A GENERALIZATION OF A CURIOUS CONGRUENCE ON HARMONIC SUMS

XIA ZHOU AND TIANXIN CAI

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ABSTRACT. Zhao established a curious harmonic congruence for prime $p > 3$:

$$\sum_{\substack{i+j+k=p \\ i,j,k>0}} \frac{1}{ijk} \equiv -2B_{p-3} \pmod{p}.$$

In this note the authors extend it to the following congruence for any prime $p > 3$ and positive integer $n \leq p - 2$:

$$\sum_{\substack{l_1+l_2+\dots+l_n=p \\ l_1,\dots,l_n>0}} \frac{1}{l_1 l_2 \cdots l_n} \equiv \begin{cases} -(n-1)! B_{p-n} \pmod{p} & \text{if } 2 \nmid n, \\ -\frac{n}{2(n+1)} n! B_{p-n-1} \pmod{p^2} & \text{if } 2 \mid n. \end{cases}$$

Other improvements on congruences of harmonic sums are also obtained.

1. INTRODUCTION

Bernoulli numbers B_k ($k = 0, 1, 2, \dots$) is defined by the series $\frac{t}{e^t-1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}$. It is well known that $B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_{2k+1} = 0$ ($k > 0$). The congruences concerning Bernoulli numbers have been studied by various authors. Using partial sum of multiple zeta value series, Jianqiang Zhao [7] first gave the congruence

$$\sum_{\substack{i+j+k=p \\ i,j,k>0}} \frac{1}{ijk} \equiv -2B_{p-3} \pmod{p}$$

for any prime $p > 3$.

Recently, Chungang Ji [1] gave a simple proof of the above congruence. Inspired by their work, we derive the following congruence.

Main Theorem. *Let $p > 3$ be a prime, $n \leq p - 2$ a positive integer,*

$$\sum_{\substack{l_1+l_2+\dots+l_n=p \\ l_1,\dots,l_n>0}} \frac{1}{l_1 l_2 \cdots l_n} \equiv \begin{cases} -(n-1)! B_{p-n} \pmod{p} & \text{if } 2 \nmid n, \\ -\frac{n}{2(n+1)} n! B_{p-n-1} \pmod{p^2} & \text{if } 2 \mid n. \end{cases}$$

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2. SEVERAL LEMMAS

Lemma 1 (Von Staudt-Clausen; cf. [6, p. 56]). *Let m be a positive integer. Then*

$$B_{2m} + \sum_{p-1|2m} \frac{1}{p} \in \mathbb{Z},$$

where the sum is taken over those primes p such that $p-1$ divides $2m$ (in particular, 2 and 3 appear in the denominator of each Bernoulli number). Consequently, pB_{2m} is p -integral for all m and all p .

Lemma 2 (cf. [2], [3], [4] or [5]). *Let p be an odd prime > 3 and let r be a positive integer, $r \leq p-3$. Then*

$$\sum_{i=1}^{p-1} \frac{1}{i^r} \equiv \begin{cases} -\frac{r(r+1)}{2(r+2)} B_{p-r-2} p^2 \pmod{p^3} & \text{if } 2 \nmid r, \\ \frac{r}{r+1} B_{p-r-1} p \pmod{p^2} & \text{if } 2 \mid r. \end{cases}$$

Lemma 3. *Let $r, \alpha_1, \dots, \alpha_n$ be positive integers, $r = \alpha_1 + \alpha_2 + \dots + \alpha_n \leq p-3$. Then*

$$\sum_{\substack{1 \leq l_1, \dots, l_n \leq p-1 \\ l_i \neq l_j}} \frac{1}{l_1^{\alpha_1} l_2^{\alpha_2} \dots l_n^{\alpha_n}} \equiv \begin{cases} (-1)^n (n-1)! \frac{r(r+1)}{2(r+2)} B_{p-r-2} p^2 \pmod{p^3} & \text{if } 2 \nmid r, \\ (-1)^{n-1} (n-1)! \frac{r}{r+1} B_{p-r-1} p \pmod{p^2} & \text{if } 2 \mid r. \end{cases}$$

Proof. If $n = 1$, the congruence is obvious by Lemma 2. We assume that the formula holds when the number of variables is less than n . Then by induction

