NEW PROOF OF THE HÖRMANDER MULTIPLIER THEOREM ON COMPACT MANIFOLDS WITHOUT BOUNDARY

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ABSTRACT. On compact manifolds (M, g) without boundary, the gradient estimates for unit band spectral projection operators χ_λ are proved for a second order elliptic differential operator L . A new proof of the Hörmander Multiplier Theorem (first proved by A. Seeger and C.D. Sogge in 1989) is given in this setting by using the gradient estimates and the Calderón-Zygmund argument.

1. INTRODUCTION

Let (M, g) be a compact boundaryless manifold of dimension $n \geq 2$, and let L be a second order elliptic differential operator which is positive and self-adjoint with respect to the C^∞ density dx . We shall consider the eigenvalue problem

$$(L + \lambda^2)u(x) = 0, \quad x \in M.$$

Recall that the spectrum of L : $0 \leq \lambda_1^2 \leq \lambda_2^2 \leq \lambda_3^2 \leq \dots \rightarrow \infty$, and let the corresponding eigenfunctions $\{e_j(x)\} \subset L^2(M)$ be an associated real orthogonal basis. Define the unit band spectral projection operators,

$$\chi_\lambda f = \sum_{\lambda_j \in [\lambda, \lambda+1)} e_j(f), \quad \text{where} \quad e_j(f)(x) = e_j(x) \int_M f(y) e_j(y) dy.$$

In [7] and [8], Sogge proved the following L^p estimates on χ_λ for $p \geq 2$, $\lambda \geq 1$:

$$(1) \quad \|\chi_\lambda f\|_p \leq C \lambda^{\sigma(p)} \|f\|_2, \quad \text{with} \quad \sigma(p) = \max \left\{ \frac{n-1}{2} - \frac{n}{p}, \frac{n-1}{2} \left(\frac{1}{2} - \frac{1}{p} \right) \right\}.$$

The special case of (1) where $p = \infty$ was proved in Hörmander [4]. The first result of this paper is the following gradient estimate for $\chi_\lambda f$.

Theorem 1.1. *There is a uniform constant C such that*

$$\|\nabla \chi_\lambda f\|_\infty \leq C \lambda^{(n+1)/2} \|f\|_2, \quad \forall \lambda \geq 1.$$

Given a function $m(\lambda) \in L^\infty(\mathbf{R})$, define the multiplier operator, $m(P)$, by

$$(2) \quad m(P)f = \sum_{j=1}^{\infty} m(\lambda_j) e_j(f),$$

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where $P = \sqrt{-L}$. Such an operator is always bounded on $L^2(M)$. However, for any other $L^p(M)$, it is known that some smoothness assumptions on the function $m(\lambda)$ are needed to ensure that $m(P)$ is bounded on $L^p(M)$ (see [9]).

Under the following assumption: *Suppose that $m \in L^\infty(\mathbf{R})$, let $L_s^2(\mathbf{R})$ denote the usual Sobolev space and fix $\beta \in C_0^\infty((1/2, 2))$ satisfying $\sum_{-\infty}^\infty \beta(2^j t) = 1$, $t > 0$, for $s > n/2$. Then there is*

$$(3) \quad \sup_{\lambda > 0} \|\beta(\cdot)m(\lambda)\|_{L_s^2}^2 < \infty.$$

By studying the parametrix of the wave kernel of $m(P)$, Seeger and Sogge [6] and Sogge [8] proved the following Hörmander Multiplier Theorem for L on (M, g) .

Theorem 1.2. *Let $m \in L^\infty(\mathbf{R})$ satisfy (3). Then there are constants C_p such that*

$$(4) \quad \|m(P)f\|_{L^p(M)} \leq C_p \|f\|_{L^p(M)}, \quad \text{for } 1 < p < \infty.$$

The second part of this paper gives a new proof for the above theorem based on the gradient estimates on χ_λ , without using the parametrix of the wave kernel of $m(P)$ as was done in [6] and [8]. As is known, the parametrix construction of the wave kernel does not work well for compact manifolds with boundary unless one assumes that the boundary is geodesic concave (see [3]). Here we use a new approach to prove Theorem 1.2, which also works for the Laplacian operator on compact manifolds with boundary (see [11] and [12]). The main idea is that we make decompositions twice to get $\{K_{\lambda,l}(x, y)\}$ for the integral kernel $K(x, y)$ of $\tilde{m}(P)$, such that $K(x, y) = \sum_j \sum_l K_{2^j, l}(x, y)$. Also, for each operator $T_{\lambda, l}(P)$ with integral kernel $K_{\lambda, l}$, there are $L^1 \rightarrow L^2$ estimates by a scaling argument using the L^∞ estimates and gradient estimates on χ_λ . With the support properties and the finite propagation speed properties, one has the relation (8) between λ and l , which is one key observation when we apply the Calderón-Zygmund decomposition to show the weak-type (1,1) estimates on $\tilde{m}(P)$. In [1], Duong-Ouhabaz-Sikora gave another proof of the above Hörmander Multiplier Theorem using heat kernel methods and the L^∞ bounds of χ_λ .

