SUBSEQUENCES OF THE HAAR BASIS CONSISTING OF FULL LEVELS IN H_p FOR $0 < p < \infty$

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ABSTRACT. We find a condition to be satisfied by strictly increasing sequences of natural numbers guaranteeing that corresponding subsequences of the Haar basis (consisting of full levels) are equivalent or span isomorphic spaces. This applies in particular to Hardy spaces for $0 < p < \infty$. We also construct a continuum of greedy nonequivalent bases in H_p .

In this paper we study (non-)isomorphic quasi-Banach and Banach spaces spanned by subsequences of the Haar basis consisting of full levels and (non-)equivalence of such subbases. We restrict the discussion to H_p spaces for $0 < p < \infty$. However, for $p > 1$, $H_p = L_p$ with equivalent norm, so in this case results presented here apply to L_p .

We will use standard Banach space notation as explained in detail in [6] or [4]. The basic reference for facts about quasi-Banach spaces is [2]. Let us only point out that $|\cdot|$ may denote, depending on the context, one of the following: absolute value of the number, cardinality of a finite set or the Lebesgue measure on $[0, 1]$.

Let $\mathcal{D}$ denote the family of all dyadic subintervals of the interval $[0, 1]$. By $\mathcal{D}_n$ we mean $\{I \in \mathcal{D} : |I| = 2^{-n}\}$, that is, the set of all dyadic intervals from $\mathcal{D}$ of length 2^{-n} . With each interval $I \in \mathcal{D}$, $I = [\frac{k}{2^n}, \frac{k+1}{2^n}]$ we associate the Haar function h_I defined by the formula

$$h_I(t) = \begin{cases} 2^{n/p} & \text{if } \frac{2k}{2^{n+1}} \leq t < \frac{2k+1}{2^{n+1}}, \\ -2^{n/p} & \text{if } \frac{2k+1}{2^{n+1}} \leq t < \frac{2(k+1)}{2^{n+1}}, \\ 0 & \text{otherwise.} \end{cases}$$

We define H_p as the space of all distributions of the form $f = \sum_{I \in \mathcal{D}} a_I h_I$ for which the following expression is finite:

$$\|f\| = \left[\int_0^1 \left(\sum_{I \in \mathcal{D}} |a_I h_I(t)|^2 \right)^{p/2} \right]^{1/p}.$$

For $1 \leq p < \infty$, $\|\cdot\|$ is actually a norm. When $0 < p < 1$ the above expression defines a quasi-norm (p norm). It is known that H^p spaces are isomorphic to classical Hardy spaces of analytic functions on the unit disc.

Let $(e_m)_{m \in M}$, $(u_n)_{n \in N}$ be basic sequences in H_p . These basic sequences are said to be equivalent if there exists a 1-1 and onto map $\phi : M \rightarrow N$ such that

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the map $e_m \mapsto u_{\phi(m)}$ extends by linearity to an isomorphism. For abbreviation we then write $(e_m)_{m \in M} \sim (u_n)_{n \in N}$.

If B is a subset of natural numbers $\mathbb{N}$, then by $\mathcal{D}(B)$ we will denote the set $\bigcup_{n \in B} \mathcal{D}_n$. Also, for $n \in \mathbb{N}$ and real C we will write $\mathcal{D}(n - C, n + C)$ instead of $\mathcal{D}(\mathbb{N} \cap (n - C, n + C))$. If $\mathcal{A} \subset \mathcal{D}$, then $H_p(\mathcal{A})$ will denote the closed linear span of $(h_I)_{I \in \mathcal{A}}$ in H_p .

Let us mention two important results.

Lemma 1. *Suppose $0 < p < 2$ and let C be a given constant. Assume that we have disjoint sets $\mathcal{A}_k \subset \mathcal{D}(n_k - C, n_k + C)$ for some strictly increasing sequence of integers $(n_k)_{k=1}^\infty$ and let*

$$x_k = \left\| \sum_{I \in \mathcal{A}_k} h_I \right\|^{-1} \sum_{I \in \mathcal{A}_k} h_I .$$

If $\inf |\text{supp } x_k| > 0$, then the sequence $(x_k)_{k=1}^\infty$ is equivalent to the unit vector basis in ℓ_2 .

