RESTRICTIONS OF BOUNDED LINEAR OPERATORS: CLOSED RANGE

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ABSTRACT. Let T be a bounded linear operator on a Banach space X , and let W be a subspace of X which is a Banach space and T -invariant. Denote by T_W the restriction of T to W . This paper explores the questions:

- If the range of T is closed, under what conditions is the range of T_W closed?
- If the range of T_W is closed, under what conditions is the range of T closed?

1. THE BASIC SETUP

Throughout, X is a Banach space with norm $\|x\|_X$, W is a subspace of X such that W is a Banach space with norm $\|w\|_W$, and W is continuously embedded in X (i.e., $\exists M > 0$ such that $M\|w\|_W \geq \|w\|_X$ for all $w \in W$). We denote the algebra of all bounded linear operators defined on a normed linear space Y by $B(Y)$. For $T \in B(Y)$, let $\mathbf{N}(T)$ and $\mathbf{R}(T)$ be the null space and range of T , respectively. When W is invariant for $T \in B(X)$ [$T(W) \subseteq W$], we denote the restriction of T to W by T_W . In this case, it is straightforward to check, using the Closed Graph Theorem, that $T_W \in B(W)$.

We are interested in the basic questions:

When W is invariant for $T \in B(X)$, (1) under what conditions is it true that $\mathbf{R}(T)$ closed in $X \implies \mathbf{R}(T_W)$ is closed in W ?; (2) under what conditions is it true that $\mathbf{R}(T_W)$ closed in $W \implies \mathbf{R}(T)$ is closed in X ? We are especially interested in conditions that give answers to either of the questions (1) or (2) for all operators of the form $\lambda - T$ and $\lambda - T_W$, $\lambda \neq 0$.

The results of this paper augment the author's paper on the spectral theory of extensions of bounded linear operators in [B1].

2. REMARKS ON THE SITUATION

WHERE W IS A CLOSED SUBSPACE OF X

In this section we assume that W is always a closed subspace of X . We start with a simple example. Let $T \in B(X)$ have the properties that $\mathbf{N}(T) \neq \{0\}$ and $\mathbf{R}(T)$ is not closed in X . Setting $W = \mathbf{N}(T)$, we have $\mathbf{R}(T_W)$ is closed, but by assumption, $\mathbf{R}(T)$ is not closed.

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Example 1. Assume that $T \in B(X)$ has the properties that $\mathbf{R}(T)$ is closed in X , but $\mathbf{R}(T^2)$ is not closed in X (we outline how to construct such an operator below). Set $W = \mathbf{R}(T)$. Then $\mathbf{R}(T_W) = \mathbf{R}(T^2)$ is not closed in W .

Let H be a separable Hilbert space with closed subspaces J and K such that $J + K$ is not closed; there is an example given in [H, section 15]. Let P and Q be the orthogonal projections with $\mathbf{R}(P) = K$ and $\mathbf{R}(Q) = J^\perp$. Since K and $J^\perp$ are both separable Hilbert spaces which are not finite dimensional, there exists a linear isometry V mapping $J^\perp$ onto K . Set $T = PVQ \in B(H)$. Now if $Tx = 0$, then $P(VQx) = 0 \implies VQx \in K^\perp \implies VQx = 0 \implies Qx = 0$ (since V is an isometry on $J^\perp$ and $Qx \in J^\perp$). Therefore, $x \in J$. This shows that $\mathbf{N}(T) = J$. Clearly, $\mathbf{R}(T) = K$. Thus, T has closed range and $\mathbf{R}(T) + \mathbf{N}(T) = K + J$, which is not closed.

Claim. $\mathbf{R}(T^2)$ is not closed.

To verify the Claim, first note that $T(\mathbf{R}(T) + \mathbf{N}(T)) = \mathbf{R}(T^2)$. Also, if $Tx \in \mathbf{R}(T^2)$, then $Tx = T^2y$ for some y . Therefore $x - Ty \in \mathbf{N}(T)$, so $x = Ty + (x - Ty) \in \mathbf{R}(T) + \mathbf{N}(T)$. This shows that the inverse image $T^{-1}[\mathbf{R}(T^2)] = \mathbf{R}(T) + \mathbf{N}(T)$. Since $\mathbf{R}(T) + \mathbf{N}(T)$ is not closed, it follows that $\mathbf{R}(T^2)$ is not closed.