In what follows we shall use the convention that C will denote a constant that is not necessarily the same at each occurrence.

2. GRADIENT ESTIMATES: THEOREM 1.1

We will apply the maximum principle to L in the cube centered at $x_0 \in M$ with length $d = (\lambda + 1)^{-1}$, following the idea of interior gradient estimates for Poisson's equation in [2]. Define the geodesic coordinates $x = (x_1, \dots, x_n)$ centered at point x_0 as follows: fix an orthogonal basis $\{v_i\}_{i=1}^n \subset T_{x_0}M$, identify $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ with the point $\exp(\sum_{i=1}^n x_i v_i) \in M$. In this coordinate, the second order elliptic operator L can be written as

$$L = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial}{\partial x_i} + c(x),$$

where $a_{ij}(x)$, $b_i(x)$, $c(x) \in C^\infty(M)$, and $c(x) \leq 0$, which comes from the fact that L is an elliptic operator. Without loss of generality, we may assume $a_{ij}(x_0) = a_{ij}(0) = \delta_{ij}$, i.e., the principle term of L at point x_0 is the Laplacian.

Denote $u(x; f) = \chi_\lambda f(x)$; we have $u \in C^\infty(M)$, and

$$Lu(x; f) = - \sum_{\lambda_j \in [\lambda, \lambda+1)} \lambda_j^2 e_j(f) := h(x; f).$$

From the L^∞ estimate (1) on χ_λ and Cauchy-Schwarz inequality, we have

$$|h(x; f)|^2 \leq C(\lambda + 1)^{n+3} \|\chi_\lambda f\|_{L^2(M)}^2.$$

Proof of Theorem 1.1. Here we estimate $|D_n u(0; f)| = |\frac{\partial}{\partial x_n} u(0; f)|$ only, and the same estimate holds for $|D_i u(0; f)|$ with $i = 1, \dots, n-1$ also. Define

$$\begin{aligned} Q &= \{x = (x_1, \dots, x_n) \in \mathbf{R}^n \mid |x_i| < d, i = 1, \dots, n\} \subset M, \\ Q' &= \{x = (x_1, \dots, x_n) \in \mathbf{R}^n \mid |x_i| < d, i = 1, \dots, n-1, 0 < x_n < d.\} \subset M. \end{aligned}$$

Consider the function

$$\varphi(x', x_n; f) = \frac{1}{2} [u(x', x_n; f) - u(x', -x_n; f)],$$

where we write $x = (x', x_n) = (x_1, \dots, x_{n-1}, x_n)$. We have that $\varphi(x', 0; f) = 0$, $\sup_{\partial Q'} |\varphi| \leq \sup_{\partial Q} |u| := A \leq C(\lambda + 1)^{(n-1)/2}$, and

$$|L\varphi| \leq \sup_Q |h| + \sup_Q |(L_{(x', x_n)} - L_{(x', -x_n)})u(x', -x_n; f)| := N, \text{ in } Q'.$$

Here we will estimate the bound of N . From Lemma 17.5.2 and Theorem 17.5.3 in [5], we have the following crude estimates on the derivatives of $u(x) = \chi_\lambda f(x)$:

$$\sup_M |D^\alpha u| \leq C_\alpha \lambda^{\frac{n}{2} + \alpha}, \quad \forall \alpha \in \mathbf{N} \cup \{0\}.$$

Applying the above estimates, we have

$$\sup_Q |(L_{(x', x_n)} - L_{(x', -x_n)})u| \leq Cd \left[\sup_Q |D^2 u| + \sup_Q |Du| + \sup_Q |u| \right] \leq Cd \lambda^{\frac{n}{2} + 2},$$

and we have $\sup_Q |h| \leq C(\lambda + 1)^{(n+3)/2}$. Hence $N \leq C[(\lambda + 1)^{(n+3)/2} + d\lambda^{\frac{n}{2} + 2}]$.

Now consider the function

$$\psi(x', x_n) = \frac{A}{d^2} [|x'|^2 + \alpha x_n (nd - (n-1)x_n)] + \beta N x_n (d - x_n), \quad \forall (x', x_n) \in Q',$$

where $\alpha \geq 1$ and $\beta \geq 1$ will be determined below. One may check $\psi(x', x_n) \geq 0$ on $x_n = 0$ and $\psi(x', x_n) \geq A$ in the remaining portion of $\partial Q'$.

$$\begin{aligned} L\psi(x) &= \frac{A}{d^2} \left[2\text{tr}(a_{ij}(x)) - (2n\alpha - 2\alpha + 1)a_{nn}(x) + 2 \sum_{i=1}^n b_i(x)x_i + b_n(x) \left(n\alpha d \right. \right. \\ &\quad \left. \left. - (2n\alpha - 2\alpha + 1)x_n \right) \right] + N\beta \left[-2a_{nn}(x) + b_n(x)(d - 2x_n) \right] + c(x)\psi(x). \end{aligned}$$

