ON UNIFORMLY GÂTEAUX SMOOTH NORMS AND NORMAL STRUCTURE

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ABSTRACT. It is shown that every separable Banach space admits an equivalent norm that is uniformly Gâteaux smooth and yet lacks asymptotic normal structure.

A Banach space is said to have the fixed point property (FPP) if for every nonempty bounded closed convex $C \subset X$ and every nonexpansive self-mapping $T: C \rightarrow C$ there is a fixed point of T in C . A Banach space is said to have the weak fixed point property (w-FPP) if for every nonempty weakly compact convex $C \subset X$ there is a fixed point for every nonexpansive $T: C \rightarrow C$. Clearly, a Banach space has w-FPP if it has FPP. The space c_0 has w-FPP but does not have FPP; see [M]. These two notions obviously coincide in reflexive spaces.

The classical results in metric fixed point theory state that a Banach space has w-FPP if its norm is uniformly Fréchet differentiable ([K]) or uniformly rotund ([B]). In fact, instead of uniformly rotund, it is sufficient to assume that the norm is only uniformly rotund in every direction (URED), [Z]. It is a natural question whether the uniform Fréchet differentiability can be weakened to uniform Gâteaux differentiability (UG), since the notion of UG is dual (in a sense) to URED. (In fact, UG is dual to weak* uniform rotundity, which is a stronger notion than URED.)

We note that in a non-separable case, a theorem of [DLT] states that for any uncountable set Γ , the non-separable space $c_0(\Gamma)$ does not have FPP under any equivalent renorming. But it is well known that for any set Γ , $c_0(\Gamma)$ has an equivalent renorming that is simultaneously locally uniformly rotund, Fréchet differentiable and UG; see e.g. [DGZ, II.7.8]. Thus even norms with rather good geometrical properties do not assure FPP.

In our note we show that the usual proofs of “UF, UR or URED implies w-FPP” cannot be adapted, since they prove the w-FPP by showing that UF, UR or URED implies that the norm has a normal structure. We show that, in contrast, if the norm of a Banach space is UG, it does not necessarily have a normal structure. Even more, every separable Banach space can be equivalently renormed to have a uniformly Gâteaux smooth norm that lacks asymptotic normal structure. This

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notion was defined by J. B. Baillon and R. Schöneberg in [BS] as a weakening of the normal structure, which is still sufficient for w-FPP.

The norm $\|\cdot\|$ on a Banach space X is said to have *asymptotic normal structure* if for every closed convex bounded set $C \subset X$ with $\text{diam } C > 0$ and every sequence $\{x_n\} \subset C$ satisfying $\lim_{n \rightarrow \infty} \|x_n - x_{n+1}\| = 0$ there exists $x \in C$ such that

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \text{diam}_{\|\cdot\|} C.$$

The norm is called *uniformly Gâteaux smooth* if the limit

$$\lim_{t \rightarrow 0} \frac{\|x + th\| - \|x\|}{t} = \|\cdot\|'(x, h)$$

is uniform in $x \in S_X$ for each $h \in S_X$, where S_X is the unit sphere of X . It follows that the derivative of the norm at $x \in X \setminus \{0\}$, i.e. $h \mapsto \|\cdot\|'(x, h)$ is an element of X^* .

Recall that a Markushevich basis of a Banach space X is a biorthogonal system $\{e_n; f_n\} \subset X \times X^*$ such that $\overline{\text{span}}\{e_n\} = X$ and $\{f_n\}$ separates the points of X (i.e. for any $x \neq y \in X$ there is $n \in \mathbb{N}$ such that $f_n(x) \neq f_n(y)$).

Theorem 1. *Let X be a separable Banach space. Then there exists an equivalent uniformly Gâteaux smooth norm lacking asymptotic normal structure.*

Proof. First, we will define a norm that lacks asymptotic normal structure. It will be done similarly as in [MS]. Let $\{e_n; f_n\}$ be a Markushevich basis of $(X, \|\cdot\|)$ such that $\|e_n\| = 1$ and $\|f_n\| \leq 20$ for all $n \in \mathbb{N}$ (see e.g. [LT, 1.f.4]). We put

$$C = \{x \in X; \|x\| \leq 2, 0 \leq f_n(x) \leq 1 \text{ for all } n \in \mathbb{N}\}.$$

This is a closed convex bounded set, $0 \in C$ and $\{e_n\} \subset C$. For an arbitrary $\beta \geq \text{diam}_{\|\cdot\|} C$, we define a new norm

$$\|x\|_\beta = \max \left\{ \|x\|, \beta \sup_{n \in \mathbb{N}} |f_n(x)| \right\},$$

which is obviously an equivalent norm on X .

