the exact sequence of sheaves

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \rightarrow \mathcal{O}^* \rightarrow 0,$$

having in mind that $\text{Pic } X \cong H^1(X, \mathcal{O}_X^*)$. $\square$

Remark 1.8. In our setting there exists a reducible divisor $A + B \in |V|$, so $\text{Pic } X$ is not generated by the hyperplane section. Therefore $n < \frac{N+2}{2}$ and $X \subset \mathbb{P}^N$ cannot be a complete intersection (from now on c.i.) if $n \geq 3$.

Denote by $\langle A \rangle \subset \mathbb{P}(V)$ (resp. $\langle B \rangle \subset \mathbb{P}(V)$) the linear span of A (resp. B) embedded by $|V|$. We can suppose

$$\dim \langle A \rangle \leq \dim \langle B \rangle$$

throughout. We will study the boundary cases in which $A \subset \langle A \rangle$ has the lowest possible codimension. More precisely we will analyze two possibilities, namely $A = \mathbb{P}^{n-1}$ and $\langle A \rangle = \mathbb{P}^n$.

2. THE CASE $A = \mathbb{P}^{n-1}$

Let $X \subset \mathbb{P}(V) = \mathbb{P}^N$ be an n -fold having a reducible divisor $A + B \in |V|$. Set $L = \langle A \rangle \cap \langle B \rangle$.

Lemma 2.1. $\dim L \geq n - 2$.

Proof. If $\dim L < n - 2$, we obtain a very ample divisor on a smooth surface which is not 1-connected by intersecting X with $n - 2$ general hyperplanes, contradicting Remark 1.4. $\square$

We start now with the case $A = \mathbb{P}^{n-1}$. Then either $\dim L = n - 2$ or $\dim L = n - 1$. Suppose $\dim L = n - 2$. Let us prove that this situation characterizes rational scrolls of $(n - 1)$ -planes.

Example 2.2. Let $0 \leq a_1 \leq \dots \leq a_n$ be integers and set $\mathcal{E} \cong \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i)$. A *rational scroll of $(n - 1)$ -planes* is the image of $\mathbb{P}(\mathcal{E})$ in $\mathbb{P}(V)$ embedded by a linear system $V \subseteq H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}(1))$, where $\mathcal{O}(1) := \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is the tautological line bundle. Its degree is $d = \sum_{i=1}^n a_i$ and it is smooth only if $a_1 > 0$. Otherwise, if $0 = a_k < a_{k+1}$, it is a cone over a smooth rational scroll of $(n - k - 1)$ -planes with vertex $\mathbb{P}^{k-1}$. We refer to a *rational normal scroll of $(n - 1)$ -planes* if it is embedded by the complete linear system $|\mathcal{O}(1)|$. In this case $\mathbb{P}(\mathcal{E}) \subset \mathbb{P}^{d+n-1}$. For a fiber $f = \mathbb{P}^{n-1}$, we have that $\xi = \mathcal{O}(1) - f$ is an effective divisor, so $f + \xi \in |\mathcal{O}(1)|$ and $f \cap \langle \xi \rangle = \mathbb{P}^{n-2}$.

Proposition 2.3. *Let $X \subset \mathbb{P}(V) = \mathbb{P}^N$ be an n -fold having a reducible divisor $A + B \in |V|$. If $A = \mathbb{P}^{n-1}$ and $\dim L = n - 2$, then X is a rational scroll of $(n - 1)$ -planes.*

Proof. If $A = \mathbb{P}^{n-1}$ and $\dim L = n - 2$, then it follows from the Grassmann formula

$$\dim\langle A + B \rangle = \dim\langle A \rangle + \dim\langle B \rangle - \dim L$$

that $\dim\langle B \rangle = N - 2$, so $|V - B|$ is a pencil of linearly equivalent divisors on X . For $\lambda \in \mathbb{P}^1$, let $R_\lambda \in |V - B|$ and $D \in |V|$. Since $A \in |V - B|$, then

$$\deg R_\lambda = R_\lambda \cdot D^{n-1} = A \cdot D^{n-1} = 1,$$

so $R_\lambda = \mathbb{P}^{n-1}$ and $X \subset \mathbb{P}^N$ is embedded as a rational scroll of $(n - 1)$ -planes. $\square$

Remark 2.4. On the other hand, if $\dim L = n - 1$ (equivalently $A \subset \langle B \rangle$), we cannot be assured that A moves on X . We can only handle this situation if $\dim\langle B \rangle \leq n$.