Since M is compact, $\text{tr}(a_{ij}(x))$ and $b_i(x)$ are bounded uniformly, $c(x) \leq 0$ and $a_{nn}(x)$ is positive, then for a large α , we can make

$$2\text{tr}(a_{ij}(x)) - (2n\alpha - 2\alpha + 1)a_{nn}(x) \leq -1.$$

Fix such an α , since $d = (\lambda + 1)^{-1}$, for large λ , we have

$$2 \sum_{i=1}^n b_i(x)x_i + b_n(x) \left(n\alpha d - (2n\alpha - 2\alpha + 1)x_n \right) < 1.$$

Then the first term of $L\psi(x)$ is negative. For the second term of $L\psi(x)$, letting β be large enough, we have

$$\beta[-2a_{nn}(x) + b_n(x)(d - 2x_n)] < -1.$$

For the third term, since $c(x) \leq 0$ and $\psi(x) \geq 0$ in Q' , we have $L\psi(x) \leq -N$ in Q' .

Now we have $L(\psi \pm \varphi) \leq 0$ in Q' and $\psi \pm \varphi \geq 0$ on $\partial Q'$, from which it follows by the maximum principle that $|\varphi(x', x_n; f)| \leq |\psi(x', x_n)|$ in Q' . Letting $x' = 0$ in the expressions for ψ and φ , then dividing by x_n and letting x_n tend to zero, we obtain

$$|D_n u(0; f)| = \lim_{x_n \rightarrow 0} \left| \frac{\varphi(0, x_n; f)}{x_n} \right| \leq \frac{\alpha n A}{d} + \beta d N.$$

The same estimate holds for $|D_i u(0)|$, $i = 1, \dots, n-1$, hence we have

$$|\nabla u(0; f)| \leq \frac{\alpha n^2 A}{d} + \beta n d N.$$

By the bounds of d , A and N , we have the estimate

$$|\nabla u(0)| \leq C(\lambda + 1)^{(n+1)/2}.$$

Since the estimate is for any $x_0 \in M$, Theorem 1.1 is proved. $\square$

Remark 2.1. Our method can apply to gradient estimates of eigenfunctions on compact manifolds with boundary as well; see details in [11] and [12]. After learning of my gradient estimates on spectral functions of Laplacian on compact manifolds, B. Xu in [10] studied the estimates of derivatives of the spectral function χ_λ on a closed manifold by studying the Hadamard parametrix of the wave operator, which gave another proof of the result of Theorem 1.1, but the method in [10] cannot apply to compact manifolds with boundary.

3. HÖRMANDER MULTIPLIER THEOREM: THEOREM 1.2

Since the complex conjugate of m satisfies the same hypotheses as (3), we need only to prove Theorem 1.2 for the exponents $1 < p \leq 2$. This will allow us to exploit orthogonality, and also reduce Theorem 1.2 to show that $m(P)$ is weak-type $(1, 1)$ by the Marcinkiewicz Interpolation Theorem, i.e.,

$$(5) \quad \mu\{x : |m(P)f(x)| > \alpha\} \leq \alpha^{-1} \|f\|_{L^1},$$

where $\mu(E)$ denotes the dx measure of $E \subset M$. Since the all eigenvalues of L are non-negative, we may assume $m(t)$ is an even function on $\mathbf{R}$. Then we have

$$m(P)f(x) = \frac{1}{2\pi} \int_{\mathbf{R}} \widehat{m}(t) e^{itP} f(x) dt = \frac{1}{\pi} \int_{\mathbf{R}_+} \widehat{m}(t) \cos(tP) f(x) dt,$$

where $P = \sqrt{-L}$, and the cosine transform $u(t, x) = \cos(tP)f(x)$ is the solution of the following Cauchy problem of the wave equation:

$$\left(\frac{\partial^2}{\partial t^2} - L\right)u(t, x) = 0, \quad u(0, x) = f(x), \quad u_t(0, x) = 0.$$

We shall use the finite propagation speed of solutions of the wave equation in Part 2 of the proof to get the key observation (8).

Proof of Theorem 1.2. The proof of the weak-type (1,1) estimate (5) will involve a splitting of $m(P)$ into two pieces: a main piece which one need carefully study, plus a remainder which has a strong (1,1) estimate by using the L^∞ estimates for χ_λ as was done in [8]. Specifically, define $\rho \in C_0^\infty(\mathbf{R})$ as

$$(6) \quad \rho(t) = 1, \quad \text{for } |t| \leq \frac{\epsilon}{2}, \quad \rho(t) = 0, \quad \text{for } |t| \geq \epsilon > 0,$$

where ϵ is a given constant related to the manifold. Write $m(P) = \tilde{m}(P) + r(P)$, where $\tilde{m}(P) = (m * \tilde{\rho})(P)$ and $r(P) = (m * (1 - \rho))(P)$. To estimate the main term and remainder, for $\lambda = 2^j$, $j = 1, 2, \dots$, define $m_\lambda(\tau) = \beta\left(\frac{\tau}{\lambda}\right)m(\tau)$.

