HOLES IN THE SPECTRUM OF FUNCTIONS GENERATING AFFINE SYSTEMS

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ABSTRACT. Given a $d \times d$ expansive dilation matrix D , a measurable set $E \subset \mathbb{R}^d$ is called a D^t -dilation generator of $\mathbb{R}^d$ if $\mathbb{R}^d$ is tiled (modulo null sets) by the collection $\{(D^t)^j E, j \in \mathbb{Z}\}$. Our main goal in this paper is to prove certain results relating the support of the Fourier transform of functions generating a wavelet or orthonormal affine system associated with the dilation D to an arbitrary set E which is a D^t -dilation generator of $\mathbb{R}^d$.

1. INTRODUCTION

Let d be a fixed positive integer. We denote by $\mathbb{Z}^d$ the set of points with integral coordinates in $\mathbb{R}^d$ and by I_d the $d \times d$ identity matrix. The Fourier transform $\hat{f}$ of a function $f \in L^1(\mathbb{R}^d)$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle \xi, x \rangle} dx, \quad \xi \in \mathbb{R}^d,$$

where $\langle \cdot, \cdot \rangle$ denotes the usual dot product in $\mathbb{R}^d$. If E is a measurable subset of $\mathbb{R}^d$, we will denote by $|E|$ its Lebesgue measure. A $d \times d$ matrix D with real entries is called a *dilation matrix* if the moduli of all its eigenvalues are greater than 1. Throughout this paper, D^t denotes the transpose of D . If B is a $d \times d$ matrix with real entries and E is a subset of $\mathbb{R}^d$, we denote by BE the set

$$BE = \{Bx, x \in E\}.$$

If $B = \lambda I_d$, we use the notation λE instead. We point out that, unless otherwise specified, relations between two measurable sets in $\mathbb{R}^d$, such as equality or inclusion, are always understood up to a subset of zero Lebesgue measure. Similarly, equality or inequality between functions is always understood in the “pointwise almost-everywhere” sense. Given a $d \times d$ dilation matrix D and any function f defined on $\mathbb{R}^d$, we define

$$\text{Supp}(f) := \{x \in \mathbb{R}^d : f(x) \neq 0\}.$$

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The *spectrum* of $f \in L^2(\mathbb{R}^d)$ is the set $\text{Supp}(\hat{f})$ which is defined up to a set of zero measure. We also associate with f the collection $\{f_{j,k}\}$ defined by

$$f_{j,k}(\cdot) := |\det D|^{\frac{1}{2}} f(D^j \cdot -k), \quad j \in \mathbb{Z}, \quad k \in \mathbb{Z}^d,$$

which we call the *affine system* with dilation D associated with f . More generally, the affine system with dilation D associated with a finite set of functions $\{f^1, \dots, f^m\}$ is the collection $\{f_{j,k}^l : 1 \leq l \leq m, j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$. Given a measurable set S in $\mathbb{R}^d$, a collection $\{S_\alpha : \alpha \in \Gamma\}$ of measurable subsets of $\mathbb{R}^d$ is called a *partition* of S if

$$\bigcup_{\alpha \in \Gamma} S_\alpha = S \quad \text{and} \quad S_{\alpha_1} \cap S_{\alpha_2} = \emptyset \quad \text{whenever} \quad \alpha_1, \alpha_2 \in \Gamma, \alpha_1 \neq \alpha_2.$$

Two measurable sets S_1 and S_2 in $\mathbb{R}^d$ are called $\mathbb{Z}^d$ -translation congruent if there exist measurable partitions $\{S_{1,k}\}_{k \in \mathbb{Z}^d}$ of S_1 and $\{S_{2,k}\}_{k \in \mathbb{Z}^d}$ of S_2 such that $S_{2,k} = S_{1,k} + k$ for all $k \in \mathbb{Z}^d$. Similarly, S_1 and S_2 are called D^t -congruent if there exist measurable partitions $\{S'_{1,j}\}_{j \in \mathbb{Z}}$ of S_1 and $\{S'_{2,j}\}_{j \in \mathbb{Z}}$ of S_2 such that $S'_{2,j} = (D^t)^j S'_{1,j}$ for all $j \in \mathbb{Z}$. It is easy to check that S_1 and S_2 are $\mathbb{Z}^d$ -translation congruent if and only if

