REGULARITY CRITERION VIA THE PRESSURE ON WEAK SOLUTIONS TO THE 3D NAVIER-STOKES EQUATIONS

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ABSTRACT. We consider the regularity of weak solutions to the Navier-Stokes equations in $\mathbb{R}^3$. Let u be a Leray-Hopf weak solution. It is proved that u becomes a regular solution if the pressure $p \in L^1(0, T; \dot{B}_{\infty, \infty}^0)$.

1. INTRODUCTION

In this paper, we consider the regularity of weak solutions to the Navier-Stokes equations in $\mathbb{R}^3 \times (0, T)$:

$$(1.1) \quad \begin{cases} u_t - \Delta u + u \cdot \nabla u + \nabla p = 0, & \text{in } \mathbb{R}^3 \times (0, T), \\ \nabla \cdot u = 0, & \text{in } \mathbb{R}^3 \times (0, T), \\ u(0) = u_0(x), & \text{in } \mathbb{R}^3, \end{cases}$$

where $u = (u^1(t, x), u^2(t, x), u^3(t, x))$ and $p = p(t, x)$ denote the unknown velocity vector and the unknown scalar pressure of the fluid at the point $(t, x) \in (0, T) \times \mathbb{R}^3$, respectively, while $u_0 = (u_0^1(x), u_0^2(x), u_0^3(x))$ is a given initial velocity vector satisfying $\nabla \cdot u_0 = 0$. Here we use the notation

$$u \cdot \nabla v = \sum_{i=1}^3 u^i \frac{\partial v}{\partial x_i}, \quad \nabla \cdot u = \sum_{i=1}^3 \frac{\partial u^i}{\partial x_i},$$

for vector functions u, v .

For $u_0 \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$, J. Leray and E. Hopf [6, 9] constructed a global weak solution $u(t, x) \in L^\infty(0, \infty; L^2(\mathbb{R}^3)) \cap L^2(0, \infty; H^1(\mathbb{R}^3))$. However, the question of regularity of weak solutions is an outstanding open problem in mathematical fluid mechanics. We are interested in the classical problem of finding sufficient conditions for weak solutions of (1.1) such that they become regular. J. Serrin [10, 11] is the pioneer in this direction, and later on, Fabes, Jones and Riviere [4], Giga [5], Sohr [14], Struwe [15] and Takahashi [16] extended and improved Serrin's regularity criterion. They showed that if u is a Leray-Hopf weak solution (see Definition 1.1) of (1.1) belonging to Serrin's class,

$$(1.2) \quad u \in L^r(0, T; L^s) \quad \text{with} \quad \frac{2}{r} + \frac{3}{s} \leq 1, \quad 3 < s \leq \infty,$$

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then u is a regular solution, i.e., $u \in C^\infty((0, T) \times \mathbb{R}^3)$. In the marginal case $s = \infty$, H. Kozono and Y. Taniuchi [7] proved the regularity of weak solutions under the condition

$$(1.3) \quad u \in L^2(0, T; BMO),$$

where BMO is the space of the bounded mean oscillation defined by

$$f \in L^1_{loc}(\mathbb{R}^3), \quad \sup_{x, R} \frac{1}{|B_R|} \int_{B_R(x)} |f(y) - \bar{f}_{B_R(x)}| dy < \infty,$$

where $\bar{f}_{B_R(x)}$ is the average of f over $B_R(x) = \{y \in \mathbb{R}^3; |x - y| < R\}$ (cf. Stein [13]). Recently, H. Kozono, T. Ogawa, and Y. Taniuchi [8] refined the condition (1.3) to

$$(1.4) \quad u \in L^2(0, T; \dot{B}^0_{\infty, \infty}).$$

Here and thereafter, $\dot{B}^s_{p, q}$ stands for the homogenous Besov space; see section 2 for definition. On the other hand, D. Chae and J. Lee [3] proposed another regularity criterion in terms of the pressure. They showed that if the pressure p satisfies