Part 1. Strong (1,1) estimate on the remainder: $\|r(P)f\|_{L^1} \leq C\|f\|_{L^1}$.

Here we follow the first part in proof of Theorem 5.3.1 in [8]. Define

$$r_\lambda(P) = (m_\lambda * (1 - \rho))(P) = \frac{1}{2\pi} \int e^{itP} (1 - \rho(t)) \hat{m}_\lambda(t) dt.$$

Note that $r_0(P) = r(P) - \sum_{j \geq 1} r_{2^j}(P)$ is a bounded and rapidly decreasing function. Hence $r_0(P)$ is bounded from L^1 to any L^p space. We need only show

$$\|r_\lambda(P)f\|_{L^2} \leq C\lambda^{n/2-s}\|f\|_{L^1}, \quad \lambda = 2^j, j = 1, 2, \dots$$

Using the L^∞ estimate (1), we have

$$\|r_\lambda(P)f\|_{L^2}^2 \leq \sum_{k=1}^{\infty} \|r_\lambda(P)\chi_k f\|_{L^2}^2 \leq C \sum_{k=1}^{\infty} \sup_{\tau \in [k, k+1]} |r_\lambda(\tau)|^2 (1+k)^{n-1} \|f\|_{L^1}^2.$$

Hence we need only show

$$\sum_{k=1}^{\infty} \sup_{\tau \in [k, k+1]} |r_\lambda(\tau)|^2 (1+k)^{n-1} \leq C\lambda^{n-2s}.$$

Note that since $m_\lambda(\tau) = 0$ for $\tau \notin [\lambda/2, 2\lambda]$, both $\tilde{m}_\lambda(\tau)$ and $r_\lambda(\tau)$ are $O((1+|\tau|+|\lambda|)^{-N})$ for any N when $\tau \notin [\lambda/4, 4\lambda]$. Hence we need only show

$$\sum_{k=\lambda/4}^{4\lambda} \sup_{\tau \in [k, k+1]} |r_\lambda(\tau)|^2 \leq C\lambda^{1-2s}.$$

By the fundamental theorem of calculus and the Cauchy-Schwartz inequality,

$$\begin{aligned} \sum_{k=\lambda/4}^{4\lambda} \sup_{\tau \in [k, k+1]} |r_\lambda(\tau)|^2 &\leq C \left(\int_{\mathbf{R}} |r_\lambda(\tau)|^2 d\tau + \int_{\mathbf{R}} |r'_\lambda(\tau)|^2 d\tau \right) \\ &= C \left(\int_{\mathbf{R}} |\hat{m}_\lambda(t)(1 - \rho(t))|^2 dt + \int_{\mathbf{R}} |t\hat{m}_\lambda(t)(1 - \rho(t))|^2 dt \right). \end{aligned}$$

Since $\rho(t) = 1$, for $|t| \leq \frac{\epsilon}{2}$, by a change variables, this is dominated by

$$\lambda^{1-2s} \int_{\mathbf{R}} |t^s \hat{m}_\lambda(t/\lambda)|^2 dt = \lambda^{1-2s} \|\beta(\cdot)m(\lambda \cdot)\|_{L_s^2}^2 \leq C\lambda^{1-2s},$$

and the second inequality comes from our assumption (3). Hence we have the estimate

$$\|r(P)f\|_{L^2} \leq C\|f\|_{L^1}.$$

Also, since our manifold is compact, we have

$$\|r(P)f\|_{L^1} \leq \text{Vol}(M)^{1/2} \|r(P)f\|_{L^2} \leq C\|f\|_{L^1}.$$

Part 2. Weak-type $(1, 1)$ estimate on the main term:

$$\mu\{x : |\tilde{m}(P)f(x)| > \alpha\} \leq \alpha^{-1} \|f\|_{L^1}.$$

The weak-type $(1, 1)$ estimate on $\tilde{m}(P)$ would follow from the integral operator

$$\begin{aligned} \tilde{m}(P)f(x) &= \frac{1}{2\pi} \int_{\mathbf{R}} \hat{m}(t)\rho(t)e^{itP}f(x)dt \\ &= \frac{1}{2\pi} \int_{\mathbf{R}} \hat{m}(t)\rho(t) \sum_{k \geq 1} e^{it\lambda_k} e_{\lambda_k}(x) \int_M e_{\lambda_k}(y)f(y)dydt \\ &= \frac{1}{2\pi} \int_M \left\{ \int_{\mathbf{R}} \hat{m}(t)\rho(t) \sum_{k \geq 1} e^{it\lambda_k} e_{\lambda_k}(x)e_{\lambda_k}(y)dt \right\} f(y)dy \end{aligned}$$

with the kernel

$$K(x, y) = \int_{\mathbf{R}} \hat{m}(t)\rho(t) \sum_{k \geq 1} e^{it\lambda_k} e_{\lambda_k}(x)e_{\lambda_k}(y)dt = \sum_{k \geq 1} (m * \check{\rho})(\lambda_k) e_{\lambda_k}(x)e_{\lambda_k}(y)$$

of weak-type $(1, 1)$. Now define the dyadic decomposition

$$K_{\lambda}(x, y) = \int_{\mathbf{R}} \hat{m}_{\lambda}(t)\rho(t) \sum_{k \geq 1} e^{it\lambda_k} e_{\lambda_k}(x)e_{\lambda_k}(y)dt.$$