$$\sum_{k \in \mathbb{Z}^d} \chi_{S_1}(\cdot - k) = \sum_{k \in \mathbb{Z}^d} \chi_{S_2}(\cdot - k),$$

while S_1 and S_2 are D^t -congruent if and only if

$$\sum_{j \in \mathbb{Z}} \chi_{S_1}((D^t)^j \cdot) = \sum_{j \in \mathbb{Z}} \chi_{S_2}((D^t)^j \cdot),$$

where χ_F denotes the characteristic function of the set F . A measurable set E in $\mathbb{R}^d$ is called a $\mathbb{Z}^d$ -translation generator if the collection $\{E + k : k \in \mathbb{Z}^d\}$ is a partition of $\mathbb{R}^d$. This is clearly equivalent to

$$\sum_{k \in \mathbb{Z}^d} \chi_E(\cdot - k) = 1.$$

Similarly, E is called a D^t -dilation generator if $\{(D^t)^j E : j \in \mathbb{Z}\}$ is a partition of $\mathbb{R}^d$ or, equivalently, if

$$\sum_{j \in \mathbb{Z}} \chi_E((D^t)^j \cdot) = 1.$$

It is clear that any two $\mathbb{Z}^d$ -translation generators are $\mathbb{Z}^d$ -translation congruent and that any two D^t -dilation generators are D^t -congruent.

A function $\psi \in L^2(\mathbb{R}^d)$ is called a *D-wavelet* if the associated affine system $\{\psi_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$. Similarly, a finite subset $\{\psi^l : 1 \leq l \leq m\}$ of $L^2(\mathbb{R}^d)$ is called a *D-wavelet collection* if the affine system $\{\psi_{j,k}^l : 1 \leq l \leq m, j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ forms an orthonormal basis for $L^2(\mathbb{R}^d)$. It was proved in [1, 3, 16] that if $\{\psi^l : 1 \leq l \leq m\}$ of $L^2(\mathbb{R}^d)$ is a $2I_d$ -wavelet collection associated with an MRA, then $m = 2^d - 1$. Further, it was also proved in [1, 16] that every $2I_d$ -wavelet collection $\{\psi^l : 1 \leq l \leq m\}$ of $L^2(\mathbb{R}^d)$, whose members satisfy a weak smoothness and decay condition on the Fourier transform side, must come from an MRA. Because of these results, the existence of single dyadic wavelets in $L^2(\mathbb{R}^d)$ with $d > 1$ was unknown until the mid-1990s. Their existence for an arbitrary dilation matrix D was finally proved by Dai, Larson and Speegle in [12] (see also [13]) via the construction of associated wavelet sets. Several

constructions of such wavelet sets can be found in [4, 5, 6, 14, 15, 17, 18, 19, 21]. For a $d \times d$ dilation matrix D , a set E in $\mathbb{R}^d$ is called a D -wavelet set for $L^2(\mathbb{R}^d)$ if ψ defined by $\hat{\psi} = \frac{\chi_E}{\sqrt{|E|}}$ is a D -wavelet. It is well known that E is a D -wavelet set if and only if E is both a $\mathbb{Z}^d$ -translation and a D^t -dilation generator of $\mathbb{R}^d$. This implies, in particular, that $|E| = 1$ if E is a D -wavelet set. Additional results on D -wavelet sets can be found in [9, 10, 12].

An at most countable subset $\{\psi^l : l \in \Gamma\}$ of $L^2(\mathbb{R}^d)$ is said to generate a *normalized tight D -wavelet frame* for $L^2(\mathbb{R}^d)$ if

$$\|f\|^2 = \sum_{l \in \Gamma} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} |\langle f, \psi_{j,k}^l \rangle|^2, \quad f \in L^2(\mathbb{R}^d).$$

The following two results ([21, Theorem 5 and 6]) due to P. Soardi and D. Weiland show that, on one hand, the support of the Fourier transform of a wavelet cannot have a large hole near the origin, while, on the other hand, if the support of its Fourier transform is not too large, it necessarily must have a hole which is an interval of length one centered at the origin. Related results concerning holes for MRA and non-MRA wavelets can also be found in [2, 6, 8].