This lemma is an immediate corollary of [7, Lemma 3.3]. Actually this lemma and the next one are formulated in [7] for $0 < p < 1$, but the proofs are valid for $p < 2$.

Lemma 2. *Suppose $0 < p < 2$. Let $x_k = \sum_{I \in \mathcal{A}_k} b_I h_I$ be a sequence of vectors in H_p such that $\|x_k\| = 1$ for all k and $\mathcal{A}_k$ are disjoint subsets of $\mathcal{D}$. If $\liminf_k |\bigcup_{I \in \mathcal{A}_k} I| = 0$, then (x_k) has a subsequence equivalent to the unit vector basis in ℓ_p .*

For any subset $\mathcal{A} \subset \mathcal{D}$ and $s = 1, 2, 3, \dots$, by $(\mathcal{A})^{(s)}$ we will denote the union of s disjoint copies of $\mathcal{A}$. Clearly it is the index set of a natural basis in $H_p(\mathcal{A})^s$.

The following proposition is our main tool and will be used in this paper many times.

Proposition 3. *Let μ_n be a strictly increasing sequence of natural numbers and let $C > 0$, $\alpha > 0$ be given constants and let $s \geq 1$ be an integer. Suppose that $\mathcal{A}_n \subset \mathcal{D}(\mu_n - C, \mu_n + C)$ and $|\mathcal{A}_n| \geq \alpha |\mathcal{D}_{\mu_n}|$. Suppose also that a 1-1 map $\pi : \bigcup \mathcal{A}_n \rightarrow \mathcal{D}^{(s)}$ establishes the equivalence of the basic sequence $\{h_I : I \in \bigcup \mathcal{A}_n\}$ with $\{h_I : I \in \pi(\bigcup \mathcal{A}_n)\}$ both considered in H_p with $p < 2$. Then for any sequence β_n tending to infinity we have*

$$(1) \quad \lim_{n \rightarrow \infty} \frac{|\{I \in \mathcal{A}_n : \pi(I) \in (\mathcal{D}(\mu_n - \beta_n, \mu_n + \beta_n))^{(s)}\}|}{|\mathcal{A}_n|} = 1 .$$

Proof. If (1) does not hold, then there exists a $\delta > 0$ such that for an infinite set of natural numbers (to save the subscripts we will write as if this set consists of all natural numbers) we have

$$(2) \quad \frac{|\{I \in \mathcal{A}_n : \pi(I) \in (\mathcal{D}(\mu_n - \beta_n, \mu_n + \beta_n))^{(s)}\}|}{|\mathcal{A}_n|} \leq 1 - \delta .$$

Clearly $|(\mathcal{D}(0, \mu_n - \beta_n))^{(s)}| \leq 2s \cdot 2^{\mu_n - \beta_n}$, so

$$(3) \quad \frac{|(\mathcal{D}(0, \mu_n - \beta_n))^{(s)}|}{|\mathcal{A}_n|} \leq \frac{2s \cdot 2^{\mu_n - \beta_n}}{\alpha 2^{\mu_n}} \rightarrow 0 .$$

From (2) and (3) we see that there are sets $\mathcal{B}_n \subset \mathcal{A}_n$ (for big n 's) such that $|\mathcal{B}_n| > \frac{\delta}{2} |\mathcal{A}_n| \geq \frac{\alpha\delta}{2} 2^{\mu_n}$, and $\pi(\mathcal{B}_n) \subset (\mathcal{D}(\mu_n + \beta_n, \infty))^{(s)}$. Let us consider the sequence of vectors

$$x_n = \left\| \sum_{I \in \mathcal{B}_n} h_I \right\|^{-1} \sum_{I \in \mathcal{B}_n} h_I.$$