We have $K(x, y) = \sum_{j=1}^{\infty} K_{2^j}(x, y) + K_0(x, y)$, where K_0 is bounded and vanishes when $\text{dist}(x, y)$ is larger than a fixed constant. In order to estimate $K_{\lambda}(x, y)$, we make a second dyadic decomposition as follows:

$$K_{\lambda, l}(x, y) = \int_{\mathbf{R}} \hat{m}_{\lambda}(t)\beta(2^{-l}\lambda|t|)\rho(t) \sum_{k \geq 1} e^{it\lambda_k} e_{\lambda_k}(x)e_{\lambda_k}(y)dt.$$

We have $K_{\lambda}(x, y) = \sum_{l=-\infty}^{\infty} K_{\lambda, l}(x, y)$. Define $T_{\lambda, l}(P)f(x) = \int_M K_{\lambda, l}(x, y)f(y)dy$. From above two dyadic decompositions, we have

$$(7) \quad \tilde{m}(P)f(x) = \sum_{k=0}^{\infty} \sum_{l=-\infty}^{\infty} T_{2^k, l}(P)f(x).$$

Note that, because of the support properties of $\rho(t)$, $K_{\lambda, l}(x, y)$ vanishes if l is larger than a fixed multiple of $\log \lambda$. Now we exploit the fact that the finite propagation speed of the wave equation mentioned before implies that the kernels of the operators $T_{\lambda, l}$, $K_{\lambda, l}$ must satisfy

$$K_{\lambda, l}(x, y) = 0, \quad \text{if } \text{dist}(x, y) \geq C(2^l\lambda^{-1}),$$

since $\cos(tP)$ will have a kernel that vanishes on this set when t belongs to the support of the integral defining $K_{\lambda, l}(x, y)$. Hence in each of the second sums of (7), there are uniform constants $c, C > 0$ such that

$$(8) \quad c\lambda \text{dist}(x, y) \leq 2^l \leq C\lambda$$

must be satisfied for each $\lambda = 2^k$. We will use this key observation later.

Now for $T_{\lambda, l}(P)$ s, we have the following estimates:

$$\begin{aligned} (a) \quad & \|T_{\lambda, l}(P)f\|_{L^2(M)} \leq C(2^l)^{-s} \lambda^{n/2} \|f\|_{L^1(M)}, \\ (b) \quad & \|T_{\lambda, l}(P)g\|_{L^2(M)} \leq C(2^l)^{-s_0} \lambda^{n/2} \left[\lambda \max_{y, y_0 \in \Omega} \text{dist}(y, y_0) \right] \|g\|_{L^1(\Omega)}, \end{aligned}$$

where $\Omega = \text{supp}(g)$, $\int_{\Omega} g(y)dy = 0$ and $n/2 < s_0 < \min\{s, n/2 + 1\}$.

We first show estimate (a). Note that $\beta(2^{-l}\lambda|t|)\rho(t) = 0$ when $|t| \leq 2^{l-1}\lambda^{-1}$. We can use the same idea to prove estimate (a) as we did to prove the estimate on the remainder $r(P)$ in Part 1, where $1 - \rho(t) = 0$, for $|t| \leq \frac{\epsilon}{2}$. Using orthogonality of χ_k for $k \in \mathbf{N}$, and the estimates (1) on χ_k for $p = \infty$, we have

$$\|T_{\lambda,l}(P)f\|_2^2 \leq \sum_{k=1}^{\infty} \|T_{\lambda,l}(P)\chi_k f\|_2^2 \leq C \sum_{k=1}^{\infty} \sup_{\tau \in [k, k+1]} |T_{\lambda,l}(\tau)|^2 (1+k)^{n-1} \|f\|_1^2.$$

Hence we need only show

$$\sum_{k=1}^{\infty} \sup_{\tau \in [k, k+1]} |T_{\lambda,l}(\tau)|^2 (1+k)^{n-1} \leq C(2^l)^{-2s} \lambda^n.$$

Note since $m_{\lambda}(\tau) = 0$, for $\tau \notin [\lambda/2, 2\lambda]$, we have $T_{\lambda,l}(\tau) = O((1+|\tau|+|\lambda|)^{-N})$ for any N when $\tau \notin [\lambda/4, 4\lambda]$. Then we have

$$\begin{aligned} \sum_{k \notin [\lambda/4, 4\lambda]} \sup_{\tau \in [k, k+1]} |T_{\lambda,l}(\tau)|^2 (1+k)^{n-1} &\leq C \sum_{k \notin [\lambda/4, 4\lambda]} (1+k+\lambda)^{-2N} (1+k)^{n-1} \\ &\leq C(1+\lambda)^{n-2N}. \end{aligned}$$