Proposition 1.1. *Let ψ be a 2-wavelet for $L^2(\mathbb{R})$. Then $|\text{Supp}(\hat{\psi}) \cap [-c, c]| > 0$ for $c > \frac{1}{2}$.*

Proposition 1.2. *Let ψ be a 2-wavelet for $L^2(\mathbb{R})$ with $|\hat{\psi}| = \chi_K$ for some $K \subset [-1, 1]$. Then $|K \cap [-\frac{1}{2}, \frac{1}{2}]| = 0$.*

Following the same general philosophy, one of us (Li) proved analogous two-dimensional results ([20, Theorem 1.1 and 1.2]). Before stating them, we will define the following sets. If $a, b > 0$ are parameters, we let

$$E^+(a, b) = \{(\xi_1, \xi_2)^t : |\xi_1| \leq a, |\xi_1 + \xi_2| \leq b\}$$

and

$$E^-(a, b) = \{(\xi_1, \xi_2)^t : |\xi_2| \leq a, |\xi_1 - \xi_2| \leq b\}.$$

Proposition 1.3. *Let $M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, and let ψ be an M -wavelet. Then, for $c > \frac{1}{2}$, we have $|\text{Supp}(\hat{\psi}) \cap E^+(c, c)| > 0$ and $|\text{Supp}(\hat{\psi}) \cap E^-(c, c)| > 0$.*

Proposition 1.4. *Let $M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and let ψ be an M -wavelet satisfying that $|\hat{\psi}| = \chi_K$ for some measurable set K .*

- (a) *If $K \subset E^+(\frac{1}{2}, 1)$, then $|K \cap E^+(\frac{1}{2}, \frac{1}{2})| = 0$.*
- (b) *If $K \subset E^-(\frac{1}{2}, 1)$, then $|K \cap E^-(\frac{1}{2}, \frac{1}{2})| = 0$.*

Note the following: the set

$$E_1 := [-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$$

is a 2-wavelet set for $L^2(\mathbb{R})$ (the so-called *Shannon wavelet-set*), while the intervals $[-1, 1]$ and $[-\frac{1}{2}, \frac{1}{2}]$ appearing in Proposition 1.2 can be expressed as

$$[-1, 1] = \bigcup_{j \leq 0} 2^j E_1, \quad [-\frac{1}{2}, \frac{1}{2}] = \bigcup_{j < 0} 2^j E_1.$$

At the same time, the interval $[-c, c]$ with $c > \frac{1}{2}$ appearing in Proposition 1.1 can be written as

$$[-c, c] = \lambda \bigcup_{j < 0} 2^j E_1,$$

with $\lambda > 1$. Similarly, both

$$E^+ = \left\{ (\xi_1, \xi_2)^t : |\xi_1| \leq \frac{1}{2}, \frac{1}{2} \leq |\xi_1 + \xi_2| \leq 1 \right\}$$

and

$$E^- = \left\{ (\xi_1, \xi_2)^t : |\xi_2| \leq \frac{1}{2}, \frac{1}{2} \leq |\xi_1 - \xi_2| \leq 1 \right\}$$

are M -wavelet sets in $L^2(\mathbb{R}^2)$ for the matrix $M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ appearing in Proposition 1.3 and Proposition 1.4. Furthermore, in relation with Proposition 1.4, we have

$$E^+(\frac{1}{2}, 1) = \bigcup_{j \leq 0} M^j E^+ \quad \text{and} \quad E^-(\frac{1}{2}, 1) = \bigcup_{j \leq 0} M^j E^-,$$

while

$$E^+(\frac{1}{2}, \frac{1}{2}) = \bigcup_{j < 0} M^j E^+ \quad \text{and} \quad E^-(\frac{1}{2}, \frac{1}{2}) = \bigcup_{j < 0} M^j E^-.$$

Similarly, for the sets appearing in Proposition 1.3, whenever $c > \frac{1}{2}$,

$$E^+(c, c) = \lambda \bigcup_{j < 0} M^{-j} E^+ \quad \text{and} \quad E^-(c, c) = \lambda \bigcup_{j < 0} M^{-j} E^-$$

for $\lambda = 2c > 1$. Inspired by these observations, it is natural for us to ask whether or not the previous results can be extended to the case of a wavelet associated with a general expansive matrix in $\mathbb{R}^d$ and of any wavelet set corresponding to the same dilation matrix. Our main goal will be to give a positive answer to that question. In fact our results make even weaker assumptions since, instead of wavelet sets, they are stated for sets that are D^t -dilation generators of measure larger (resp. smaller) than or equal to one. The main results of this paper are stated below. We formulate the two next theorems for finitely generated wavelet systems as opposed to singly generated ones. Note that, in the first one of these, the dilation matrix is assumed to be *integral*, i.e. its entries must be integers.