$$\begin{aligned} & \sum_{\substack{1 \leq l_1, \dots, l_n \leq p-1 \\ l_i \neq l_j}} \frac{1}{l_1^{\alpha_1} l_2^{\alpha_2} \dots l_n^{\alpha_n}} \\ &= \sum_{\substack{1 \leq l_1, \dots, l_{n-1} \leq p-1 \\ l_i \neq l_j}} \frac{1}{l_1^{\alpha_1} l_2^{\alpha_2} \dots l_{n-1}^{\alpha_{n-1}}} \left(\sum_{l_n=1}^{p-1} \frac{1}{l_n^{\alpha_n}} - \frac{1}{l_1^{\alpha_n}} - \dots - \frac{1}{l_{n-1}^{\alpha_n}} \right) \\ &= \sum_{\substack{1 \leq l_1, \dots, l_{n-1} \leq p-1 \\ l_i \neq l_j}} \frac{1}{l_1^{\alpha_1} l_2^{\alpha_2} \dots l_{n-1}^{\alpha_{n-1}}} \sum_{l_n=1}^{p-1} \frac{1}{l_n^{\alpha_n}} \\ &\quad - \sum_{\substack{1 \leq l_1, \dots, l_{n-1} \leq p-1 \\ l_i \neq l_j}} \frac{1}{l_1^{\alpha_1 + \alpha_n} l_2^{\alpha_2} \dots l_{n-1}^{\alpha_{n-1}}} \\ &\quad - \dots - \sum_{\substack{1 \leq l_1, \dots, l_{n-1} \leq p-1 \\ l_i \neq l_j}} \frac{1}{l_1^{\alpha_1 + \alpha_n} l_2^{\alpha_2} \dots l_{n-1}^{\alpha_{n-1} + \alpha_n}}. \end{aligned}$$

From Lemma 1, Lemma 2 and the above assumption, we have

$$\sum_{\substack{1 \leq l_1, \dots, l_{n-1} \leq p-1 \\ l_i \neq l_j}} \frac{1}{l_1^{\alpha_1} l_2^{\alpha_2} \dots l_{n-1}^{\alpha_{n-1}}} \sum_{l_n=1}^{p-1} \frac{1}{l_n^{\alpha_n}} \equiv \begin{cases} 0 \pmod{p^3} & \text{if } 2 \nmid r, \\ 0 \pmod{p^2} & \text{if } 2 \mid r. \end{cases}$$

If r is odd, then,

$$\begin{aligned} \sum_{\substack{1 \leq l_1, \dots, l_n \leq p-1 \\ l_i \neq l_j}} \frac{1}{l_1^{\alpha_1} l_2^{\alpha_2} \dots l_n^{\alpha_n}} &\equiv -(n-1) \sum_{\substack{1 \leq l_1, \dots, l_{n-1} \leq p-1 \\ l_i \neq l_j}} \frac{1}{l_1^{\alpha_1 + \alpha_n} l_2^{\alpha_2} \dots l_{n-1}^{\alpha_{n-1}}} \\ &\equiv -(n-1)(-1)^{n-1}(n-2)! \frac{r(r+1)}{2(r+2)} B_{p-r-2} p^2 \\ &\equiv (-1)^n (n-1)! \frac{r(r+1)}{2(r+2)} B_{p-r-2} p^2 \pmod{p^3}, \end{aligned}$$

and if r is even, we can similarly derive that

$$\sum_{\substack{1 \leq l_1, \dots, l_n \leq p-1 \\ l_i \neq l_j}} \frac{1}{l_1^{\alpha_1} l_2^{\alpha_2} \dots l_n^{\alpha_n}} \equiv (-1)^{n-1} (n-1)! \frac{r}{r+1} B_{p-r-1} p \pmod{p^2}.$$

The proof of Lemma 3 is complete by induction.

3. THE PROOF OF THE MAIN THEOREM

Since $n \leq p-2$, it is easy to see that

$$\begin{aligned} \sum_{\substack{l_1 + l_2 + \dots + l_n = p \\ l_1, \dots, l_n > 0}} \frac{1}{l_1 l_2 \dots l_n} \\ = \frac{1}{p} \sum_{\substack{l_1 + l_2 + \dots + l_n = p \\ l_1, \dots, l_n > 0}} \frac{l_1 + l_2 + \dots + l_n}{l_1 l_2 \dots l_n} = \frac{n}{p} \sum_{\substack{l_1 + l_2 + \dots + l_{n-1} < p \\ l_1, \dots, l_{n-1} > 0}} \frac{1}{l_1 l_2 \dots l_{n-1}}. \end{aligned}$$

Writing $u_{n-1} = l_1 + l_2 + \dots + l_{n-1}$, so that $1 \leq l_1 + l_2 + \dots + l_{n-2} < u_{n-1} \leq p-1$, we find that

$$\frac{1}{l_1 l_2 \dots l_{n-1}} = \frac{l_1 + l_2 + \dots + l_{n-1}}{l_1 l_2 \dots l_{n-1} u_{n-1}} = \frac{1}{u_{n-1}} \left(\frac{1}{l_1 \dots l_{n-2}} + \dots + \frac{1}{l_2 \dots l_{n-1}} \right)$$

and hence

$$\sum_{\substack{l_1 + l_2 + \dots + l_{n-1} < p \\ l_1, \dots, l_{n-1} > 0}} \frac{1}{l_1 l_2 \dots l_{n-1}} = (n-1) \sum_{\substack{1 \leq l_1 + l_2 + \dots + l_{n-2} < u_{n-1} \leq p-1 \\ l_1, \dots, l_{n-2} > 0}} \frac{1}{l_1 l_2 \dots l_{n-2} u_{n-1}}.$$