Since $2^l \leq C\lambda$ from our observation (8) above, we need only show

$$\begin{aligned} &\sum_{k=\lambda/4}^{4\lambda} \sup_{\tau \in [k, k+1]} |T_{\lambda,l}(\tau)|^2 (1+k)^{n-1} \leq C(2^l)^{-2s} \lambda^n, \\ \text{i.e.} \quad &\sum_{k=\lambda/4}^{4\lambda} \sup_{\tau \in [k, k+1]} |T_{\lambda,l}(\tau)|^2 \leq C(2^l)^{-2s} \lambda. \end{aligned}$$

Using the same argument as in Part 1, note that $\beta(2^{-l}\lambda|t|)\rho(t) = 0$ when $|t| \leq 2^{l-1}\lambda^{-1}$; we have the estimate (a).

Next we prove the estimate (b). Given function $g \in L^1(M)$ with $\Omega = \text{supp}(g)$ and $\int_M g(y)dy = 0$, fix a point $y_0 \in \Omega$. Using the cancellation of g , we have

$$\begin{aligned} &\|T_{\lambda,l}(P)g\|_{L^2}^2 \\ &= \int_M \left| \int_{\Omega} K_{\lambda,l}(x, y) g(y) dy \right|^2 dx \\ &= \int_M \left| \int_{\Omega} [K_{\lambda,l}(x, y) - K_{\lambda,l}(x, y_0)] g(y) dy \right|^2 dx \\ &= \int_M \left| \int_{\Omega} \sum_{k \geq 1} T_{\lambda,l}(\lambda_k) e_{\lambda_k}(x) [e_{\lambda_k}(y) - e_{\lambda_k}(y_0)] g(y) dy \right|^2 dx \\ &= \sum_{k \geq 1} \int_M \left| \int_{\Omega} \sum_{\lambda_j \in [k, k+1)} \left\{ T_{\lambda,l}(\lambda_j) e_{\lambda_j}(x) [e_{\lambda_j}(y) - e_{\lambda_j}(y_0)] \right\} g(y) dy \right|^2 dx, \end{aligned}$$

where we use the orthogonality of $\{e_{\lambda_j}(x)\}$, which is dominated by

$$\begin{aligned} &\sum_{k \geq 1} \int_M \max_{y \in \Omega} \left| \sum_{\lambda_j \in [k, k+1)} \left\{ T_{\lambda,l}(\lambda_j) e_{\lambda_j}(x) [e_{\lambda_j}(y) - e_{\lambda_j}(y_0)] \right\} \right|^2 dx \int_{\Omega} |g(y)| dy^2 \\ &= \|g\|_{L^1}^2 \sum_{k \geq 1} \int_M \left| \sum_{\lambda_j \in [k, k+1)} \left\{ T_{\lambda,l}(\lambda_j) e_{\lambda_j}(x) [e_{\lambda_j}(y_1) - e_{\lambda_j}(y_0)] \right\} \right|^2 dx, \end{aligned}$$

where the maximum achieves at some point $y_1 \in \Omega$,

$$\begin{aligned}
&= \|g\|_{L^1}^2 \sum_{k \geq 1} \int_M \left| (\nabla_y \sum_{\lambda_j \in [k, k+1)} T_{\lambda, l}(\lambda_j) e_{\lambda_j}(x) e_{\lambda_j}(\bar{y}), y_1 - y_0) \right|^2 dx \\
&= \|g\|_{L^1}^2 \sum_{k \geq 1} \int_M \left| \left\{ \sum_{\lambda_j \in [k, k+1)} T_{\lambda, l}(\lambda_j) e_{\lambda_j}(x) (\nabla e_{\lambda_j}(\bar{y}), y_1 - y_0) \right\} \right|^2 dx \\
&= \|g\|_{L^1}^2 \sum_{k \geq 1} \sum_{\lambda_j \in [k, k+1)} \left| T_{\lambda, l}(\lambda_j) (\nabla e_{\lambda_j}(\bar{y}), y_1 - y_0) \right|^2 \\
&\leq \|g\|_{L^1}^2 \sum_{k \geq 1} \max_{\tau \in [k, k+1)} |T_{\lambda, l}(\tau)|^2 \left\{ \sum_{\lambda_j \in [k, k+1)} |\nabla e_{\lambda_j}(\bar{y})|^2 \text{dist}(y_1, y_0)^2 \right\} \\
&\leq \|g\|_{L^1}^2 \left[\max_{y, y_0 \in \Omega} \text{dist}(y, y_0) \right]^2 \sum_{k \geq 1} \max_{\tau \in [k, k+1)} |T_{\lambda, l}(\tau)|^2 \left\{ \sum_{\lambda_j \in [k, k+1)} |\nabla e_{\lambda_j}(\bar{y})|^2 \right\} \\
&\leq C \|g\|_{L^1}^2 \left[\max_{y, y_0 \in \Omega} \text{dist}(y, y_0) \right]^2 \sum_{k \geq 1} \max_{\tau \in [k, k+1)} |T_{\lambda, l}(\tau)|^2 (1+k)^{n+1}.
\end{aligned}$$