Fact 2. *For all $n \in \mathbb{N}$, $\|e_n\|_\beta = \beta$ and $\|f_n\|_\beta^* = 1/\beta$.*

Proof of Fact 2.

$$\|e_n\|_\beta = \max \left\{ \|e_n\|, \beta \sup_{k \in \mathbb{N}} |f_k(e_n)| \right\} = \max\{1, \beta\} = \beta.$$

Regarding f_n , we have

$$\|f_n\|_\beta^* \geq f_n \left(\frac{e_n}{\beta} \right) = \frac{1}{\beta},$$

and, on the other hand,

$$\begin{aligned} \|f_n\|_\beta^* &= \sup \left\{ f_n \left(\frac{\sum_{k=1}^N a_k e_k}{\left\| \sum_{k=1}^N a_k e_k \right\|_\beta} \right); N \geq n, a_1, \dots, a_N \in \mathbb{R} \right\} \\ &= \sup_{a_n \neq 0} \frac{a_n}{\left\| \sum_{k=1}^N a_k e_k \right\|_\beta} \leq \frac{a_n}{\beta a_n} = \frac{1}{\beta}, \end{aligned}$$

where the inequality holds because, by the definition,

$$\left\| \sum_{k=1}^N a_k e_k \right\|_{\beta} \geq \beta f_n \left(\sum_{k=1}^N a_k e_k \right) = \beta a_n.$$

□

Fact 3. $\text{diam}_{\|\cdot\|_{\beta}} C = \beta$.

Proof of Fact 3. First, $\text{diam}_{\|\cdot\|_{\beta}} C \geq \|e_1 - 0\|_{\beta} = \beta$. On the other hand, if $x, y \in C$, then (as $f_n(x), f_n(y) \in [0, 1]$) $|f_n(x - y)| \leq 1$ and thus

$$\|x - y\|_{\beta} = \max \left\{ \|x - y\|, \beta \sup_{n \in \mathbb{N}} |f_n(x - y)| \right\} \leq \beta.$$

□

Now we define a norm $\|\cdot\|_{\beta}^*$ on X^* by a formula

$$(\|f\|_{\beta}^*)^2 = (\|f\|_{\beta})^2 + \sum_{n=1}^{\infty} \frac{1}{2^n} f^2(e_n).$$

By a standard convexity argument (see [DGZ, Fact II.2.3]), the norm $\|\cdot\|_{\beta}^*$ is W^*UR . Since $\|\cdot\|_{\beta}^*$ is weak*-lsc, it is a dual norm. Let $\|\cdot\|_{\beta}$ be the norm on X that is predual to $\|\cdot\|_{\beta}^*$. By a standard duality argument (see [DGZ, Thm. II.6.7]), the norm $\|\cdot\|_{\beta}$ is uniformly Gâteaux smooth.

Fact 4. a) $\lim_{n \rightarrow \infty} \|f_n\|_{\beta}^* = 1/\beta$,
 b) $\lim_{n \rightarrow \infty} \|e_n\|_{\beta} = \beta$,
 c) $\text{diam}_{\|\cdot\|_{\beta}} C = \beta$.