Since

$$|\text{supp } x_n| \geq \frac{1}{2C} |\mathcal{B}_n| 2^{-(\mu_n + C)} \geq \frac{\alpha\delta}{4C} 2^{-C}$$

from Lemma 1 we infer that $(x_n)_{n>N}$ is equivalent to the unit vector basis in ℓ_2 . On the other hand,

$$|\text{supp } T_\pi(x_n)| \leq |\mathcal{B}_n| 2^{-(\mu_n + \beta_n)} \leq |\mathcal{A}_n| 2^{-(\mu_n + \beta_n)} \leq 2 \cdot 2^{\mu_n + C} 2^{-(\mu_n + \beta_n)} \rightarrow 0,$$

so from Lemma 2 we easily get that $(T_\pi(x_n))$ has a subsequence equivalent to the unit vector basis in ℓ_p . This contradiction completes the proof. $\square$

In the rest of the paper we will be dealing with two sequences of natural numbers (k_n) and (l_n) . By $\mathbf{k}$ (resp. $\mathbf{l}$) we will always denote the set of elements of the sequence (k_n) (resp. (l_n)).

Theorem 1. *Suppose that $(k_n)_{n=1}^\infty$ and $(l_n)_{n=1}^\infty$ are two increasing sequences of natural numbers and assume that $\lim_{n \rightarrow \infty} k_{n+1} - k_n = \infty$. If basic sequences $\{h_I : I \in \mathcal{D}(\mathbf{l})\}$ and $\{h_I : I \in \mathcal{D}(\mathbf{k})\}$ are equivalent in some H_p , $0 < p < \infty$, $p \neq 2$, then the symmetric difference $\mathbf{l} \div \mathbf{k}$ is finite.*

Proof. Since the case $2 < p < \infty$ follows by duality from the case $1 < p < 2$ we will consider only $0 < p < 2$. Let us fix a map $\pi : \mathcal{D}(\mathbf{k}) \rightarrow \mathcal{D}(\mathbf{l})$ which establishes the equivalence. Let us prove two lemmas.

Lemma 4. *Under the assumptions of Theorem 1 there exists a constant C such that $\mathbf{l} \subset \bigcup_{n=1}^\infty (k_n - C, k_n + C)$.*

Proof. If such a C does not exist, we can find a subsequence (l_{n_s}) such that $\text{dist}(l_{n_s}, \mathbf{k})$ tends to infinity. Applying Proposition 3 for $\mathcal{A}_s = \mathcal{D}_{l_{n_s}} \mu_s = l_{n_s}$, $C = 1/2$ and $\beta_s = 0.3 \text{dist}(l_{n_s}, \mathbf{k})$ and the map $\pi^{-1}|_{\bigcup_{s=1}^\infty \mathcal{D}_{l_{n_s}}}$ we get the contradiction, because in this situation the set in the numerator of (1) is empty. $\square$

Lemma 5. *Under the assumptions of Theorem 1,*

$$(4) \quad \lim_{n \rightarrow \infty} \frac{|\mathcal{D}(\mathbf{l} \cap \bigcup_{n=1}^\infty (k_n - C, k_n + C))|}{|\mathcal{D}_{k_n}|} = 1$$

where C is the constant from Lemma 4.

Proof. We apply Proposition 3 twice, once to $\mathcal{A}_n = \mathcal{D}(\mathbf{l} \cap (k_n - C, k_n + C))$, $\beta_n = 0.3 \min(k_{n+1} - k_n, k_n, k_{n-1})$ and the map π^{-1} yielding

$$(5) \quad \lim_{n \rightarrow \infty} \frac{|\{I \in \mathcal{A}_n : \pi^{-1}(I) \in \mathcal{D}_{k_n}\}|}{|\mathcal{A}_n|} = 1$$

and the second time to $\mathcal{A}_n = \mathcal{D}_{k_n}$, β_n as above and the map π yielding

$$(6) \quad \lim_{n \rightarrow \infty} \frac{|\{I \in \mathcal{D}_{k_n} : \pi(I) \in \mathcal{D}(\mathbf{l} \cap (k_n - C, k_n + C))\}|}{|\mathcal{D}_{k_n}|} = 1.$$

Since π is 1-1 (4) follows when we divide (6) by (5). $\square$

To complete the proof of Theorem 1, by Lemma 4, it suffices to observe that (4) is possible only if $\mathbf{l} \cap (k_n - C, k_n + C) = \{k_n\}$ except for the finite number of n 's. $\square$

In the case $0 < p < 1$ we will prove the same conclusion under apparently weaker assumptions.