We note the basic criterion: $T \in B(X)$ has closed range in X if and only if $\exists \gamma > 0$ such that $\|T(x)\|_X \geq \gamma\|x + \mathbf{N}(T)\|_q$ for all $x \in X$ (here $\|x + \mathbf{N}(T)\|_q$ is the usual quotient norm on $X/\mathbf{N}(T)$). Also, we use the standard terminology: $T \in B(X)$ is bounded below if $\exists \gamma > 0$ such that $\|T(x)\|_X \geq \gamma\|x\|_X$ for all $x \in X$. As is well known, T is bounded below exactly when $\mathbf{N}(T) = \{0\}$ and $\mathbf{R}(T)$ is closed. Clearly, when $T \in B(X)$ is bounded below and W is a closed T -invariant subspace of X , then T_W is bounded below.

3. THE MAIN RESULTS

In this section we give some answers to the questions stated in the introductory section, section 1.

Proposition 2. Assume that $S \in B(X)$ and W is S -invariant. The statements (a) and (b) are equivalent:

- (a) $S^{-1}[W] \equiv \{x \in X : S(x) \in W\} = W + \mathbf{N}(S)$;
- (b) $\mathbf{R}(S_W) = \mathbf{R}(S) \cap W$.

When (b) holds, $\mathbf{R}(S)$ closed $\implies \mathbf{R}(S_W)$ is closed.

Proof. Suppose that (a) holds. Now always $\mathbf{R}(S_W) \subseteq \mathbf{R}(S) \cap W$. We prove the reverse inclusion. Suppose $y \in X$ and $S(y) = w \in W$. Then by (a), $y = v + z$ where $v \in W$ and $z \in \mathbf{N}(S)$. Then $w = S(v)$, so $w \in \mathbf{R}(S_W)$.

Now suppose that (b) is true. If $y \in X$ and $S(y) = w \in W$, then there exists $v \in W$ such that $S(v) = S(y)$. Then $y = v + (y - v)$ and $(y - v) \in \mathbf{N}(S)$. Thus, (a) holds.

Assume that (b) holds and that $\mathbf{R}(S)$ is closed. Suppose that $\{w_n\} \subseteq W$, $w_0 \in W$, and $\|S(w_n) - w_0\|_W \rightarrow 0$. By our standing hypothesis, $\|S(w_n) - w_0\|_X \rightarrow 0$. Then $w_0 \in \mathbf{R}(S) \cap W$. By (b), $w_0 = S(v)$ for some $v \in W$. This proves that $\mathbf{R}(S_W)$ is closed. $\square$

Let $P = \{S \in B(X) : S(W) \subseteq W \text{ and for some } n \geq 1, S^n(X) \subseteq W\}$.

Proposition 3. Assume that $T \in P$, and fix n , $n \geq 1$, such that $T^n(X) \subseteq W$. Then for all $\lambda \neq 0$, $(\lambda - T)^{-1}[W] = W$. It follows that

(#) for all $\lambda \neq 0$, $\mathbf{R}(\lambda - T)$ closed $\implies \mathbf{R}(\lambda - T_W)$ is closed.

Proof. Fix $\lambda \neq 0$. Set $S = \lambda^{-1}T$, so $S^n(X) \subseteq W$. Suppose $x \in X$ and $(I - S)x = w \in W$. Now $I - S^n = (I - S)p(S)$, where $p(S) = (I + S + \dots + S^{n-1})$. Therefore, $(I - S^n)x = p(S)(I - S)x = p(S)w \in W$. Since $S^n x \in W$, we have $x \in W$. This proves $(\lambda - T)^{-1}[W] = (I - S)^{-1}[W] = W$.

Then the assertion (#) follows from Proposition 2. $\square$

Note that Example 1 shows that Proposition 3 need not hold for the case where $\lambda = 0$.

Proposition 4. Assume that $S \in B(X)$, W is S -invariant, and $S(X) \subseteq W$. If $\lambda \neq 0$ and $Z \equiv \mathbf{R}(\lambda - S_W)$ is closed in W , then $\mathbf{R}(\lambda - S)$ is closed.