Corollary 2.5. *Let $X \subset \mathbb{P}(V) = \mathbb{P}^N$ be an n -fold having a reducible divisor $A + B \in |V|$. If $A = \mathbb{P}^{n-1}$ and $\dim\langle B \rangle \leq n$, then one of the following holds:*

- i) $X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \subset \mathbb{P}^4$ or $X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3}) \subset \mathbb{P}^5$;
- ii) $n = 2$ and $X \subset \mathbb{P}^3$.

Proof. If $\dim L = n - 2$ and $\dim\langle B \rangle = n$, then we get

$$\dim\langle A + B \rangle = \dim\langle A \rangle + \dim\langle B \rangle - \dim L = n + 1.$$

So $X \subset \mathbb{P}^{n+2}$ is a rational scroll of $(n - 1)$ -planes by Proposition 2.3. Since an n -fold $X \subset \mathbb{P}^{n+2}$ other than the projected Veronese surface in $\mathbb{P}^4$ is linearly normal by Zak's theorem (see [9], II Cor. 2.17), we deduce that $d = \deg X = 3$ (cf. Example 2.2) and case i) occurs.

Finally, if either $\dim L = n - 1$ or $\dim\langle B \rangle = n - 1$, then $X \subset \mathbb{P}^{n+1}$ is a hypersurface and $n = 2$ follows from Remark 1.8. $\square$

3. THE CASE $\langle A \rangle = \mathbb{P}^n$

Now suppose $\langle A \rangle = \mathbb{P}^n$, that is, A is a hypersurface. Then $n - 2 \leq \dim L \leq n$ by Lemma 2.1. If $\dim L < n$, we prove that A moves on X and we give a precise description of $X \subset \mathbb{P}^N$.

Example 3.1 (Veronese surface). Embed $\mathbb{P}^2$ in $\mathbb{P}^5$ by $V = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$. Then there exists a divisor $A + B \in |V|$ with $A, B \in |\mathcal{O}_{\mathbb{P}^2}(1)|$. Moreover $A \cdot B = 1$. The same holds for the projected Veronese surface, that is, $\mathbb{P}^2$ embedded in $\mathbb{P}^4$ by a general subspace of $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ of codimension one.

Example 3.2. Using the notation of Example 2.2, let $\mathbb{P}(\mathcal{E}) \subset \mathbb{P}(V)$ be a rational scroll of n -planes embedded by $V \subseteq H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}(1))$. Consider a smooth effective divisor X on $\mathbb{P}(\mathcal{E})$ embedded by $|V_X|$. Set $A = f|_X$ and $B \in |(V - f)|_X|$. Then $X \subset \mathbb{P}(V|_X)$ and $A + B \in |V_X|$. Furthermore $\langle A \rangle \cap \langle B \rangle = \mathbb{P}^{n-1}$ if $\langle A \rangle = f = \mathbb{P}^n$.

Proposition 3.3. *Let $X \subset \mathbb{P}(V) = \mathbb{P}^N$ be an n -fold having a reducible divisor $A + B \in |V|$. Suppose $\langle A \rangle = \mathbb{P}^n$ and $\dim\langle B \rangle \geq n$. Set $L = \langle A \rangle \cap \langle B \rangle$. Then*

- a) $X \subset \mathbb{P}^5$ is the Veronese surface if $\dim L = n - 2$;
- b) $X \subset \mathbb{P}^N$ is a divisor on a rational scroll of n -planes if $\dim L = n - 1$.