Theorem 2. *Suppose that $(k_n)_{n=1}^\infty$ and $(l_n)_{n=1}^\infty$ are two increasing sequences of natural numbers and assume that $\lim_{n \rightarrow \infty} k_{n+1} - k_n = \infty$. If $0 < p < 1$ and spaces $H_p(\mathcal{D}(\mathbf{k}))$ and $H_p(\mathcal{D}(\mathbf{l}))$ are isomorphic, then the symmetric difference $\mathbf{l} \div \mathbf{k}$ is finite.*

Remark. By a theorem of Gamlen-Gaudet ([1]) and [5, Theorem 1] it is essential to assume in Theorem 2 that $0 < p < 1$.

Before we proceed let us state an almost obvious consequence of [7, Theorem 2.9].

Theorem 3. *Let $0 < p < 1$ and let $\mathcal{A}, \mathcal{B} \subset \mathcal{D}$ be such that $H_p(\mathcal{A})$ is isomorphic to $H_p(\mathcal{B})$. Then there exist two subsets $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$ and 1-1 maps $\psi_i : \mathcal{A}_i \rightarrow \mathcal{B}$ $i = 1, 2$ such that*

- 1) *each ψ_i establishes an equivalence between basic sequences $\{h_I : I \in \mathcal{A}_i\}$ and $\{h_I : I \in \psi_i(\mathcal{A}_i)\}$,*
- 2) $\mathcal{A}_1 \cup \mathcal{A}_2 = \mathcal{A}$,
- 3) $\psi_1(\mathcal{A}_1) \cup \psi_2(\mathcal{A}_2) = \mathcal{B}$.

Proposition 6. *Under the assumptions of Theorem 2 the set of all natural numbers can be partitioned into four sets $\mathbb{N}_1, \mathbb{N}_2, \mathbb{N}_3, \mathbb{N}_4$ so that*

- 1) *there exists a constant C such that $\mathbf{l} \subset \bigcup_{n=1}^\infty (k_n - C, k_n + C)$,*
- 2) *if $n \in \mathbb{N}_1$, then $\mathbf{l} \cap (k_n - C, k_n + C) = k_n + 1$,*
- 3) *if $n \in \mathbb{N}_2$, then $k_n \in \mathbf{l} \cap (k_n - C, k_n + C) \subset (k_n - C, k_n]$,*
- 4) *if $n \in \mathbb{N}_3$, then $k_n - 1 \in \mathbf{l} \cap (k_n - C, k_n + C) \subset (k_n - C, k_n - 1]$,*
- 5) *the set $\mathbb{N}_4$ is finite.*

The proof of this proposition uses Proposition 3 much in the same way as the proof of Theorem 1 but instead of equivalence we will use Theorem 3.

Proof. From Theorem 3 we get two subsets $\mathcal{A}_1, \mathcal{A}_2 \subset \bigcup_{n=1}^\infty \mathcal{D}_{k_n}$ such that $\mathcal{A}_1 \cup \mathcal{A}_2 = \bigcup_{n=1}^\infty \mathcal{D}_{k_n}$ and maps $\psi_i : \mathcal{A}_i \rightarrow \bigcup_{n=1}^\infty \mathcal{D}_{k_n}$ which establish equivalence between basic sequences $\{h_I : I \in \mathcal{A}_i\}$ and $\{h_I : I \in \psi_i(\mathcal{A}_i)\}$. Moreover $\psi_1(\mathcal{A}_1) \cup \psi_2(\mathcal{A}_2) = \mathcal{D}(\mathbf{l})$. Clearly we can split $\mathbf{l}$ into two subsets $\mathbf{l}_1$ and $\mathbf{l}_2$ so that $|\psi_i(\mathcal{A}_i) \cap \mathcal{D}_{l_s}| \geq \frac{1}{2}|\mathcal{D}_{l_s}|$ for $l_s \in \mathbf{l}_i$. So 1) follows from Proposition 3 applied to each of the sequences separately.

