A COUNTEREXAMPLE TO BUELER'S CONJECTURE

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ABSTRACT. We give a counterexample to the following conjecture of E. Bueler:
 “The de Rham cohomology of any complete Riemannian manifold is isomor-
 phic to a weighted L^2 cohomology where the weight is the heat kernel.”

1. INTRODUCTION

1.1. Weighted L^2 cohomology. We first describe weighted L^2 cohomology and Bueler's conjecture. For more details we refer to E. Bueler's paper ([2] see also [6]).

Suppose (M, g) is a complete Riemannian manifold and $h \in C^\infty(M)$ is a smooth function. We introduce the measure μ ,

$$d\mu(x) = e^{2h(x)} d\text{vol}_g(x),$$

and the space of L^2_μ differential forms,

$$L^2_\mu(\Lambda^k T^*M) = \{\alpha \in L^2_{loc}(\Lambda^k T^*M), \|\alpha\|_\mu^2 := \int_M |\alpha|^2(x) d\mu(x) < \infty\}.$$

Let $d_\mu^* = e^{-2h} d^* e^{2h}$ be the formal adjoint of the operator $d : C_0^\infty(\Lambda^k T^*M) \rightarrow L^2_\mu(\Lambda^{k+1} T^*M)$. The k^{th} space of (reduced) L^2_μ cohomology is defined by

$$\mathbb{H}_\mu^k(M, g) = \frac{\{\alpha \in L^2_\mu(\Lambda^k T^*M), d\alpha = 0\}}{dC_0^\infty(\Lambda^{k-1} T^*M)} = \frac{\{\alpha \in L^2_\mu(\Lambda^k T^*M), d\alpha = 0\}}{d\mathcal{D}_\mu^{k-1}(d)},$$

where we take the L^2_μ closure and $\mathcal{D}_\mu^{k-1}(d)$ is the domain of d , that is, the space of forms $\alpha \in L^2_\mu(\Lambda^{k-1} T^*M)$ such that $d\alpha \in L^2_\mu(\Lambda^k T^*M)$. Also if $\mathcal{H}_\mu^k(M) = \{\alpha \in L^2_\mu(\Lambda^k T^*M), d\alpha = 0, d_\mu^* \alpha = 0\}$, then we also have $\mathcal{H}_\mu^k(M) \simeq \mathbb{H}_\mu^k(M)$. Moreover if the manifold is compact (without boundary), then the celebrated Hodge-de Rham theorem tells us that these two spaces are isomorphic to $H^k(M, \mathbb{R})$, the real cohomology groups of M . Concerning a complete Riemannian manifold, E. Bueler had made the following interesting conjecture [2].

Conjecture. *Assume that (M, g) is a connected oriented complete Riemannian manifold with Ricci curvature bounded from below. Also, consider for $t > 0$ and $x_0 \in M$, the heat kernel $\rho_t(x, x_0)$ and the heat kernel measure $d\mu(x) = \rho_t(x, x_0) d\text{vol}_g(x)$.*

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Then 0 is an isolated eigenvalue of the self-adjoint operator $\Delta_\mu := dd_\mu^* + d_\mu^*d$ and for any k we have an isomorphism

$$\mathcal{H}_\mu^k(M) \simeq H^k(M, \mathbb{R}).$$

E. Bueler has verified this conjecture in degree $k = 0$, and according to [4] it also holds in degree $k = \dim M$. About the topological interpretation of some weighted L^2 cohomology, there are results of Z.M. Ahmed and D. Strook and more optimal results of N. Yeganefar ([1], [6]). Here we will show that we cannot hope for more.

Theorem 1.1. *In any dimension n , there is a connected oriented manifold M , such that for any complete Riemannian metric on M and any smooth positive measure μ , the natural map*

$$\mathcal{H}_\mu^k(M) \rightarrow H^k(M, \mathbb{R})$$

is not surjective for $k \neq 0, n$.

Actually our example is quite simple: take a compact surface S of genus $g \geq 2$ and let

$$\Gamma \simeq \mathbb{Z} \rightarrow \hat{S} \rightarrow S$$

be a cyclic cover of S . For arbitrary dimension n case, do consider $M = \mathbb{T}^{n-2} \times \hat{S}$, the product of a $(n-2)$ torus with $\hat{S}$.

