A REMARK ON THE SEMI-CLASSICAL MEASURE FROM $-\frac{h^2}{2}\Delta + V$ WITH A DEGENERATE POTENTIAL V

YIFENG YU

(Communicated by Mikhail Shubin)

ABSTRACT. This note is motivated by Evans (2004) and Anantharaman (2004). We study the semiclassical measure arising from the operator $P(h) = -\frac{h^2}{2}\Delta + V(x)$ when the potential V has degenerate minimum points. We will use the technique of integration by parts and some identities of Evans to derive information on the support of the measure.

1. INTRODUCTION

In this paper, we denote $\mathbb{T}^n$ as the n -dimensional flat Torus and assume that $V \in C^\infty(\mathbb{T}^n)$. We will study the support of semiclassical measures associated with principle eigenfunctions of the operator $P(h) = -\frac{h^2}{2}\Delta + V(x)$. Precisely speaking, let ϕ_h be the positive principle eigenfunction for

$$(1.1) \quad \begin{cases} -\frac{h^2}{2}\Delta\phi_h + V\phi_h = E_h\phi_h & \text{in } \mathbb{T}^n \\ \phi_h \text{ is } \mathbb{T}^n\text{-periodic,} \end{cases}$$

where E_h is the principle eigenvalue. Moreover, we assume

$$\int_{\mathbb{T}^n} \phi_h^2 dx = 1$$

and

$$\min_{x \in \mathbb{T}^n} V = 0.$$

Upon passing if necessary to a subsequence, we assume that

$$\phi_h^2 = \sigma_h \rightharpoonup \sigma \quad \text{as Radon measures in } \mathbb{T}^n, \text{ as } h \rightarrow 0.$$

It is well known that

$$(1.2) \quad \text{spt}\sigma \subset \{x \in \mathbb{T}^n \mid V(x) = 0\} = \Gamma.$$

We are interested in knowing more information about $\text{spt}\sigma$. By using WKB approximation (see [H]), it is not difficult to prove that if V has no degenerate minimum points, i.e., for any $x \in \Gamma$ we have that the Hessian matrix $[D^2V(x)] > 0$, then

$$(1.3) \quad \text{spt}\sigma \subset \{x \in \mathbb{T}^n \mid V(x) = 0 \text{ and } H_V(x) = \min_{\{y \in \mathbb{T}^n \mid V(y)=0\}} H_V(y)\}$$

Received by the editors December 19, 2005.

2000 *Mathematics Subject Classification*. Primary 35P20.

©2006 American Mathematical Society
 Reverts to public domain 28 years from publication

and

$$\lim_{h \rightarrow 0} \frac{E_h}{h} = \min_{\{y \in \mathbb{T}^n \mid V(y)=0\}} H_V(y),$$

where

$$H_V(y) = \sum_{i=0}^n \sqrt{\lambda_i(y)}$$

and $\{\lambda_i(y)\}$ are the eigenvalues of the Hessian matrix $[D^2V(y)]$. If $n \geq 2$ and V has degenerate minimum points, i.e., the Hessian matrices at those minimum points have eigenvalue 0, then the WKB method fails. Instead of assuming that V has no degenerate minimum points, Anantharaman in [A] proved the above result under assumption A1 in [A], which is necessary for the Laplace method. However, assumption A1 in [A] in general is not easy to verify. We will discuss more about Anantharaman's result in section 3. The main goal of this paper is to use the technique of integration by parts and some interesting identities developed in Evans [E1] to prove (1.3) when the potential V has at least one second-order degenerate minimum point. See section 3 for the motivation of considering such a case. The following is our main result:

Theorem 1.1. *If there exists $x_0 \in \mathbb{T}^n$ such that $V(x_0) = 0$ and $\Delta V(x_0) = 0$, then*

$$\text{spt } \sigma \subset \{x \in \mathbb{T}^n \mid V(x) = 0 \text{ and } \Delta V(x) = 0\}$$

and

$$\lim_{h \rightarrow 0} \frac{E_h}{h} = 0.$$

Remark 1.2. In Theorem 1.1, we made no assumptions about the structure of the set $\{x \in \mathbb{T}^n \mid V(x) = 0\}$ and the other minimum points except x_0 . Also, $V(x) = 0$ and $\Delta V(x) = 0$ imply that the Hessian matrix $[D^2V(x)] = 0$, which is equivalent to saying that all the eigenvalues of the Hessian matrix $[D^2V(x)]$ are 0. Hence under the assumptions of Theorem 1.1, (1.3) still holds. We do not know whether (1.3) is true for any smooth potential V .

Our paper is organized as follows: in section 2, we prove Theorem 1.1. In section 3, we will discuss the connections with [E1] and [A].