Now let us assume that

$$|\psi_1(\mathcal{A}_1) \cap \mathcal{D}(\mathbf{l} \cap (k_n - C, k_n + C))| \geq \frac{1}{2}|\mathcal{D}(\mathbf{l} \cap (k_n - C, k_n + C))|$$

for each n , where C is the constant from 1). To be precise we can once more split the integers into two sets so that this would hold for ψ_1 on one set and for ψ_2 on the other. To save on indexing we will work only on one set.

Applying Proposition 3 to ψ_1^{-1} and $\beta_n = 0.3 \min(k_{n+1} - k_n, k_n - k_{n-1})$ we get

$$\lim_{n \rightarrow \infty} \frac{|\{I \in \psi_1(\mathcal{A}_1) \cap \mathcal{D}(k_n - C, k_n + C) : \psi_1^{-1}(I) \in \mathcal{D}_{k_n}\}|}{|\psi_1(\mathcal{A}_1) \cap \mathcal{D}(k_n - C, k_n + C)|} = 1.$$

This gives

$$\liminf_{n \rightarrow \infty} \frac{|\mathcal{D}_{k_n}|}{\frac{1}{2}|\mathcal{D}(\mathbf{1} \cap (k_n - C, k_n + C))|} \geq 1.$$

From this we see that only in a finite number of cases is a number s , $k_n + 2 \leq s \leq k_n + C$, in $\mathbf{1}$ and also that if $k_n + 1 \in \mathbf{1}$, then (except for a finite number of cases) no other integer from $(k_n - C, k_n + C)$ belongs to $\mathbf{1}$. This allows us to define $\mathbb{N}_1$ and $\mathbb{N}_2$ so that 2) and 3) hold. To show 4) let us assume (once more in general we need to work with two subsequences) that $|\mathcal{A}_1 \cap \mathcal{D}_{k_n}| \geq \frac{1}{2}|\mathcal{D}_{k_n}|$ for all n . We apply Proposition 3 to ψ_1 and we obtain

$$\lim_{n \rightarrow \infty} \frac{|\{I \in \mathcal{A}_1 \cap \mathcal{D}_{k_n} : \psi_1(I) \in \mathcal{D}(\mathbf{1} \cap (k_n - C, k_n + C))\}|}{|\mathcal{A}_1 \cap \mathcal{D}_{k_n}|} = 1,$$

which gives

$$\liminf_{n \rightarrow \infty} \frac{|\mathcal{D}(\mathbf{1} \cap (k_n - C, k_n + C))|}{\frac{1}{2}|\mathcal{A}_1 \cap \mathcal{D}_{k_n}|} \geq 1.$$

If $\mathbf{1} \cap [k_n, k_n + C) = \emptyset$ (i.e., we are not already in the considered cases 2) and 3)), then this implies that $(k_n - 1) \in \mathbf{1} \cap (k_n - C, k_n + C)$ except for a finite number of cases, so we can define $\mathbb{N}_3$. Obviously the set $\mathbb{N}_4$ consists of the finite number of exceptional cases discussed above. $\square$

Lemma 7. *Let $(l_n)_{n=1}^\infty$ be a strictly increasing sequence of natural numbers. Then*

$$H_p(\mathcal{D}(\{l_n\}_{n=1}^\infty)) \oplus H_p(\mathcal{D}(\{l_n\}_{n=1}^\infty)) \sim H_p(\mathcal{D}(\{l_n + 1\}_{n=1}^\infty)).$$

This lemma follows by a simple linear change of variables.