Theorem 1.5. *Let D be a $d \times d$ integral dilation matrix, let E be a D^t -dilation generator with $|E| \geq 1$, and let $\{\psi^l : 1 \leq l \leq m\}$ be a D -wavelet collection. Then, to every τ with $\tau > m^{\frac{1}{d}}$, there corresponds an l with $1 \leq l \leq m$ such that*

$$\left| \text{Supp}(\hat{\psi}^l) \cap \left(\tau \bigcup_{j < 0} (D^t)^j E \right) \right| > 0.$$

Obviously, the previous remarks show that Proposition 1.1 and Proposition 1.3 are particular cases of Theorem 1.5. We can also use it to obtain the following one-dimensional result which generalizes Proposition 1.1.

Corollary 1.6. *If ψ is a 2-wavelet for $L^2(\mathbb{R})$, then $|\text{Supp}(\hat{\psi}) \cap [\alpha, \beta]| > 0$ for $-1 < \alpha < 0$, and $\beta > 1 + \alpha$.*

Indeed, if we take $\gamma = \frac{\beta}{\beta-\alpha}$ and $E = [2\gamma-2, \gamma-1] \cup [\gamma, 2\gamma]$, then E is a 2-wavelet set for $L^2(\mathbb{R})$ since it is both a $\mathbb{Z}$ -translation and a 2-dilation generator of $\mathbb{R}$. Note that $[\alpha, \beta] = \tau \bigcup_{j \leq 0} 2^j E$ with $\tau = \beta - \alpha > 1$. It follows from Theorem 1.5 that $|\text{Supp}(\hat{\psi}) \cap [\alpha, \beta]| > 0$.

The following result concerns orthonormal affine systems having more than one generator and which might not generate the whole space $L^2(\mathbb{R}^d)$.

Theorem 1.7. *Let D be a $d \times d$ dilation matrix, let E be a D^t -dilation generator of measure not greater than 1, and let $\{\psi_{j,k}^l : 1 \leq l \leq m, j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ be an orthonormal affine system with dilation D in $L^2(\mathbb{R}^d)$. Suppose, furthermore, that $m > 1$. Then,*

$$\left| \text{Supp}(\hat{\psi}^l) \cap \left(\bigcup_{j \geq 0} (D^t)^j E \right) \right| > 0,$$

for some l with $1 \leq l \leq m$.

Theorem 1.7 shows that, for any D^t -dilation generator of measure not greater than 1, in particular, for any D -wavelet set E , and any D -wavelet collection $\{\psi^l : 1 \leq l \leq m\}$ with $m > 1$, it is impossible that $\text{Supp}(\hat{\psi}^l)$ be a subset of $\bigcup_{j \leq 0} (D^t)^j E$ for every $1 \leq l \leq m$. Interestingly, this last result can also be obtained from a result of M. Bownik ([7, Theorem 3.7]) on generalized multiresolution analyses (GMRA) applied to the situation where $\hat{V}_1 = L^2(\bigcup_{j \leq 0} (D^t)^j E)$. (See more particularly the remark following Corollary 3.8 in [7].) The following theorem shows that Theorem 1.7 is also true when $m = 1$, except in the case where E is actually a D -wavelet set.

Theorem 1.8. *Let D be a $d \times d$ dilation matrix, let E be a D^t -dilation generator of measure not greater than 1, and let $\{\psi_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ be an orthonormal affine system with dilation D in $L^2(\mathbb{R}^d)$. Suppose, furthermore, that the inclusion $\text{Supp}(\hat{\psi}) \subset \bigcup_{j \leq 0} (D^t)^j E$ holds. Then, E is a D -wavelet set and $|\hat{\psi}| = \chi_E$.*

In relation with this last result, we point out that if a function $\psi \in L^2(\mathbb{R}^d)$ satisfies $|\hat{\psi}| = \chi_E$, for some D -wavelet set E , then ψ is a D -wavelet for $L^2(\mathbb{R}^d)$.

2. PROOFS OF THE MAIN RESULTS

We will need the following lemma.

Lemma 2.1. *Let $\tau > 0$, let D be a $d \times d$ dilation matrix, and let E be a D^t -dilation generator. Define $\phi^{(s)}, s \in \mathbb{Z}^d$, by*

$$\hat{\phi}^{(s)} = \chi_{\tau E \cap ([0, 1]^d + s)}.$$

Then, the collection $\{\phi^{(s)} : s \in \mathbb{Z}^d\}$ generates a normalized tight D -wavelet frame for $L^2(\mathbb{R}^d)$.