2. PROOFS

To prove Theorem 1.1, we follow the change of variables in [E1] to create a quadratic term which is very useful in our estimates. Let $v_h = -h \log \phi_h$. For convenience we drop the lower index h and write v_h as v . It is easy to check that v satisfies the following PDE:

$$(2.1) \quad \begin{cases} -\frac{h}{2} \Delta v + \frac{1}{2} |Dv|^2 - V = -E_h & \text{in } \mathbb{T}^n, \\ v \text{ is } \mathbb{T}^n\text{-periodic.} \end{cases}$$

Lemma 2.1.

$$\int_{\mathbb{T}^n} |DV|^2 d\sigma_h = \frac{h}{2} \int_{\mathbb{T}^n} \sum_{1 \leq i, k \leq n} v_{x_k x_i} V_{x_k x_i} d\sigma_h.$$

Proof. Differentiating (2.1) gives

$$-\frac{h}{2}\Delta v_{x_k} + \sum_{i=1}^n v_{x_i} v_{x_i x_k} - V_{x_k} = 0.$$

Then

$$V_{x_k}^2 = -\frac{h}{2}\Delta v_{x_k} V_{x_k} + \sum_{i=1}^n v_{x_i} v_{x_i x_k} V_{x_k}.$$

Integrating by parts, we get

$$\begin{aligned} \frac{h}{2} \int_{\mathbb{T}^n} \Delta v_{x_k} V_{x_k} d\sigma_h &= \sum_{i=1}^n \frac{h}{2} \int_{\mathbb{T}^n} v_{x_k x_i x_i} V_{x_k} e^{\frac{-2v}{h}} dx \\ &= \sum_{i=1}^n \left(-\frac{h}{2} \int_{\mathbb{T}^n} v_{x_k x_i} V_{x_k x_i} d\sigma_h + \int_{\mathbb{T}^n} v_{x_k x_i} V_{x_k} v_{x_i} d\sigma_h \right). \end{aligned}$$

So

$$\int_{\mathbb{T}^n} |DV|^2 d\sigma_h = \frac{h}{2} \int_{\mathbb{T}^n} \sum_{1 \leq i, k \leq n} v_{x_k x_i} V_{x_k x_i} d\sigma_h.$$

□

In [E1], Evans discovered a lot of amazing identities. The following is one of them in our special case. For the reader's convenience, we present the proof below.

Lemma 2.2.

$$\int_{\mathbb{T}^n} |D^2 v|^2 d\sigma_h = \int_{\mathbb{T}^n} \Delta V d\sigma_h.$$

Proof. Differentiating (2.1) twice, we have

$$-\frac{h}{2}\Delta v_{x_k x_k} + \sum_{i=1}^n (v_{x_i x_k}^2 + v_{x_i} v_{x_i x_k x_k}) - V_{x_k x_k} = 0.$$

Integrating by parts, we have

$$\int_{\mathbb{T}^n} \frac{h}{2} \Delta v_{x_k x_k} \sigma_h = \sum_{i=1}^n \int_{\mathbb{T}^n} v_{x_k x_k x_i} v_{x_i} d\sigma_h.$$

Hence

$$\int_{\mathbb{T}^n} \Delta V d\sigma_h = \int_{\mathbb{T}^n} |D^2 v|^2 d\sigma_h.$$

□

Proof of Theorem 1.1. We first use a perturbation argument to show that

$$\lim_{h \rightarrow 0} \frac{E_h}{h} = 0.$$

Choose $\psi \in C^\infty(\mathbb{T}^n)$ such that

$$0 < \psi(x) \leq 1 \quad \text{for } x \neq x_0$$

and

$$\psi(x) = \frac{1}{2}|x - x_0|^2$$

in a neighbourhood of x_0 . Now let

$$V_\epsilon = V + \epsilon\psi.$$

Since x_0 is the only minimum point of V_ϵ and it is nondegenerate, we have

$$\lim_{h \rightarrow 0} \frac{E_h^\epsilon}{h} = H_{V_\epsilon}(x_0) = n\sqrt{\epsilon},$$

where E_h^ϵ is the principle eigenvalue of equation (1.1) with V replaced by V_ϵ . Since

$$\begin{aligned} 0 \leq E_h &= \min_{\|u\|_{L^2(\mathbb{T}^n)}=1} \frac{h^2}{2} \int_{\mathbb{T}^n} |Du|^2 dx + \int_{\mathbb{T}^n} V u^2 dx \\ &\leq \min_{\|u\|_{L^2(\mathbb{T}^n)}=1} \frac{h^2}{2} \int_{\mathbb{T}^n} |Du|^2 dx + \int_{\mathbb{T}^n} V_\epsilon u^2 dx \\ &\leq E_h^\epsilon. \end{aligned}$$