$$(1.5) \quad p \in L^r(0, T; L^s) \quad \text{with} \quad \frac{2}{r} + \frac{3}{s} < 2, \quad \text{for } s > \frac{3}{2},$$

then u is a regular solution. More recently, L. C. Berselli and G. P. Galdi [2] have extended the range of r and s to $\frac{2}{r} + \frac{3}{s} \leq 2, s > \frac{3}{2}$. Moreover, they also showed that if the gradient of the pressure satisfies the condition

$$(1.6) \quad \nabla p \in L^r(0, T; L^s(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{r} + \frac{3}{s} = 3, \quad \text{for } s \in [\frac{9}{7}, 3],$$

then u is also a regular solution. In particular, their result implies the regularity of weak solutions under the condition

$$(1.7) \quad p \in L^1(0, T; L^\infty \cup D^{1,3}),$$

where the function space $D^{1,p}$ for $1 \leq p \leq \infty$ is defined by

$$D^{1,p} \equiv \left\{ \phi \in L^1_{loc}(\mathbb{R}^3) \quad \text{such that} \quad \int_{\mathbb{R}^3} |\nabla \phi|^p dx < \infty \right\}.$$

Very recently, Y. Zhou [18] showed that if the gradient of the pressure satisfies the condition

$$(1.8) \quad \nabla p \in L^r(0, T; L^s(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{r} + \frac{3}{s} \leq 3, \quad \text{for } s \in [1, \infty],$$

then the weak solution u is actually regular.

The aim of this paper is to consider the regularity condition in terms of $p \in L^1(0, T; \dot{B}^0_{\infty, \infty})$. Making use of the Littlewood-Paley decomposition, we first establish the following a priori estimate:

$$(1.9) \quad \sup_{0 \leq t \leq T} \|u(t)\|_s \leq C(\|u_0\|_s + (CT)^{\frac{1}{s}} + e)^{\exp\left(C \int_0^T \|p(t)\|_{\dot{B}^0_{\infty, \infty}} dt\right)},$$

and then use the standard continuation argument for the local smooth solution. Before stating our result, let us recall the definition of Leray-Hopf weak solution.

Definition 1.1. Let $u_0 \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$. The function $u(t, x)$ will be called a Leray-Hopf weak solution of (1.1) in $(0, T)$ if u satisfies the following properties:

- (1) $u \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3))$ (Leray-Hopf class).
- (2) $u_t - \Delta u + u \cdot \nabla u + \nabla p = 0$ in $\mathcal{D}'((0, T) \times \mathbb{R}^3)$.
- (3) $\nabla \cdot u = 0$ in $\mathcal{D}'((0, T) \times \mathbb{R}^3)$.
- (4) (The Energy Inequality)

$$\|u(t)\|_2^2 + 2 \int_0^t \|\nabla u(\tau)\|_2^2 d\tau \leq \|u_0\|_2^2, \quad \forall \quad 0 \leq t \leq T.$$

Our result on the regularity criterion of weak solutions now reads as follows.

Theorem 1.2 (Regularity criterion). *Let $u_0 \in L^2(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ for some $q > 3$, and $\nabla \cdot u_0 = 0$. Suppose $u(t, x)$ is a Leray-Hopf weak solution of (1.1) in $(0, T)$. If the pressure p satisfies the condition*

$$(1.10) \quad \int_0^T \|p(t)\|_{\dot{B}_{\infty, \infty}^0} dt < \infty,$$

then u is a regular solution in $(0, T]$.

As an application of Theorem 1.2, we also obtain the following regularity criterion of weak solutions in terms of the gradient of the pressure.

Corollary 1.3. *Let u, u_0 be as in Theorem 1.2. Suppose that the pressure p satisfies*

$$(1.11) \quad \int_0^T \|\nabla p(t)\|_{\dot{B}_{3, \infty}^0} dt < \infty.$$

Then u is a regular solution in $(0, T]$.