Now using the same computation as in the estimate (a), for some constant s_0 satisfying $n/2 < s_0 < \min\{s, n/2 + 1\}$, we have

$$\sum_{k \geq 1} \max_{\tau \in [k, k+1)} |T_{\lambda, l}(\tau)|^2 (1+k)^{n+1} \leq C(2^l)^{-2s_0} \lambda^{n+2}.$$

Combining the above two estimates, we prove the estimate (b):

$$\|T_{\lambda, l}(P)g\|_{L^2(M)} \leq C(2^l)^{-s_0} \lambda^{n/2} \left[\lambda \max_{y, y_0 \in \Omega} \text{dist}(y, y_0) \right] \|g\|_{L^1(\Omega)}.$$

Next we use the estimates (a) and (b) to show that

$$\tilde{m}(P)f(x) = \int_M K(x, y)f(y)dy$$

is weak-type (1,1). We let $f(x) = g(x) + \sum_{k=1}^{\infty} b_k(x) := g(x) + b(x)$ be the Calderón-Zygmund decomposition of $f \in L^1(M)$ at the level α . Let $Q_k (\supset \text{supp}(b_k))$ be the cube associated to b_k on M , and we have

$$\|g\|_{L^1} + \sum_{k=1}^{\infty} \|b_k\|_{L^1} \leq 3\|f\|_{L^1}; \quad |g(x)| \leq 2^n \alpha \quad \text{almost everywhere,}$$

and for certain non-overlapping cubes Q_k ,

$$b_k(x) = 0 \quad \text{for } x \notin Q_k \quad \text{and} \quad \int_M b_k(x)dx = 0; \quad \sum_{k=1}^{\infty} \mu|Q_k| \leq \alpha^{-1} \|f\|_{L^1}.$$

Since $\int_M |g|^2 dx \leq 2^n \alpha \int_M |g| dx$ and

$$\{x : |\tilde{m}(P)f(x)| > \alpha\} \subset \left(\{x : |\tilde{m}(P)g(x)| > \frac{\alpha}{2}\} \cup \{x : |\tilde{m}(P)b(x)| > \frac{\alpha}{2}\} \right),$$

by L^2 boundedness of $\tilde{m}(P)$ and Tchebyshev's inequality, we get

$$\mu\{x : |\tilde{m}(P)g(x)| > \alpha/2\} \leq C\alpha^{-2} \|g\|_{L^2}^2 \leq C'\alpha^{-1} \|f\|_{L^1}.$$

Let Q_k^* be the cube with the same center as Q_k but twice the side length. After possibly making a translation, we may assume that $Q_k = \{x : \max |x_j| \leq R\}$. Let $\mathbf{O}^* = \bigcup Q_k^*$; we have $\mu|\mathbf{O}^*| \leq 2^n \alpha^{-1} \|f\|_{L^1}$ and

$$\mu\{x \notin \mathbf{O}^* : |\tilde{m}(P)b(x)| > \alpha/2\} \leq 2\alpha^{-1} \sum_{k=1}^{\infty} \int_{x \notin Q_k^*} |\tilde{m}(P)b_k(x)| dx.$$

Hence we need only show

$$\int_{x \notin Q_k^*} |\tilde{m}(P)b_k(x)| dx = \int_{x \notin Q_k^*} \left| \int_{Q_k} K(x, y) b_k(y) dy \right| dx \leq C \int_M |b_k| dx.$$

From the double dyadic decomposition (7), we show two estimates of $T_{\lambda, l}(P)b_k(x)$ on the set $\{x \in M : x \notin \mathbf{O}^*\}$:

$$\begin{aligned} \text{(I)} \quad & \|T_{\lambda, l}(P)b_k\|_{L^1(x \notin \mathbf{O}^*)} \leq C(2^l)^{n/2-s} \|b_k\|_{L^1(Q_k)}, \\ \text{(II)} \quad & \|T_{\lambda, l}(P)b_k\|_{L^1(x \notin \mathbf{O}^*)} \leq C(2^l)^{n/2-s_0} \left[\lambda \max_{y, y_0 \in Q_k} \text{dist}(y, y_0) \right] \|b_k\|_{L^1(Q_k)}, \end{aligned}$$

From our observation (8), it suffices to show that for all geodesic balls $B_{R_{\lambda, l}}$ of radius $R_{\lambda, l} = 2^l \lambda^{-1}$, one has the bounds

$$\begin{aligned} \text{(I)'} \quad & \|T_{\lambda, l}(P)b_k\|_{L^1(\{x \notin \mathbf{O}^*\} \cap \mathbf{B}_{R_{\lambda, l}})} \leq C(2^l)^{\frac{n}{2}-s} \|b_k\|_{L^1(Q_k)}, \\ \text{(II)'} \quad & \|T_{\lambda, l}(P)b_k\|_{L^1(\{x \notin \mathbf{O}^*\} \cap \mathbf{B}_{R_{\lambda, l}})} \leq C(2^l)^{\frac{n}{2}-s_0} \left[\lambda \max_{y, y_0 \in Q_k} \text{dist}(y, y_0) \right] \|b_k\|_{L^1(Q_k)}, \end{aligned}$$