So

$$(2.2) \quad 0 \leq \lim_{h \rightarrow 0} \frac{E_h}{h} \leq \lim_{h \rightarrow 0} \frac{E_h^\epsilon}{h} = n\sqrt{\epsilon}.$$

Sending $\epsilon \rightarrow 0$, we derive

$$\lim_{h \rightarrow 0} \frac{E_h}{h} = 0.$$

Integrating (2.1),

$$\begin{aligned} E_h &= \int_{\mathbb{T}^n} V d\sigma_h - \frac{1}{2} \int_{\mathbb{T}^n} |Dv|^2 d\sigma_h + \frac{h}{2} \int_{\mathbb{T}^n} \Delta v d\sigma_h \\ &= \int_{\mathbb{T}^n} V d\sigma_h - \frac{1}{2} \int_{\mathbb{T}^n} |Dv|^2 d\sigma_h + \int_{\mathbb{T}^n} |Dv|^2 d\sigma_h \\ &= \int_{\mathbb{T}^n} V d\sigma_h + \frac{1}{2} \int_{\mathbb{T}^n} |Dv|^2 d\sigma_h \\ &\geq \frac{1}{2} \int_{\mathbb{T}^n} |Dv|^2 d\sigma_h. \end{aligned}$$

Therefore

$$(2.3) \quad \lim_{h \rightarrow 0} \frac{1}{h} \int_{\mathbb{T}^n} |Dv|^2 d\sigma_h = 0.$$

Integrating by parts,

$$\begin{aligned} (2.4) \quad \int_{\mathbb{T}^n} \Delta V d\sigma_h &= \frac{2}{h} \int_{\mathbb{T}^n} DV \cdot Dv d\sigma_h \\ &\leq \frac{2}{h} \left(\int_{\mathbb{T}^n} |DV|^2 d\sigma_h \right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^n} |Dv|^2 d\sigma_h \right)^{\frac{1}{2}}. \end{aligned}$$

According to Lemma 2.1 and Lemma 2.2,

$$\begin{aligned} (2.5) \quad \int_{\mathbb{T}^n} |DV|^2 d\sigma_h &= \frac{h}{2} \int_{\mathbb{T}^n} \sum_{1 \leq i, k \leq n} v_{x_k x_i} V_{x_k x_i} d\sigma_h \\ &\leq \frac{h}{2} \left(\int_{\mathbb{T}^n} |D^2 v|^2 d\sigma_h \right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^n} |D^2 V|^2 d\sigma_h \right)^{\frac{1}{2}} \\ &= \frac{h}{2} \left(\int_{\mathbb{T}^n} \Delta V d\sigma_h \right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^n} |D^2 V|^2 d\sigma_h \right)^{\frac{1}{2}} \\ &\leq Ch, \end{aligned}$$

where C is some constant depending only on V . Combining (2.3), (2.4) and (2.5), we derive

$$(2.6) \quad \int_{\mathbb{T}^n} \Delta V d\sigma = \lim_{h \rightarrow 0} \int_{\mathbb{T}^n} \Delta V d\sigma_h = 0.$$

Owing to (1.2), we have

$$(2.7) \quad \Delta V \geq 0 \quad \text{on } \text{spt}(\sigma).$$

According to (2.6) and (2.7),

$$\Delta V = 0 \quad \text{on } \text{spt}(\sigma).$$

Hence Theorem 1.1 holds. $\square$

3. APPLICATIONS IN EVANS'S CONSTRUCTION OF QUASIMODES

In [E1], Evans constructed a quasimode ψ of the following stationary Schrödinger equation:

$$-\frac{h^2}{2}\Delta\psi - V\psi = E\psi \quad \text{in } \mathbb{T}^n.$$

The construction is as follows: for any $P \in \mathbb{R}^n$, let w and w^* be positive principle eigenfunctions for the following dual eigenvalue problems:

$$\begin{cases} -\frac{h^2}{2}\Delta w + hP \cdot Dw + Vw = E^0 w & \text{in } \mathbb{T}^n, \\ w \text{ is } \mathbb{T}^n\text{-periodic}, \end{cases}$$

$$\begin{cases} -\frac{h^2}{2}\Delta w^* - hP \cdot Dw^* + Vw^* = E^0 w^* & \text{in } \mathbb{T}^n, \\ w^* \text{ is } \mathbb{T}^n\text{-periodic}, \end{cases}$$

where E^0 is the principle eigenvalue. Moreover, we assume

$$\int_{\mathbb{T}^n} w^* w dx = 1.$$

Define

$$\begin{cases} v := -h \log w, \\ v^* := h \log w^*. \end{cases}$$

It is easy to see that v and v^* satisfy the PDE

$$(3.1) \quad \begin{cases} -\frac{h}{2}\Delta v + \frac{1}{2}|P + Dv|^2 - V = \bar{H}_h(P) & \text{in } \mathbb{T}^n, \\ v \text{ is } \mathbb{T}^n\text{-periodic}, \end{cases}$$

$$(3.2) \quad \begin{cases} \frac{h}{2}\Delta v^* + \frac{1}{2}|P + Dv^*|^2 - V = \bar{H}_h(P) & \text{in } \mathbb{T}^n, \\ v^* \text{ is } \mathbb{T}^n\text{-periodic}. \end{cases}$$