Proof. Observing that

$$\hat{\phi}_{j,k}^{(s)}(\xi) = |\det D|^{-\frac{j}{2}} \hat{\phi}^{(s)}((D^t)^{-j} \xi) e^{-2\pi i \langle k, (D^t)^{-j} \xi \rangle}, \quad \xi \in \mathbb{R}^d,$$

we have, for $f \in L^2(\mathbb{R}^d)$

$$\begin{aligned}
& \sum_{s \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \left| \langle f, \phi_{j,k}^{(s)} \rangle \right|^2 \\
&= \sum_{s \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} |\det D|^{-j} \left| \int_{\mathbb{R}^d} \hat{f}(\xi) \overline{\hat{\phi}^{(s)}((D^t)^{-j}\xi)} e^{2\pi i \langle k, (D^t)^{-j}\xi \rangle} d\xi \right|^2 \\
&= \sum_{s \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} |\det D|^j \left| \int_{\mathbb{R}^d} \hat{f}((D^t)^j \xi) \overline{\hat{\phi}^{(s)}(\xi)} e^{2\pi i \langle k, \xi \rangle} d\xi \right|^2 \\
&= \sum_{s \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} |\det D|^j \left| \int_{\tau E \cap ([0, 1]^d + s)} \hat{f}((D^t)^j \xi) e^{2\pi i \langle k, \xi \rangle} d\xi \right|^2.
\end{aligned}$$

Applying Parseval's theorem on the interval $[0, 1]^d + s$, we can rewrite this last expression as

$$\begin{aligned}
& \sum_{s \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}} |\det D|^j \int_{\tau E \cap ([0, 1]^d + s)} \left| \hat{f}((D^t)^j \xi) \right|^2 d\xi \\
&= \sum_{j \in \mathbb{Z}} |\det D|^j \int_{\tau E} \left| \hat{f}((D^t)^j \xi) \right|^2 d\xi \\
&= \sum_{j \in \mathbb{Z}} \int_{\tau (D^t)^j E} \left| \hat{f}(\xi) \right|^2 d\xi = \int_{\mathbb{R}^d} \left| \hat{f}(\xi) \right|^2 d\xi = \|f\|^2,
\end{aligned}$$

which proves the lemma. $\square$

Proof of Theorem 1.5. Suppose that, for some $\tau > 0$,

$$(2.1) \quad \left| \text{Supp}(\hat{\psi}^l) \cap \left(\tau \bigcup_{j < 0} (D^t)^j E \right) \right| = 0, \quad 1 \leq l \leq m.$$

It suffices to show that $\tau \leq m^{\frac{1}{d}}$ under these conditions. Let $\phi^{(s)}$ be defined as in Lemma 2.1. Then

$$\|\phi^{(s)}\|^2 = \|\hat{\phi}^{(s)}\|^2 = |\tau E \cap ([0, 1]^d + s)|$$

and

$$\sum_{s \in \mathbb{Z}^d} \|\phi^{(s)}\|^2 = |\tau E| = \tau^d |E|.$$

Since $\{\psi^l : 1 \leq l \leq m\}$ is a D -wavelet collection, we have

$$\begin{aligned}
 \tau^d |E| &= \sum_{s \in \mathbb{Z}^d} \|\phi^{(s)}\|^2 = \sum_{s \in \mathbb{Z}^d} \sum_{l=1}^m \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \left| \langle \phi^{(s)}, \psi_{j,k}^l \rangle \right|^2 \\
 &= \sum_{s \in \mathbb{Z}^d} \sum_{l=1}^m \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \left| \int_{\mathbb{R}^d} \hat{\phi}^{(s)}(\xi) |\det D|^{-\frac{j}{2}} \overline{\hat{\psi}^l((D^t)^{-j} \xi)} e^{2\pi i \langle k, (D^t)^{-j} \xi \rangle} d\xi \right|^2 \\
 &= \sum_{s \in \mathbb{Z}^d} \sum_{l=1}^m \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} |\det D|^j \left| \int_{\mathbb{R}^d} \hat{\phi}^{(s)}((D^t)^j \xi) \overline{\hat{\psi}^l(\xi)} e^{2\pi i \langle k, \xi \rangle} d\xi \right|^2 \\
 &= \sum_{s \in \mathbb{Z}^d} \sum_{l=1}^m \sum_{j \leq 0} \sum_{k \in \mathbb{Z}^d} |\det D|^j \left| \int_{\mathbb{R}^d} \hat{\phi}^{(s)}((D^t)^j \xi) \overline{\hat{\psi}^l(\xi)} e^{2\pi i \langle k, \xi \rangle} d\xi \right|^2,
 \end{aligned}$$