Remark 1.1. From Theorem 1.2 and Corollary 1.3, it follows that if the pressure p satisfies

$$(1.12) \quad p \in L^1(0, T; \dot{B}_{\infty, \infty}^0 \cup \dot{B}_{3, \infty}^1),$$

then u is a regular solution. In particular, by the embedding $L^p \subsetneq \dot{B}_{p, \infty}^0$ for any $p \in [1, \infty]$, this implies the regularity of weak solutions under the condition $p \in L^1(0, T; L^\infty \cup D^{1,3})$.

Remark 1.2. Although we have the pressure-velocity relation

$$p = \sum_{i,j=1}^3 R_i R_j (u^i u^j),$$

and the Riesz transforms are bounded on $\dot{B}_{\infty, \infty}^0$ (see section 2), the regularity conditions (1.4) and (1.10) are independent, since we do not know whether the inequality

$$\|u^i u^j\|_{\dot{B}_{\infty, \infty}^0} \leq C \|u^i\|_{\dot{B}_{\infty, \infty}^0} \|u^j\|_{\dot{B}_{\infty, \infty}^0}$$

holds.

Remark 1.3. Our regularity criterion allows a function which has a singularity-like logarithmic function, since by the well-known fact, $\log|x|^{-1} \in BMO \subsetneq \dot{B}_{\infty, \infty}^0$.

Remark 1.4. It is not known if the Leray-Hopf weak solution is regular under the following more general condition:

$$(1.13) \quad \int_0^T \|p(t)\|_{\dot{B}_{s, \infty}^0}^r dt < \infty \quad \text{with} \quad \frac{2}{r} + \frac{3}{s} \leq 2, \quad \frac{3}{2} < s \leq \infty.$$

Throughout this paper, we use the letter C to denote the constants which may change from line to line. If we need to specify the constants, we shall denote them by $C_1, C_2, \dots$.

2. PRELIMINARIES

We first introduce the Littlewood-Paley decomposition. Let $\mathcal{S}(\mathbb{R}^3)$ be the Schwartz class of rapidly decreasing function. Given $f \in \mathcal{S}(\mathbb{R}^3)$, its Fourier transform $\mathcal{F}f = \hat{f}$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^3} e^{-ix \cdot \xi} f(x) dx,$$

and its inverse Fourier transform $\mathcal{F}^{-1}f = f^\vee$ is defined by

$$f^\vee(x) = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{ix \cdot \xi} f(\xi) d\xi.$$

Let us choose a nonnegative radial function $\phi \in \mathcal{S}(\mathbb{R}^3)$ such that

$$0 \leq \hat{\phi}(\xi) \leq 1, \quad \hat{\phi}(\xi) = \begin{cases} 1, & \text{if } |\xi| \leq 1, \\ 0, & \text{if } |\xi| \geq 2, \end{cases}$$

and let

$$\psi(x) = \phi(x) - 2^{-3}\phi(x/2), \quad \phi_j(x) = 2^{3j}\phi(2^j x), \quad \psi_j(x) = 2^{3j}\psi(2^j x), \quad j \in \mathbb{Z}.$$

For $j \in \mathbb{Z}$, the Littlewood-Paley projection operators S_j and Δ_j are respectively defined by

$$(2.1) \quad S_j f = \phi_j * f,$$

$$(2.2) \quad \Delta_j f = \psi_j * f.$$

Informally, Δ_j is a frequency projection to the annulus $\{|\xi| \sim 2^j\}$, while S_j is a frequency projection to the ball $\{|\xi| \lesssim 2^j\}$. Observe that $\Delta_j = S_j - S_{j-1}$. Also, if f is an L^2 function, then $S_j f \rightarrow 0$ in L^2 as $j \rightarrow -\infty$ and $S_j f \rightarrow f$ in L^2 as $j \rightarrow +\infty$ (this is an easy consequence of Parseval's theorem). By telescoping the series, we thus have the Littlewood-Paley decomposition

$$(2.3) \quad f = \sum_{j=-\infty}^{+\infty} \Delta_j f,$$

for all $f \in L^2$, where the summation is in the L^2 sense. Note that

$$\Delta_j f = \sum_{l=j-2}^{j+2} \Delta_l (\Delta_j f) = \sum_{l=j-2}^{j+2} \psi_l * \psi_j * f.$$

Then from the Young inequality, it follows that

$$(2.4) \quad \|\Delta_j f\|_q \leq C 2^{3j(\frac{1}{p} - \frac{1}{q})} \|\Delta_j f\|_p \leq C 2^{3j(\frac{1}{p} - \frac{1}{q})} \|f\|_p,$$

where $1 \leq p \leq q \leq \infty$, and C is a constant independent of f, j .