Proof of Theorem 2. We define the following spaces:

$$\begin{aligned} X &= H_p\left(\bigcup_{n \in \mathbb{N}_1} \mathcal{D}_{k_n}\right), \\ Y &= H_p\left(\bigcup_{n \in \mathbb{N}_2} \mathcal{D}_{k_n}\right), \\ V &= H_p\left(\bigcup_{n \in \mathbb{N}_3} \mathcal{D}_{k_n-1}\right), \\ Z_Y &= H_p\left(\bigcup_{n \in \mathbb{N}_2} \mathcal{D}(\mathbf{1} \cap (k_n - C, k_n - 1])\right), \\ Z_V &= H_p\left(\bigcup_{n \in \mathbb{N}_3} \mathcal{D}(\mathbf{1} \cap (k_n - C, k_n - 2])\right). \end{aligned}$$

We denote $Z = Z_Y \oplus Z_V$. To prove Theorem 2 we will need one more result.

Lemma 8. *Let $0 < p < 2$. If $\mathcal{A} \cup \mathcal{B}$ is an infinite family of dyadic intervals and the set $\mathcal{A} \div \mathcal{B}$ is finite, then $(h_I)_{I \in \mathcal{A}} \sim (h_J)_{J \in \mathcal{B}}$ and*

$$H_p(\mathcal{A}) \sim H_p(\mathcal{B}).$$

Proof. By Lemma 2 we can find $\mathcal{C} \subset \mathcal{A} \cap \mathcal{B}$ such that $(h_I)_{I \in \mathcal{C}}$ is equivalent to the unit vector basis in ℓ_p . Let π be a bijection of $\mathcal{C}' = \mathcal{C} \cup \mathcal{A} \setminus \mathcal{B}$ on $\mathcal{C} \cup \mathcal{B} \setminus \mathcal{A}$. We define a bijection $\tau : \mathcal{A} \rightarrow \mathcal{B}$ which is the identity on $(\mathcal{A} \cap \mathcal{B}) \setminus \mathcal{C}'$ and equals π on $\mathcal{C}'$. Linear extension of the function $h_I \mapsto h_{\tau(I)}$ is an isomorphism of $H_p(\mathcal{A})$ on $H_p(\mathcal{B})$ since H_p is p -convex, so $(h_I)_{I \in \mathcal{A}} \sim (h_J)_{J \in \mathcal{B}}$. $\square$

Since $\mathbb{N}_4$ is finite from Lemmas 7 and 8 and Proposition 6 we get

$$\begin{aligned} H_p(\mathcal{D}(\mathbf{l})) &\sim X^2 \oplus Y \oplus V \oplus Z, \\ H_p(\mathcal{D}(\mathbf{k})) &\sim X \oplus Y \oplus V^2, \end{aligned}$$

so from our assumption we get

$$(7) \quad X^2 \oplus Y \oplus V \oplus Z \sim X \oplus Y \oplus V^2.$$

Using (7) we get

$$\begin{aligned} &X \oplus Y \oplus V^4 \\ &\sim (X \oplus Y \oplus V^2) \oplus V^2 \sim (X^2 \oplus Y \oplus V \oplus Z) \oplus V^2 \\ &\sim (X \oplus Y \oplus V^2) \oplus X \oplus Z \oplus V \sim (X^2 \oplus Y \oplus V \oplus Z) \oplus X \oplus Z \oplus V \\ &\sim (X \oplus Y \oplus V^2) \oplus X^2 \oplus Z^2 \sim (X^2 \oplus Y \oplus V \oplus Z) \oplus X^2 \oplus Z^2 \\ &\sim X^4 \oplus Y \oplus V \oplus Z^3. \end{aligned}$$