The idea of the proof is the following. Let $k \neq 0, n$. Then on the one hand, by elliptic regularity, the growth of Δ_μ -harmonic forms is governed by the geometry, and on the other hand, for this manifold there are cohomology classes with arbitrary growth in $H^k(M, \mathbb{R})$; hence the natural map $\mathcal{H}_\mu^k(M) \rightarrow H^k(M, \mathbb{R})$ is not surjective.

2. A TECHNICAL POINT: THE GROWTH OF HARMONIC FORMS

Here we consider a complete Riemannian manifold (M^n, g) and a positive smooth measure $d\mu = e^{2h}d\text{vol}_g$ on it.

Proposition 2.1. *Let $o \in M$ be a fixed point, for $x \in M$, let $r(x) = d(o, x)$ be the geodesic distance between o and x , let $R(x)$ be the maximum of the absolute value of sectional curvature of planes in $T_x M$ and define*

$$m(R) = \max_{r(x) \leq R} \{|\nabla dh|(x) + R(x)\}.$$

There is a constant C_n depending only of the dimension such that if $\alpha \in \mathcal{H}_\mu^k(M)$, then on the ball $r(x) \leq R$

$$e^{h(x)}|\alpha|(x) \leq C_n \frac{e^{C_n m(2R)R^2}}{\sqrt{\text{vol}(B(o, 2R))}} \|\alpha\|_\mu.$$

Remark 1. The referee kindly pointed out to us that there is a constant $C(R)$ which only depends on the metric g and of the function h on $B(o, 2R)$ such that if $\alpha \in \mathcal{H}_\mu^k(M)$, then on the ball $r(x) \leq R$

$$|\alpha|(x) \leq C(R)\|\alpha\|_\mu.$$

This is a consequence of standard elliptic regularity, and in fact this estimate is all that we need.

Proof. If we let $\theta(x) = e^{h(x)}\alpha(x)$, then θ satisfies the equation (see Theorem 4.2 in [2])

$$(dd^* + d^*d)\theta + |dh|^2\theta - 2\nabla dh(\theta) - (\Delta h)\theta = 0,$$

where the Hessian of h acts on k forms by

$$\nabla dh(\theta) = \sum_{i,j} \theta_j \wedge \nabla dh(e_i, e_j) \text{int}_{e_i} \theta,$$

where $\{e_i\}_i$ is a local orthonormal frame and $\{\theta_i\}_i$ is the dual frame. There is $\mathcal{R}$ an expression in the curvature operator of (M, g) , such that we get the Bochner-Weitzenböck formula ([3]) $(dd^* + d^*d)\theta = \nabla^*\nabla\theta + \mathcal{R}(\theta)$. The Kato formula says that the function $u(x) = |\theta|(x)$ satisfies (in the distribution sense) the subharmonic estimate

$$\Delta u \leq |\nabla^*\nabla\theta|,$$

hence we obtain for a certain constant C_n depending only on n ,

$$(1) \quad \Delta u(x) \leq C_n(R(x) + |\nabla dh|(x)) u(x).$$

Now according to L. Saloff-Coste (Theorem 10.4 in [5]), on $B(o, 2R) = \{r(x) < 2R\}$, the ball of radius $2R$, we have a Sobolev inequality: $\forall f \in C_0^\infty(B(o, 2R))$

$$(2) \quad \|f\|_{L^{\frac{2\nu}{\nu-2}}}^2 \leq C_n \frac{R^2 e^{c_n \sqrt{k_R} R}}{(\text{vol}(B(o, 2R)))^{2/\nu}} \|df\|_{L^2}^2,$$

where $-k_R < 0$ is a lower bound for the Ricci curvature on the ball $B(o, 4R)$ and $\nu = \max(3, n)$. With (1) and (2), the Moser iteration scheme implies that for $x \in B(o, R)$,

$$(3) \quad u(x) \leq C_n \frac{e^{C_n m(2R) R^2}}{\sqrt{\text{vol}(B(o, 2R))}} \|u\|_{L^2(B(o, 2R))},$$

from which we easily infer the desired estimate.