Proof of Fact 4. a) Follows directly from Fact 2.

b) Since $\|f\|_{\beta}^* \geq \|f\|_{\beta}$ for all $f \in X^*$, we have $\|x\|_{\beta} \leq \|x\|_{\beta}^*$ for all $x \in X$ and thus $\|e_n\|_{\beta} \leq \beta$. On the other hand

$$\liminf_{n \rightarrow \infty} \|e_n\|_{\beta} \geq \liminf_{n \rightarrow \infty} \frac{f_n(e_n)}{\|f_n\|_{\beta}^*} = \beta.$$

c) As above, we get $\text{diam}_{\|\cdot\|_{\beta}} C \leq \text{diam}_{\|\cdot\|_{\beta}} C = \beta$. On the other hand,

$$\text{diam}_{\|\cdot\|_{\beta}} C \geq \|e_n\|_{\beta} \rightarrow \beta.$$

□

Now we are ready to prove that $\|\cdot\|_{\beta}$ does not have asymptotic normal structure. Indeed, we define the sequence $\{x_n\} \subset C$ by

$$x_n = \begin{cases} (1 - j2^{-2k})e_k + e_{k+1}, & \text{where } n = 2^{2k} + j, j = 1, \dots, 2^{2k}, \\ e_{k+1} + j2^{-2k-1}e_{k+2}, & \text{for } n = 2^{2k+1} + j, j = 1, \dots, 2^{2k+1}. \end{cases}$$

Clearly, $x_n \in C$ and

$$\lim_{n \rightarrow \infty} \|x_n - x_{n+1}\|_{\beta} = 0.$$

Choose $x \in C$. For any $\varepsilon > 0$ let $N \in \mathbb{N}$ and $y = \sum_{l=1}^N a_l e_l$ be such that $\|x - y\|_\beta < \varepsilon$. Then, for all $k > N$ and all $n = 2^{2k+i} + j$, $j = 1, \dots, 2^{2k+i}$, $i = 0, 1$,

$$\|x - x_n\|_\beta > \|y - x_n\|_\beta - \varepsilon \geq \frac{f_{k+1}(y - x_n)}{\|f_{k+1}\|_\beta^*} - \varepsilon = \frac{1}{\|f_{k+1}\|_\beta^*} - \varepsilon.$$

Thus,

$$\beta \geq \liminf_{n \rightarrow \infty} \|x - x_n\|_\beta \geq \beta - \varepsilon,$$

and consequently $\lim_{n \rightarrow \infty} \|x - x_n\|_\beta = \beta = \text{diam}_{\|\cdot\|_\beta} C$. $\square$

Remark. Note that in the proof we could take $C = \overline{\text{conv}}\{0, e_n, e_n + e_{n+1}; n \in \mathbb{N}\}$. If the basis $\{e_n\}$ is weakly null, then by Krein's theorem C is weakly compact, and hence we have an example of a weakly compact convex set without asymptotic normal structure.

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REFERENCES

- [BS] J.B. Baillon and R. Schöneberg, *Asymptotic normal structure and fixed points of non-expansive mappings*, Proc. Amer. Math. Soc. **81** (1981), no. 2, 257–264. MR0593469 (82c:47068)
- [B] F.E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci. U.S.A. **54** (1965), 1041–1044. MR0187120 (32:4574)
- [DGZ] R. Deville, G. Godefroy and V. Zizler, *Smoothness and Renormings in Banach Spaces*, Monographs and Surveys in Pure and Applied Mathematics **64**, Pitman, 1993. MR1211634 (94d:46012)
- [DLT] P.N. Dowling, C.J. Lennard and B. Turett, *Asymptotically isometric copies of c_0 in Banach spaces*, J. Math. Anal. Appl. **219** (1998), 377–391. MR1606342 (98m:46023)
- [K] M.A. Khamsi, *Uniform smoothness implies super-normal structure property*, Nonlinear Anal. **19** (1992), 1063–1069. MR1194145 (93k:46012)
- [M] B. Maurey, *Points fixes des contractions sur un convexe fermé de L_1* , Seminaire d'Analyse Fonctionnelle **80–81**, Ecole Polytechnique Palaiseau, 1981. MR0659309 (83h:47041)
- [MS] S.A. Mariadoss and P.M. Soardi, *A remark on asymptotic normal structure in Banach spaces*, Rend. Sem. Mat. Univ. Politec. Torino **44** (1987), no. 3, 393–395. MR0932426 (89e:46017)
- [LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I. Sequence Spaces*, Springer-Verlag, 1977. MR0500056 (58:17766)
- [Z] V. Zizler, *On some rotundity and smoothness properties of Banach spaces*, Dissertationes Math. Rozprawy Mat. **87** (1971), 33 pp. MR0300060 (45:9108)

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