Proof. First note that $S \in B(X, W)$ (this follows from the Closed Graph Theorem and the easily established fact that $S : X \rightarrow W$ is closed). We may assume that $\lambda = 1$, so our hypothesis is that $Z \equiv \mathbf{R}(I - S_W)$ is closed in W . For convenience, set $\mathbf{N} = \mathbf{N}(I - S_W)$, and also note that $\mathbf{N} = \mathbf{N}(I - S)$ (this follows from $S(X) \subseteq W$). Define $(I - S_W) : W/\mathbf{N} \rightarrow Z$ by $(I - S_W)(w + \mathbf{N}) = (I - S_W)w$. Let $L : Z \rightarrow W/\mathbf{N}$ be the bounded inverse of $(I - S_W)$ (the Open Mapping Theorem), so

$$L(I - S_W)(w + \mathbf{N}) = w + \mathbf{N} \quad \text{for all } w \in W.$$

Now we complete the proof that $\mathbf{R}(I - S)$ is closed by showing that the linear map $(I - S)x \rightarrow x + \mathbf{N}$ from $\mathbf{R}(I - S)$ onto $X/\mathbf{N}$ is continuous. Assume that $\{x_n\} \subseteq X$ and $\|x_n - S(x_n)\|_X \rightarrow 0$. Then $\|(I - S_W)(S(x_n))\|_W = \|S(x_n) - S(x_n)\|_W \rightarrow 0$. Thus, $S(x_n) + \mathbf{N} = L(I - S_W)(S(x_n) + \mathbf{N}) \rightarrow 0$ in $W/\mathbf{N}$, and therefore also in $X/\mathbf{N}$. It follows that

$$x_n + \mathbf{N} = [x_n - S(x_n) + \mathbf{N}] + [S(x_n) + \mathbf{N}] \rightarrow 0 \text{ in } X/\mathbf{N}. \quad \square$$

We note that [B2, Theorem 5] could be used to prove Proposition 4. Here we have chosen to give a direct self-contained argument.

Lemma 5. Assume that $T \in B(X)$ and $T(W) \subseteq W$. Set $V = T^{-1}[W]$, so $W \subseteq V \subseteq X$. Define a norm on V by $\|v\|_V = \|v\|_X + \|Tv\|_W$.

- (a) $\|v\|_V$ is a complete norm on V ;
- (b) W is continuously embedded in V , and V is continuously embedded in X ;
- (c) V is T -invariant and $T(V) \subseteq W$.

Proof. The proofs of the assertions (a), (b), and (c) are all straightforward. We provide the proof of (a).

Assume that $\{v_n\} \subseteq V$ is a Cauchy sequence in the norm $\|\cdot\|_V$. Then $\{v_n\}$ is Cauchy in $\|\cdot\|_X$ and $\{T(v_n)\}$ is Cauchy in $\|\cdot\|_W$. There exist $v_0 \in X$ and $w_0 \in W$ such that

$$\|v_n - v_0\|_X \rightarrow 0 \text{ and } \|T(v_n) - w_0\|_W \rightarrow 0.$$

Therefore, $\|T(v_n) - T(v_0)\|_X \rightarrow 0$ and $\|T(v_n) - w_0\|_X \rightarrow 0$. Thus, $T(v_0) = w_0$. This proves that $\|v_n - v_0\|_V = \|v_n - v_0\|_X + \|T(v_n - v_0)\|_W \rightarrow 0$. $\square$

Theorem 6. Assume that $T \in P$, and fix n , $n \geq 1$, such that $T^n(X) \subseteq W$. For all $\lambda \neq 0$:

- (1) $\mathbf{R}(\lambda - T)$ closed $\iff \mathbf{R}(\lambda - T_W)$ is closed;
- (2) $\mathbf{N}(\lambda - T) = \mathbf{N}(\lambda - T_W)$;
- (3) $\lambda - T$ is bounded below $\iff \lambda - T_W$ is bounded below.

Proof. For convenience, set $V_0 = W$, $V_n = X$, and $\|w\|_0 = \|w\|_W$ for $w \in W$.