Using Lemma 8 we can represent the space $X \oplus Y \oplus V^4$ as the space spanned by full levels of the Haar system. More precisely using definitions of our spaces we see that $X \oplus Y \oplus V^4$ is isomorphic with

$$(8) \quad H_p\left(\bigcup_{n \in \mathbb{N}_1} \mathcal{D}_{k_n}\right) \oplus H_p\left(\bigcup_{n \in \mathbb{N}_2} \mathcal{D}_{k_n}\right) \oplus H_p\left(\bigcup_{n \in \mathbb{N}_3} \mathcal{D}_{k_n+1}\right)$$

while $X^4 \oplus Y \oplus V \oplus Z^3$ is isomorphic with

$$(9) \quad H_p\left(\bigcup_{n \in \mathbb{N}_1} \mathcal{D}_{k_n+2}\right) \oplus H_p\left(\bigcup_{n \in \mathbb{N}_2} \mathcal{D}_{k_n}\right) \oplus H_p\left(\bigcup_{n \in \mathbb{N}_3} \mathcal{D}_{k_n-1}\right) \oplus Z^3.$$

We view this space as a subspace of $H_p \oplus (H_p)^3$ spanned by full levels of the Haar system. If the spaces given in (8) and (9) are isomorphic and the set $\mathbb{N}_1$ is infinite, then from Theorem 3 there exists an infinite subset $\mathbb{N}'$ of $\mathbb{N}_1$ and subsets $\mathcal{A}_n \subset \mathcal{D}_{k_n+2}$ for $n \in \mathbb{N}'$ such that $|\mathcal{A}_n| \geq \frac{1}{2}|\mathcal{D}_{k_n+2}|$ and a map ψ which establishes the equivalence of $\{h_I : I \in \bigcup_{n \in \mathbb{N}'} \mathcal{A}_n\}$ with a subset of a natural basis of the space given in (8). But this contradicts Proposition 3. Thus the space X is finite dimensional and (7) takes the form

$$(10) \quad Y \oplus V \oplus Z_Y \oplus Z_V \sim Y \oplus V^2.$$

But $Y \oplus V^2 \sim H_p(\bigcup_{n \in \mathbb{N}_2 \cup \mathbb{N}_3} \mathcal{D}_{k_n})$, so from Proposition 6 point 2) applied to $(k_n)_{n \in \mathbb{N}_2 \cup \mathbb{N}_3}$ (so that $V^2 \sim H_p(\bigcup_{n \in \mathbb{N}_1} \mathcal{D}_{k_n})$ with respect to the new partition of $\mathbb{N}$) we infer that Z_V is finite dimensional, so (10) takes the form

$$(11) \quad Y \oplus V \oplus Z_Y \sim Y \oplus V^2.$$

From this by easy decomposition we get

$$(12) \quad Y \oplus V^k \sim Y \oplus V \oplus Z_Y^{k-1}$$

for each $k = 1, 2, \dots$. If we take $k = 2^{C+2} + 1$ on the right-hand side we get a space spanned by full levels of the Haar system and from Theorem 3 and Proposition 6 we get Z_Y is finite dimensional. If we take $k = 4$ from Theorem 3 and Proposition 6 we get that V is finite dimensional. Thus the only remaining part is Y which proves the theorem. $\square$

Corollary 9. *Let $0 < p < 1$. For every strictly increasing sequence of natural numbers such that*

$$\sup_n (k_{n+1} - k_n) = \infty$$

we have

$$H_p\left(\bigcup_{n=1}^{\infty} \mathcal{D}_{k_n}\right) \not\sim H_p\left(\bigcup_{n=1}^{\infty} \mathcal{D}_{k_n}\right) \oplus H_p\left(\bigcup_{n=1}^{\infty} \mathcal{D}_{k_n}\right).$$

Proof. Let $X = H_p\left(\bigcup_{n=1}^{\infty} \mathcal{D}_{k_n}\right)$. If $X \sim X \oplus X$, then $X \sim X^8$. Since

$$\limsup_n (k_{n+1} - k_n) = \infty$$

hence there is an infinite strictly increasing subsequence $l_1, l_2, l_3, \dots$ of (k_n) subject to the condition

$$\lim_{n \rightarrow \infty} (l_{2n} - l_{2n-1}) = \infty,$$

where $l_{2n-1} = k_{m-1}$ if and only if $l_{2n} = k_m$, for all $m \geq 2$. Since by Lemma 7,

$$X \sim X^8 \sim H_p\left(\bigcup_n \mathcal{D}_{k_n+3}\right),$$

by an argument similar to the one used in the proof of Lemma 4, the sequence $\{l_{(2n-1)+3} : n \in \mathbb{N}\}$ must be finite, so $(k_{n+1} - k_n)$ is bounded. $\square$