where in the last equality we used (2.1) and the fact that

$$\text{Supp}(\hat{\phi}^{(s)}((D^t)^j \cdot)) \subset (D^t)^{-j} \tau E, \quad j \in \mathbb{Z}.$$

Also, taking into account the fact that D is an integral matrix and that

$$\mathbb{Z}^d = \{k, k \in \mathbb{Z}^d\} \subset \{D^j k, k \in \mathbb{Z}^d\}, \quad \text{for any } j \leq 0,$$

leads to

$$\begin{aligned}
 \tau^d |E| &\leq \sum_{s \in \mathbb{Z}^d} \sum_{l=1}^m \sum_{j \leq 0} \sum_{k \in \mathbb{Z}^d} |\det D|^j \left| \int_{\mathbb{R}^d} \hat{\phi}^{(s)}((D^t)^j \xi) \overline{\hat{\psi}^l(\xi)} e^{2\pi i \langle D^j k, \xi \rangle} d\xi \right|^2 \\
 &\leq \sum_{s \in \mathbb{Z}^d} \sum_{l=1}^m \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} |\det D|^j \left| \int_{\mathbb{R}^d} \hat{\phi}^{(s)}((D^t)^j \xi) \overline{\hat{\psi}^l(\xi)} e^{2\pi i \langle k, (D^t)^j \xi \rangle} d\xi \right|^2 \\
 &= \sum_{l=1}^m \sum_{s \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \left| \langle \phi_{j,k}^{(s)}, \psi^l \rangle \right|^2.
 \end{aligned}$$

It follows that $\tau^d \leq \sum_{l=1}^m \|\psi^l\|^2 = m$ by Lemma 2.1. The proof is completed. $\square$

Proof of Theorem 1.7. We will argue by contradiction. Suppose that

$$(2.2) \quad \text{Supp}(\hat{\psi}^l) \subset \left(\bigcup_{j \leq 0} (D^t)^j E \right), \quad 1 \leq l \leq m.$$

Let $\|\cdot\|_d$ denote the usual matrix norm of a $d \times d$ matrix. Due to the fact that $\lim_{j \rightarrow -\infty} \|(D^t)^j\|_d^{-\frac{1}{j}} < 1$, there exists a constant $c_0 \geq 1$ such that $\|(D^t)^j\|_d \leq c_0$ for j with $j \leq 0$. Let $c := 2c_0$ and define the sets S_p by

$$S_p = \left[-\frac{1}{2c}, \frac{1}{2c} \right]^d + \frac{p}{c}, \quad p \in \mathbb{Z}^d.$$

Then, $\{S_p : p \in \mathbb{Z}^d\}$ is a partition of $\mathbb{R}^d$, and $(D^t)^j S_p$ is $\mathbb{Z}^d$ -translation congruent to a subset of $[-\frac{1}{2}, \frac{1}{2}]^d$ for every j with $j \leq 0$. Define ψ_E by $\hat{\psi}_E = \chi_E$. Then observing that the collection $\{\hat{\psi}_{j,k}^l : 1 \leq l \leq m, j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ is also an