To show (I)', using the estimate (a) and Hölder inequality, we get

$$\begin{aligned} \|T_{\lambda, l}(P)b_k\|_{L^1(\{x \notin \mathbf{O}^*\} \cap \mathbf{B}_{R_{\lambda, l}})} & \leq \text{Vol}(B_{R_{\lambda, l}})^{1/2} \|T_{\lambda, l}(P)b_k\|_{L^2} \\ & \leq C(2^l \lambda^{-1})^{n/2} (2^l)^{-s} \lambda^{n/2} \|b_k\|_{L^1} \\ & = C(2^l)^{n/2-s} \|b_k\|_{L^1}. \end{aligned}$$

To show (II)', using the cancellation property $\int_{Q_k} b_k(y) dy = 0$, the estimate (b), and Hölder inequality, we have

$$\begin{aligned} & \|T_{\lambda, l}(P)b_k\|_{L^1(\{x \notin \mathbf{O}^*\} \cap \mathbf{B}_{R_{\lambda, l}})} \\ & \leq \text{Vol}(B_{R_{\lambda, l}})^{1/2} \|T_{\lambda, l}(P)b_k\|_{L^2} \\ & \leq C \left(\frac{2^l}{\lambda} \right)^{\frac{n}{2}} (2^l)^{-s_0} \lambda^{\frac{n}{2}} \left[\lambda \max_{y, y_0 \in Q_k} \text{dist}(y, y_0) \right] \|b_k\|_{L^1(Q_k)} \\ & = C(2^l)^{\frac{n}{2}-s_0} \left[\lambda \max_{y, y_0 \in Q_k} \text{dist}(y, y_0) \right] \|b_k\|_{L^1(Q_k)}. \end{aligned}$$

From our observation (8), and estimates (I), we have

$$\begin{aligned} \sum_{l=-\infty}^{\infty} \|T_{\lambda, l}(P)b_k\|_{L^1(x \notin \mathbf{O}^*)} & \leq C \sum_{2^l \geq c\lambda \text{dist}(x, y)} (2^l)^{n/2-s} \|b_k\|_{L^1(Q_k)} \\ & \leq C_s (\lambda \text{dist}(x, y))^{n/2-s} \|b_k\|_{L^1(Q_k)} \\ & \leq C_s (\lambda R)^{n/2-s} \|b_k\|_{L^1(Q_k)}, \end{aligned}$$

and from $\max_{y, y_0 \in Q_k} \text{dist}(y, y_0) \leq CR$, estimate (II), and $\frac{n}{2} < s_0 < \min\{s, \frac{n}{2} + 1\}$,

$$\begin{aligned} & \sum_{l=-\infty}^{\infty} \|T_{\lambda, l}(P)b_k\|_{L^1(x \notin \mathbf{O}^*)} \\ & \leq C \sum_{2^l \geq c\lambda \text{dist}(x, y)} (2^l)^{\frac{n}{2}-s_0} \left[\lambda \max_{y, y_0 \in Q_k} \text{dist}(y, y_0) \right] \|b_k\|_{L^1(Q_k)} \\ & \leq C_{s_0} (\lambda \text{dist}(x, y))^{\frac{n}{2}-s_0} \left[\lambda \max_{y, y_0 \in Q_k} \text{dist}(y, y_0) \right] \|b_k\|_{L^1(Q_k)} \\ & \leq C_{s_0} (\lambda R)^{\frac{n}{2}+1-s_0} \|b_k\|_{L^1(Q_k)}. \end{aligned}$$

Therefore, we combine the above two estimates to conclude that

$$\begin{aligned} \int_{x \notin Q_k^*} |\tilde{m}(P)b_k(x)| dx & \leq \sum_{j=0}^{\infty} \sum_{l=-\infty}^{\infty} \|T_{2^j, l}(P)b_k\|_{L^1(x \notin \mathbf{O}^*)} \\ & \leq C_s \sum_{2^j R > 1} (2^j R)^{\frac{n}{2}-s} \|b_k\|_1 + C_{s_0} \sum_{2^j R \leq 1} (2^j R)^{\frac{n}{2}+1-s_0} \|b_k\|_1 \\ & \leq C_s \|b_k\|_1. \end{aligned}$$

Hence we have the weak-type $(1, 1)$ estimate on the main term

$$\mu\{x : |\tilde{m}(P)f(x)| > \alpha\} \leq \alpha^{-1} \|f\|_{L^1}.$$

Combining Case 1 and Case 2, we have the weak-type estimate of $m(P)$ and we finish the proof of Theorem 1.2. $\square$

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