Let $V_1 = T^{-1}[V_0]$ be equipped with the complete norm $\|v\|_1 = \|v\|_X + \|Tv\|_0$. By Lemma 5, $V_0 \subseteq V_1 \subseteq X$, the embeddings are continuous, and $T(V_1) \subseteq V_0$. Continuing inductively in this fashion, using Lemma 5, construct V_k , $1 \leq k \leq n-1$, with norm $\|v\|_k = \|v\|_X + \|Tv\|_{k-1}$, having the properties:

- (a) $(V_k, \|v\|_k)$ is a Banach space, $1 \leq k \leq n-1$;
- (b) $W = V_0 \subseteq V_1 \subseteq V_2 \subseteq \dots \subseteq V_{n-1} \subseteq V_n = X$, and each of the embeddings are continuous;
- (c) $T(V_k) \subseteq V_{k-1}$ for $1 \leq k \leq n$.

Let T_k be the restriction of T to V_k , $0 \leq k \leq n$. It follows from Propositions 3 and 4 that for all $\lambda \neq 0$ and $1 \leq k \leq n$, $\mathbf{R}(\lambda - T_{k-1})$ closed $\iff \mathbf{R}(\lambda - T_k)$ is closed. Therefore, $\mathbf{R}(\lambda - T)$ closed $\iff \mathbf{R}(\lambda - T_W)$ is closed. This proves (1).

To verify (2), first note that $\mathbf{N}(\lambda - T_W) \subseteq \mathbf{N}(\lambda - T)$. Now suppose that $\lambda \neq 0$ and $x \in \mathbf{N}(\lambda - T)$. Then $Tx = \lambda x$, $T^n x = \lambda^n x$, and since $T^n(X) \subseteq W$, $x \in W$. Therefore, $\mathbf{N}(\lambda - T) = \mathbf{N}(\lambda - T_W)$.

(3) follows from (1) and (2). $\square$

For $S \in B(X)$, let $\sigma_{ap}(S) = \{\lambda : \exists \{x_n\} \subseteq X \text{ with } \|x_n\|_X = 1 \text{ for all } n, \text{ and } \|(\lambda - S)x_n\|_X \rightarrow 0\}$. It is well known that $\sigma_{ap}(S) = \{\lambda : \lambda - S \text{ is not bounded below}\}$; see [L:T, p. 282]. We set $\sigma'_{ap}(S) = \sigma_{ap}(S) \setminus \{0\}$ (set difference).

Corollary 7. Assume that $T \in P$. Then $\sigma'_{ap}(T) = \sigma'_{ap}(T_W)$.

We note that for $T \in P$, many of the results in [B1] apply to T . This follows by using the results in [B1] in combination with the construction of the spaces V_k with the properties (a), (b), and (c) as in the proof of Theorem 6. For example: Let $\Phi(X)$ denote the set of all Fredholm operators in $B(X)$. Using [B1, Theorem 4] and the construction mentioned above we have for all $\lambda \neq 0$,

$$\lambda - T \in \Phi(X) \iff \lambda - T_W \in \Phi(W).$$

4. GENERALIZED INVERSES

An operator $S \in B(X)$ has a g-inverse (generalized inverse) $G \in B(X)$ if $SGS = S$. There is a well-known characterization of when S has a g-inverse [L:T, Theorem 12.9, p. 251]:

$S \in B(X)$ has a g-inverse $\iff \mathbf{R}(S)$ is closed and has a closed complement in X , and $\mathbf{N}(S)$ has a closed complement.

In particular, when X is a Hilbert space, $S \in B(X)$ has a g-inverse exactly when $\mathbf{R}(S)$ is closed.

Theorem 8. Assume that $T \in P$. For $\lambda \neq 0$, if $\lambda - T$ has a g-inverse in $B(X)$, then $\lambda - T_W$ has a g-inverse in $B(W)$. Conversely, if W is dense in X , and $\lambda - T_W$ has a g-inverse in $B(W)$, then $\lambda - T$ has a g-inverse in $B(X)$.

Proof. First we verify that it suffices to prove the result when $\lambda = 1$. Note that for $\lambda \neq 0$, $\lambda^{-1}T \in P$, and that $(I - \lambda^{-1}T)$ has a g-inverse G , so

$$(I - \lambda^{-1}T)G(I - \lambda^{-1}T) = (I - \lambda^{-1}T),$$

if and only if $(\lambda - T)(\lambda^{-1}G)(\lambda - T) = (\lambda - T)$.