Using this method step by step and denoting $l_1 + \cdots + l_k = u_k$ we have

$$\begin{aligned}
 & \sum_{\substack{l_1+l_2+\cdots+l_n=p \\ l_1, \dots, l_n > 0}} \frac{1}{l_1 l_2 \cdots l_n} \frac{n}{p} \sum_{\substack{l_1+l_2+\cdots+l_{n-1} < p \\ l_1, \dots, l_{n-1} > 0}} \frac{1}{l_1 l_2 \cdots l_{n-1}} \\
 &= \frac{n(n-1)}{p} \sum_{\substack{1 \leq l_1+l_2+\cdots+l_{n-2} < u_{n-1} \leq p-1 \\ l_1, \dots, l_{n-2} > 0}} \frac{1}{l_1 l_2 \cdots l_{n-2} u_{n-1}} \\
 &= \cdots \\
 &= \frac{n!}{p} \sum_{1 \leq u_1 < u_2 < \cdots < u_{n-1} < p} \frac{1}{u_1 u_2 \cdots u_{n-1}} \\
 &= \frac{n!}{p} \frac{1}{(n-1)!} \sum_{\substack{1 \leq u_1, \dots, u_{n-1} \leq p-1 \\ u_i \neq u_j}} \frac{1}{u_1 u_2 \cdots u_{n-1}} \\
 &= \frac{n}{p} \sum_{\substack{1 \leq u_1, \dots, u_{n-1} \leq p-1 \\ u_i \neq u_j}} \frac{1}{u_1 u_2 \cdots u_{n-1}}.
 \end{aligned}$$

From Lemma 3,

$$\begin{aligned}
 & \sum_{\substack{1 \leq u_1, \dots, u_{n-1} \leq p-1 \\ u_i \neq u_j}} \frac{1}{u_1 u_2 \cdots u_{n-1}} \\
 & \equiv \begin{cases} (-1)^{n-1} (n-1)! \frac{n}{2(n+1)} B_{p-n-1} p^2 \pmod{p^3} & \text{if } 2 \nmid (n-1), \\ (-1)^{n-2} (n-1)! \frac{1}{n} B_{p-n} p \pmod{p^2} & \text{if } 2 \mid (n-1), \end{cases}
 \end{aligned}$$

we have

$$\sum_{\substack{l_1+l_2+\cdots+l_n=p \\ l_1, \dots, l_n > 0}} \frac{1}{l_1 l_2 \cdots l_n} \equiv \begin{cases} -n! \frac{n}{2(n+1)} B_{p-n-1} p \pmod{p^2} & \text{if } 2 \mid n, \\ -(n-1)! B_{p-n} \pmod{p} & \text{if } 2 \nmid n. \end{cases}$$

We have completed the proof of the Main Theorem.

Corollary (Zhao). *For any prime $p > 3$,*

$$\sum_{\substack{i+j+k=p \\ i, j, k > 0}} \frac{1}{ijk} \equiv -2B_{p-3} \pmod{p}.$$

Remark. In [8] Zhao also considered another congruence on harmonic sum. For any positive integer n and α , $n\alpha \leq p-3$, he obtains

$$\sum_{1 \leq l_1 < l_2 < \cdots < l_n \leq p-1} \frac{1}{l_1^\alpha l_2^\alpha \cdots l_n^\alpha} \equiv 0 \begin{cases} \pmod{p^2} & \text{if } 2 \nmid n\alpha, \\ \pmod{p} & \text{if } 2 \mid n\alpha. \end{cases}$$

By using Lemma 3, we improve the above congruence to

$$\sum_{1 \leq l_1 < l_2 < \cdots < l_n \leq p-1} \frac{1}{l_1^\alpha l_2^\alpha \cdots l_n^\alpha} \equiv \begin{cases} (-1)^n \frac{\alpha(n\alpha+1)}{2(n\alpha+2)} B_{p-n\alpha-2} p^2 \pmod{p^3} & \text{if } 2 \nmid n\alpha, \\ (-1)^{n-1} \frac{\alpha}{n\alpha+1} B_{p-n\alpha-1} p \pmod{p^2} & \text{if } 2 \mid n\alpha. \end{cases}$$

for any positive integers n and α with $n\alpha \leq p-3$, since the sum is over ordered l_i 's.

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DEPARTMENT OF MATHEMATICS, ZHEJIANG UNIVERSITY, HANGZHOU 310027, PEOPLE'S REPUBLIC OF CHINA

E-mail address: unitqq@zju.edu.cn

DEPARTMENT OF MATHEMATICS, ZHEJIANG UNIVERSITY, HANGZHOU 310027, PEOPLE'S REPUBLIC OF CHINA

E-mail address: txcai@mail.hz.zj.cn