orthonormal system in $L^2(\mathbb{R}^d)$ leads to

$$\begin{aligned}
1 &\geq \|\hat{\psi}_E\|^2 = \sum_{p \in \mathbb{Z}^d} \int_{\mathbb{R}^d} \left| \hat{\psi}_E(\xi) \chi_{S_p}(\xi) \right|^2 d\xi \\
&\geq \sum_{p \in \mathbb{Z}^d} \sum_{l=1}^m \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \left| \langle \hat{\psi}_E \chi_{S_p}, \hat{\psi}_{j,k}^l \rangle \right|^2 \\
&= \sum_{p \in \mathbb{Z}^d} \sum_{l=1}^m \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \left| \int_{E \cap S_p} |\det D|^{-\frac{j}{2}} \overline{\hat{\psi}^l((D^t)^{-j} \xi)} e^{2\pi i \langle k, (D^t)^{-j} \xi \rangle} d\xi \right|^2 \\
&= \sum_{p \in \mathbb{Z}^d} \sum_{l=1}^m \sum_{j \in \mathbb{Z}} |\det D|^j \sum_{k \in \mathbb{Z}^d} \left| \int_{(D^t)^{-j}(E \cap S_p)} \overline{\hat{\psi}^l(\xi)} e^{2\pi i \langle k, \xi \rangle} d\xi \right|^2 \\
&= \sum_{p \in \mathbb{Z}^d} \sum_{l=1}^m \sum_{j \geq 0} |\det D|^j \sum_{k \in \mathbb{Z}^d} \left| \int_{(D^t)^{-j}(E \cap S_p)} \overline{\hat{\psi}^l(\xi)} e^{2\pi i \langle k, \xi \rangle} d\xi \right|^2,
\end{aligned}$$

where (2.2) was used in the last equality. Applying Plancherel's formula gives

$$\begin{aligned}
1 &\geq \sum_{p \in \mathbb{Z}^d} \sum_{l=1}^m \sum_{j \geq 0} |\det D|^j \int_{(D^t)^{-j}(E \cap S_p)} \left| \hat{\psi}^l(\xi) \right|^2 d\xi \\
(2.3) \quad &= \sum_{j \geq 0} |\det D|^j \sum_{l=1}^m \int_{(D^t)^{-j}E} \left| \hat{\psi}^l(\xi) \right|^2 d\xi.
\end{aligned}$$

Because of (2.2) and the fact that E is a D^t -dilation generator, we have

$$1 = \frac{1}{m} \sum_{l=1}^m \|\hat{\psi}^l\|^2 = \sum_{j \geq 0} \frac{1}{m} \sum_{l=1}^m \int_{(D^t)^{-j}E} \left| \hat{\psi}^l(\xi) \right|^2 d\xi.$$

This together with (2.3) leads to

$$(2.4) \quad \sum_{j \geq 0} \left(|\det D|^j - \frac{1}{m} \right) \sum_{l=1}^m \int_{(D^t)^{-j}E} \left| \hat{\psi}^l(\xi) \right|^2 d\xi \leq 0.$$

Since $|\det D| > 1$ and $m \geq 2$, it follows immediately that

$$\left| \text{Supp}(\hat{\psi}^l) \cap \left(\bigcup_{j \leq 0} (D^t)^j E \right) \right| = 0$$

for l with $1 \leq l \leq m$. Again using (2.2), we conclude that $\hat{\psi}^l = 0$ a.e. on $\mathbb{R}^d$ for l with $1 \leq l \leq m$, which yields a contradiction. The proof is completed. $\square$

Proof of Theorem 1.8. Proceeding as in the proof of Theorem 1.7, we see that, when $m = 1$, (2.4) becomes

$$\sum_{j \geq 0} (|\det D|^j - 1) \int_{(D^t)^{-j}E} \left| \hat{\psi}(\xi) \right|^2 d\xi \leq 0.$$

Since $|\det D| > 1$, we obtain immediately that

$$\int_{(D^t)^j E} \left| \hat{\psi}(\xi) \right|^2 d\xi = 0$$

for all $j \in \mathbb{Z}$ with $j < 0$, which implies, using our hypothesis on the spectrum of ψ , that $\text{Supp}(\hat{\psi}) \subset E$. Since the sequence $\{\psi(\cdot - n)\}_{n \in \mathbb{Z}}$ is orthonormal, it follows that

$$(2.5) \quad \sum_{k \in \mathbb{Z}^d} \left| \hat{\psi}(\cdot + k) \right|^2 = 1.$$

This implies that $|\hat{\psi}| \leq \chi_E$ almost everywhere. Integrating both sides of this last inequality, we obtain

$$1 = \int_{\mathbb{R}^d} \left| \hat{\psi}(\xi) \right|^2 d\xi \leq \int_{\mathbb{R}^d} \chi_E(\xi) d\xi = |E|.$$

Since $|E| \leq 1$ by hypothesis, it follows that $|\hat{\psi}| = \chi_E$. This fact, together with (2.5), shows that the $\mathbb{Z}^d$ -translates of E tiles $\mathbb{R}^d$ and E is thus a D -wavelet set. This completes the proof. $\square$

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