Proof. Suppose first that $\dim L = n - 2$. We start by considering $n = 2$. Then $\dim L = 0$ and hence $A \cdot B = 1$, so X is not 2-connected. Moreover

$$\dim\langle A + B \rangle = \dim\langle A \rangle + \dim\langle B \rangle - \dim L \geq 4,$$

so $X \subset \mathbb{P}^5$ is the Veronese surface by Theorem 1.5 since $A = \mathbb{P}^1$ in the scroll case. If $n \geq 3$, we reduce to $n = 2$ by cutting X with $n - 2$ general hyperplanes. To

prove a) just note that the Veronese surface is not a hyperplane section of a smooth 3-fold.

Consider now $\dim L = n-1$. Then $\dim \langle B \rangle = N-2$, so there is a pencil $h_\lambda \subset \mathbb{P}^{N*}$ of hyperplanes containing $\langle B \rangle$. We can assume that L is not a fixed component of $|V - B|$ (otherwise we argue with $|V - B - L|$).

Claim 1. A general $R_\lambda \in |V - B|$ is smooth in codimension one.

By Theorem 1.1, a general R_λ is smooth away from the base locus $\text{Bs } |V - B|$ of $|V - B|$. So, if $\text{Bs } |V - B| = \emptyset$, then a general R_λ is smooth. Assume now that $\text{Bs } |V - B| \neq \emptyset$ and take a point $x \in \text{Bs } |V - B|$. Suppose by contradiction that x is a singular point of a general $R_\lambda \in |V - B|$. Since this is a closed condition, we deduce that x is a singular point of R_λ for every $\lambda \in \mathbb{P}^1$. From Remark 1.2, it follows that h_λ is a pencil of tangent hyperplanes at x . Therefore the embedded tangent n -plane $\mathbb{T}_x X$ to X at x is contained in the base locus $\langle B \rangle$ of h_λ . In particular x is a singular point of $A = R_{\lambda_0}$, so

$$\langle A \rangle = \mathbb{T}_x X \subset \langle B \rangle,$$

contradicting the hypothesis about L .

Claim 2. R_λ is a hypersurface for every $\lambda \in \mathbb{P}^1$.

Let $K \subset \mathbb{P}^N$ be a general linear subspace of codimension $n-2$. For a general $\lambda \in \mathbb{P}^1$ we have that $R_\lambda \cap K$ is a smooth curve by claim 1, so

$$g(R_\lambda \cap K) = p_a(R_\lambda \cap K) = p_a(A \cap K),$$

where g and p_a denote respectively the geometric and arithmetic genus. The second equality follows from $A \in |R_\lambda|$. Since $A \cap K$ is a plane curve, then $R_\lambda \cap K$ is also a plane curve in view of the Castelnuovo bound. Therefore a general (and hence every) R_λ is also a hypersurface.

Thus $X \subset \mathbb{P}^N$ contains a pencil of hypersurfaces, so X is a divisor on a rational scroll of n -planes and b) is proved. $\square$

For $\dim L = n$, as in Corollary 2.5, we can only determine X if B is also a hypersurface.

Theorem 3.4. *Let $X \subset \mathbb{P}(V) = \mathbb{P}^N$ be an n -fold having a reducible divisor $A+B \in |V|$. If $\langle A \rangle = \mathbb{P}^n$ and $\langle B \rangle = \mathbb{P}^n$, then one of the following holds:*

- i) $X \subset \mathbb{P}^5$ is the Veronese surface or its general projection to $\mathbb{P}^4$;
- ii) $n = 2$ and $X \subset \mathbb{P}^4$ is either a c.i. or linked to a 2-plane by a c.i. of a quadric cone of rank 4 and another hypersurface of $\mathbb{P}^4$;
- iii) $n = 2$ and $X \subset \mathbb{P}^4$ is linked to a 2-plane by a c.i. of a quadric cone of rank 3 and another hypersurface of $\mathbb{P}^4$;
- iv) $n = 3$ and $X \subset \mathbb{P}^5$ is linked to a 3-plane by a c.i. of a quadric cone of rank 4 and another hypersurface of $\mathbb{P}^5$;
- v) $n = 2$, $X \subset \mathbb{P}^3$ and $\deg X \geq 4$.