Furthermore,

$$(3.3) \quad \int_{\mathbb{T}^n} e^{\frac{(v^*-v)}{h}} dx = 1.$$

Here we write $\bar{H}_h(P) = \frac{|P|^2}{2} - E^0(P)$, and we set $a = e^{\frac{v^*-v}{2h}}$ and $u = P \cdot x + \frac{v+v^*}{2}$. The quasimode in [E1] is given by

$$\psi = ae^{\frac{iu}{h}}.$$

See [E1] for details.

Let R denote the error term $-\frac{h^2}{2}\Delta\psi - V\psi - \bar{H}_h(P)\psi$. In [E1], Evans showed $R = O(h)$ in $L^2(\mathbb{T}^n)$. It is an open problem whether the error estimate can be improved to $o(h)$ under suitable conditions. For $P = 0$, which is the case we are considering in this paper, using Theorem 1.1, we can show that $R = o(h)$ if and only if there exists $x_0 \in \mathbb{T}^n$ such that $V(x_0) = 0$ and $\Delta V(x_0) = 0$. See Yu [Y] for details.

Anantharaman considered in [A] a somewhat similar approach within a probabilistic framework. Upon passing if necessary to a subsequence, we assume that

$$ww^* \rightharpoonup \sigma \quad \text{as Radon measures in } \mathbb{T}^n, \text{ as } h \rightarrow 0.$$

Under several technical assumptions of the dynamics (i.e., assumptions A1, A2 and A3 in [A]), Anantharaman proved that σ is a projected Mather measure with respect to the Hamiltonian $H(p, x) = \frac{1}{2}|p + P|^2 - V(x)$ and it minimizes some entropy among all the projected Mather measures. A corollary of her result is that if $P = 0$, then (1.3) holds under assumption A1 which requires that any minima of the action

$$\mathcal{A}(\gamma) = \int_0^t \frac{1}{2}|\dot{\gamma}|^2 + V(\gamma) ds$$

is nondegenerate and the number of minima of $\mathcal{A}$ connecting two fixed endpoints is finite. We refer to [A] for precise statements of these technical assumptions.

The author would like to thank Professor L.C. Evans and Professor M. Zworski for very helpful discussions and comments. Also, I want to give my gratitude to Hans Christianson for helping me find references.

REFERENCES

- [A] N. Anantharaman, *On the zero-temperature or vanishing viscosity limit for certain Markov processes arising from Lagrangian dynamics.*, J. Eur. Math. Soc. (JEMS) 6 (2004), no. 2, 207–276. MR2055035 (2005i:82004)
- [D-S] M. Dimassi, J. Sjöstrand, *Spectral Asymptotics in the Semi-Classical Limit*, Cambridge University Press, 1999. MR1735654 (2001b:35237)
- [E1] L.C. Evans, *Toward a quantum analogue of weak KAM theory*, Comm. Math. Phys. 244 (2004), no. 2, 311–334. MR2031033 (2005c:81063)
- [E2] L.C. Evans, *Some new PDE methods for weak KAM theory*, Calc. Var. Partial Differential Equations 17 (2003), no. 2, 159–177. MR1986317 (2004e:37097)
- [E3] L.C. Evans, *Effective Hamiltonians and quantum states*, Seminaire Equations aux Dérivées Partielles, Ecole Polytechnique (2000-2001). MR1860693 (2002j:81049)
- [E-Z] L.C. Evans, M. Zworski, *Lectures on semiclassical analysis*, Lecture note.
- [H] B. Helffer, *Semi-classical analysis for the Schrödinger operator and applications*, Springer Lect. Notes in Math., 1336 (1988). MR0960278 (90c:81043)
- [J-K-M] H.R. Jauslin, H.O. Kreiss, J. Moser, *On the forced Burgers equation with periodic boundary conditions*, Proc. Symposia in Pure Math 65, 133-153, 1999. MR1662751 (99m:35208)
- [Y] Y. Yu, *L^∞ variational problems, Aronsson equations and weak KAM theory*, Ph.D. dissertation, U.C. Berkeley(2005).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS AT AUSTIN, AUSTIN, TEXAS 78712
E-mail address: yifengyu@math.berkeley.edu