Now fix m , $m \geq 1$, such that $T^m(X) \subseteq W$. Assume that $I - T$ has a g-inverse $G \in B(X)$. By [B3, Prop. 2], there exists a polynomial $p(x)$ with $p(0) = 0$ such that $F = I + p(T) + T^mGT^m$ is a g-inverse of $I - T$. Clearly, $F(W) \subseteq W$ (since

$T^m(X) \subseteq W$). Conversely, assume that W is dense in X , and $I - T_W$ has a g -inverse $G \in B(W)$. Then as noted above, $F = I + p(T_W) + T_W^m G T_W^m \in B(W)$ is a g -inverse of $I - T_W$. Now $T^m \in B(X, W)$, so $T^m G T^m \in B(X, W)$. Then as W is continuously embedded in X , F has an extension $\bar{F} = I + p(T) + T^m G T^m \in B(X)$. Note that since W is dense in X and $(I - T_W)F(I - T_W) = (I - T_W)$ on W , then $(I - T)\bar{F}(I - T) = (I - T)$ on X . $\square$

Corollary 9. *Assume that $T \in P$, and $\lambda \neq 0$. Consider the following four properties:*

- (1) $\mathbf{R}(\lambda - T)$ is closed;
- (2) $\lambda - T$ has a g -inverse in $B(X)$;
- (3) $\lambda - T_W$ has a g -inverse in $B(W)$;
- (4) $\mathbf{R}(\lambda - T_W)$ is closed.

If X is a Hilbert space, or W is dense in X and W is a Hilbert space, then all four conditions are equivalent.

5. SOME EXAMPLES

In this section we present some samples of standard types of integral operators to which the results of section 3 apply.

Example 10. When Ω is a locally compact Hausdorff space, let $BC(\Omega)$ denote the space of all functions $f : \Omega \rightarrow \mathbf{C}$ such that f is bounded and continuous on Ω , equipped with the complete norm, $\|f\|_u = \sup\{|f(\omega)| : \omega \in \Omega\}$. Let μ be a regular Borel measure on Ω . The Banach space $L^1(\Omega, \mu) \cap BC(\Omega)$ is an important space in the theory of integral operators that act on spaces of continuous functions; see [J, 10.1 and 12.4].

Now let G be a locally compact Hausdorff topological group. We assume that G is unimodular and has a fixed Haar measure. For $f \in L^1(G)$, define T_f on any $g \in L^p(G)$ by $T_f(g) = f * g$ (convolution). Using the convolution results summarized in [H:R, pp. 297-298], we have:

- (1) If $f \in L^1(G) \cap L^2(G)$, then $T_f(L^2(G)) \subseteq L^2(G) \cap BC(G)$; and
- (2) If $f \in L^1(G) \cap L^2(G)$, then $f * f \in L^1(G) \cap BC(G)$, and $T_f^2 = T_{f*f}$ so $T_f^2(L^1(G)) \subseteq L^1(G) \cap BC(G)$.

Thus, the results of this paper apply to the convolution operators T_f in these two cases. For example, in (2) we can take $X = L^1(G)$ and $W = L^1(G) \cap BC(G)$. For $f \in L^1(G) \cap L^2(G)$, we have $T_f^2(X) \subseteq W$.

Example 11. For $n \geq 1$, $BC^n(0, \infty)$ is the set of all functions $f : (0, \infty) \rightarrow \mathbf{C}$ such that for $0 \leq k \leq n$, $f^{(k)}$ (the k^{th} derivative of f ; $f^{(0)} = f$) exists and $f^{(k)} \in BC(0, \infty)$. The natural complete norm on $BC^n(0, \infty)$ is $\|f\| = \sum_{k=0}^n \|f^{(k)}\|_u$.

Fix $n \geq 1$, and assume that $g \in BC^{n-1}(0, \infty) \cap L^1(0, \infty)$, $k \in BC^n(0, \infty)$. Set

$$T(f)(x) = k(x) \int_0^x g(t)f(t)dt, \quad f \in BC(0, \infty), \quad x > 0.$$

If $f \in BC^m(0, \infty)$ for some m , $0 \leq m < n$ (here $BC^0(0, \infty) = BC(0, \infty)$), then it is easily verified that $T(f) \in BC^{m+1}(0, \infty)$. Set $W = BC^n(0, \infty)$. We have $T^n(BC(0, \infty)) \subseteq W$, so the results of this paper apply to the operator T .

The same statements apply to much more general Volterra-type operators on $BC(0, \infty)$.

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