Proof. By Lemma 2.1, it follows that $n-2 \leq \dim L \leq n$. If $n-2 \leq \dim L \leq n-1$, then, by Proposition 3.3, either $X \subset \mathbb{P}^5$ is the Veronese surface or $X \subset \mathbb{P}^{n+2}$ is a divisor on a rational scroll of n -planes. Except for the projected Veronese surface in $\mathbb{P}^4$, an n -fold $X \subset \mathbb{P}^{n+2}$ is linearly normal by Zak's theorem ([9], II Cor. 2.17), so either case i) holds or X is a divisor on a rational normal scroll of n -planes of degree 2 (cf. Example 2.2).

So we suppose $X \subset \mathbb{P}^{n+2}$ to be a smooth divisor on a quadric cone Q of rank 3 or 4. Therefore X is either a c.i. or linked to an n -plane by a c.i. (see [4], Th. 2.1). Note that from Remark 1.8 it follows that $n \leq 3$. If $n = 2$ and $X \subset \mathbb{P}^4$ is a c.i., the vertex W of the cone cannot intersect X . Otherwise X is singular along $X \cap W$. This fact together with the above considerations gives the cases ii) and iii). If $n = 3$, then X cannot be a c.i. by Remark 1.8 and, to get iv), we have to exclude the rank 3 case. By contradiction, suppose $X \subset \mathbb{P}^5$ to be contained in a quadric cone Q of rank 3, that is, $X \subset \mathbb{P}(\mathcal{E})$ for $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$. The vertex $W = \mathbb{P}^2$ of Q is contained in X by [4], Cor. 2.7. Moreover, W corresponds to $\tilde{W} := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 3}) \subset \mathbb{P}(\mathcal{E})$ and $X \cap \tilde{W}$ is isomorphic to $W = \mathbb{P}^2$. This implies that $X \cap \tilde{W}$ is a fiber of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 3}) \cong \mathbb{P}^1 \times \mathbb{P}^2$, which is impossible since X must intersect any fiber $\mathbb{P}^2$ of $\tilde{W}$ for dimensional reasons. Finally, if $\dim L = n$, then $X \subset \mathbb{P}^{n+1}$ is a hypersurface. So we get $n = 2$ by Remark 1.8 and necessarily $\deg X \geq 4$. $\square$

Since $X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \subset \mathbb{P}^4$ (resp. $X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3}) \subset \mathbb{P}^5$) is linked to a 2-plane (resp. 3-plane) by the c.i. of two quadric hypersurfaces, we can summarize our work in the following way.

Corollary 3.5. *Let $X \subset \mathbb{P}(V) = \mathbb{P}^N$ be an n -fold having a reducible divisor $A+B \in |V|$. If $A \subset \mathbb{P}^n$ and $B \subset \mathbb{P}^n$, then one of the following holds:*

- i) $X \subset \mathbb{P}^5$ is the Veronese surface or its general projection to $\mathbb{P}^4$;
- ii) $n = 2$ and $X \subset \mathbb{P}^4$ is either a c.i. or linked to a 2-plane by a c.i. of a quadric cone of rank 4 and another hypersurface of $\mathbb{P}^4$;
- iii) $n = 2$ and $X \subset \mathbb{P}^4$ is linked to a 2-plane by a c.i. of a quadric cone of rank 3 and another hypersurface of $\mathbb{P}^4$;
- iv) $n = 3$ and $X \subset \mathbb{P}^5$ is linked to a 3-plane by a c.i. of a quadric cone of rank 4 and another hypersurface of $\mathbb{P}^5$;
- v) $n = 2$ and $X \subset \mathbb{P}^3$.

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