Let us explain quickly how we obtain this last estimate. First an integration by parts and a short computation yields that for any Lipschitz function ϕ with compact support and for any $p \geq 2$ we get

$$\int_M \left| d(\phi u^{p/2}) \right|^2 d\text{vol}_g \leq \int_M \left[\frac{p}{2} \phi^2 C_n (R + |\nabla dh|) + |d\phi|^2 \right] u^p d\text{vol}_g.$$

For $R < s < s' \leq 2R$, we apply this inequality for the function ϕ defined as follows:

$$\phi(x) = \begin{cases} 1 & \text{on } B(o, s), \\ (s' - r(x))/(s' - s) & \text{on } B(o, s') \setminus B(o, s), \\ 0 & \text{on } M \setminus B(o, s'), \end{cases}$$

and we obtain

$$(4) \quad \left(\int_{B(o, s)} u^p \frac{\nu}{\nu-2} \right)^{1-\frac{2}{\nu}} \leq C_n \frac{R^2 e^{c_n \sqrt{k_R} R}}{(\text{vol}(B(o, 2R)))^{2/\nu}} \left[\frac{p}{2} C_n m(2R) + \frac{1}{(s' - s)^2} \right] \int_{B(o, s')} u^p.$$

For $k \geq 0$, let

$$R_l := 2R - \sum_{j=1}^{l-1} \frac{R}{2^j} \text{ and } \kappa = \frac{\nu-2}{\nu}.$$

The inequality (4) applied to $s' = R_{l-1}$, $s = R_l$, and $p = 2\kappa^{1-l}$ yields the following inequality:

$$H_l \leq 2^{l\kappa^{l-1}} A^{\kappa^{l-1}} H_{l-1},$$

where

$$H_l = \|u\|_{L^{\frac{2}{\kappa^l}}(B(o, R_l))}$$

and

$$A^2 = C_n \frac{R^2 e^{c_n \sqrt{\kappa} R}}{(\text{vol}(B(o, 2R)))^{2/\nu}} \left[C_n m(2R) + \frac{1}{R^2} \right] \leq a_n \frac{e^{b_n m(2R) R^2}}{(\text{vol}(B(o, 2R)))^{2/\nu}}$$

for certain constants a_n and b_n depending only on the dimension n . Hence we get

$$H_l \leq 2^{\sum_{j=1}^l j \kappa^{j-1}} A^{\sum_{j=1}^l \kappa^{j-1}} H_0.$$

Letting $l \rightarrow \infty$, we obtain the estimation (3). $\square$

3. JUSTIFICATION OF THE EXAMPLE AND FURTHER COMMENTS

3.1. Justification. Now, we consider the manifold $M = \mathbb{T}^{n-2} \times \hat{S}$ which is a cyclic cover of $\mathbb{T}^{n-2} \times S$; let γ be a generator of the covering group. We assume M is endowed with a complete Riemannian metric and a smooth measure $d\mu = e^{2h} d\text{vol}_g$. For every $k \in \{1, \dots, n-1\}$ we have a k -cycle c such that $\gamma^l(c) \cap c = \emptyset$ for any $l \in \mathbb{Z} \setminus \{0\}$ and a closed k -form ψ with compact support such that $\int_c \psi = 1$ and such that $(\text{support } \psi) \cap (\text{support } (\gamma^l)^* \psi) = \emptyset$ for any $l \in \mathbb{Z} \setminus \{0\}$. Let $a = (a_p)_{p \in \mathbb{N}}$ be a nonzero sequence of real number. Then the k -form $\psi_a = \sum_{p \in \mathbb{N}} a_p (\gamma^p)^* \psi$ represents a nonzero k cohomology class of M , indeed $\int_{\gamma^p c} \psi_a = a_p$. We define $R_p = \max\{r(\gamma^l(x)), x \in c, l = 0, \dots, p\}$. Then if the de Rham cohomology class of ψ_a contains $\alpha \in \mathcal{H}_\mu^k(M)$, then according to (2.1), we will have

$$|a_p| = \left| \int_{\gamma^p c} \alpha \right| \leq M_p \|\alpha\|_\mu,$$

where

$$M_p = \text{vol}_g(\gamma^p(c)) C_n \frac{e^{C_n m(2R_p) R_p^2}}{\sqrt{\text{vol}(B(o, 2R_p))}} \max_{r(x) \leq R_p} e^{-h(x)}.$$

As a consequence, for the sequence defined by $a_p = (M_p + 1)2^p$, ψ_a cannot be represented by an element of $\mathcal{H}_\mu^k(M)$.

3.2. Further comments. Our counterexample does not exclude that this conjecture holds for a complete Riemannian metric with bounded curvature, positive injectivity radius on the interior of a compact manifold with boundary.

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