Corollary 10. *Let $(m_n)_n$ be a strictly increasing sequence of natural numbers. If $X = H_p\left(\bigcup_n \mathcal{D}_{m_n}\right)$ and $X \sim X \oplus X$, then X is isomorphic to H_p .*

Proof. Since

$$X = H_p\left(\bigcup_n \mathcal{D}_{m_n}\right)$$

and

$$X \sim X \oplus X \sim H_p\left(\bigcup_n \mathcal{D}_{m_n+1}\right),$$

$\sup_n |m_{n+1} - m_n| < \infty$ by Corollary 9. Now [7, Theorem 3.6] gives the claim. $\square$

Recall that a biorthogonal system in H_p is a family $(x_n, x_n^*)_{n \in F} \subset H_p \oplus H_p^*$ such that $x_n^*(x_m)$ equals 0 if $n \neq m$ and equals 1 if $n = m$; here F is any countable index set. We fix a biorthogonal system $(x_n, x_n^*)_{n \in F}$ such that $\text{span}(x_n)_{n \in F} = H_p$, $\inf_{n \in F} \|x_n\| > 0$ and $\sup_{n \in F} \|x_n^*\| < \infty$. This implies that for each $x \in H_p$ we have $\lim_{n \rightarrow \infty} x_n^*(x) = 0$. For each $x \in H_p$ and $m = 1, 2, \dots$ we define

$$\mathcal{G}_m(x) = \sum_{n \in A} x_n^*(x) x_n$$

where $A \subset F$ is a set of cardinality m such that $|x_n^*(x)| \geq |x_k^*(x)|$ whenever $n \in A$ and $k \notin A$. Let us recall ([3]) the definition of the best m -term approximation. For $x \in H_p$ and $m = 1, 2, \dots$ we put

$$\sigma_m(x) = \inf \left\{ \|x - \sum_{n \in A} a_n x_n\| : |A| \leq m \text{ and } a_n \text{'s are scalars} \right\}.$$

A basis $(x_n, x_n^*)_{n \in F}$ is called greedy if there exists a constant C such that for every $x \in H_p$ we have

$$\|x - \mathcal{G}_m(x)\| \leq C \sigma_m(x).$$

We call a normalized basis $(x_n, x_n^*)_{n \in F}$ democratic if there is a constant $C > 0$ such that for each $m \in \mathbb{N}$ and for all subsets P, Q of F , $|P| = |Q| = m$ and

$$\left\| \sum_{n \in P} x_n \right\| \leq C \left\| \sum_{n \in Q} x_n \right\|.$$

We mention the characterization of greedy bases given in [3]: *a normalized basis of Banach space is greedy if and only if it is democratic and unconditional*. The Haar basis is greedy in H_p for $1 \leq p < \infty$ [3]. Let us focus our attention on sets $\mathbf{k}^{(\alpha)}$ of natural numbers $k_n^{(\alpha)} = [n^\alpha]$ for real $\alpha > 1$. By a theorem of Gamlen-Gaudet [1] and [5, Theorem 1], for $1 \leq p < \infty$, there is an isomorphism $i^{(\alpha)}$ of $H_p(\mathcal{D}(\mathbf{k}^{(\alpha)}))$ onto H_p so that $(i^{(\alpha)}(h_I))_{I \in \mathcal{D}(\mathbf{k}^{(\alpha)})}$ is a basis in H_p equivalent to $(h_I)_{I \in \mathcal{D}(\mathbf{k}^{(\alpha)})}$. On the other hand, $\mathbf{k}^{(\alpha)} \div \mathbf{k}^{(\beta)}$ is not finite for $1 < \alpha < \beta$; thus by Theorem 1 we have proved

Theorem 4. *For $1 \leq p \neq 2 < \infty$ there is a continuum of greedy non-equivalent bases in H_p .*

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