Let $s \in \mathbb{R}, p, q \in [1, \infty]$. The homogenous Besov space $\dot{B}_{p,q}^s$ is defined by the full-dyadic decomposition such as

$$\dot{B}_{p,q}^s = \{f \in \mathcal{Z}'(\mathbb{R}^3) : \|f\|_{\dot{B}_{p,q}^s} < \infty\},$$

where $\|f\|_{\dot{B}_{p,q}^s} = (\sum_{j=-\infty}^{\infty} 2^{jsq} \|\Delta_j f\|_p^q)^{\frac{1}{q}}$ and $\mathcal{Z}'(\mathbb{R}^3)$ denotes the dual space of $\mathcal{Z}(\mathbb{R}^3) = \{f \in \mathcal{S}(\mathbb{R}^3); D^\alpha \hat{f}(0) = 0; \forall \alpha \in \mathbb{N}^3 \text{ multi-index}\}$ and can be identified by the quotient space of $\mathcal{S}'/\mathcal{P}$ with the polynomials space $\mathcal{P}$. For $p = q = 2$, $\dot{B}_{p,q}^s \cong \dot{H}^s$, where $\dot{H}^s$ is the homogenous Sobolev space. We refer to [1, 17] for more detailed properties.

Now we recall the following result due to Y. Giga [5] that will be used in the proof of Theorem 1.2.

Proposition 2.1 ([5]). *Suppose $u_0 \in L^s(\mathbb{R}^3)$, $s \geq 3$; then there exists T and a unique classical solution $u \in BC([0, T]; L^s(\mathbb{R}^3))$. Moreover, let $(0, T_*)$ be the maximal interval such that u solves (1.1) in $C((0, T_*); L^s(\mathbb{R}^3))$, $s > 3$. Then for any $t \in (0, T_*)$,*

$$(2.5) \quad \|u(t)\|_s \geq \frac{C}{(T_* - t)^{\frac{s-3}{2s}}},$$

with the constant C independent of T_* and s .

Finally we recall the well-known pressure-velocity relation in $\mathbb{R}^3$, given by

$$(2.6) \quad p = \sum_{i,j=1}^3 R_i R_j (u^i u^j),$$

where $(R_j)_{j=1}^3$ are the Riesz transform in $\mathbb{R}^3$, i.e. $\widehat{R_j f}(\xi) = i \frac{\xi_j}{|\xi|} \hat{f}(\xi)$. (This can be obtained formally by taking divergence of (1.1), and solving the resulting equation for p .) From the L^γ boundedness of Riesz transform (cf. Stein [12]), it follows that

$$(2.7) \quad \|p\|_\gamma \leq C \|u\|_{2\gamma}^2, \quad 1 < \gamma < \infty.$$

3. PROOF OF THEOREM 1.2

In this section, we prove Theorem 1.2.

Proof of Theorem 1.2. First we derive a priori estimates for smooth solutions of (1.1). More precisely, let $3 < s \leq 4$; we will show the following a priori estimate:

$$(3.1) \quad \sup_{0 \leq t \leq T} \|u(t)\|_s \leq C(\|u_0\|_s + (CT)^{\frac{1}{s}} + e)^{\exp\left(C \int_0^T \|p(t)\|_{\dot{B}_{\infty,\infty}^0} dt\right)}.$$

Multiplying (1.1) by $|u|^{s-2}u$ and integrating by parts, we have

$$(3.2) \quad \begin{aligned} & \frac{1}{s} \frac{d}{dt} \|u(t)\|_s^s + \int_{\mathbb{R}^3} |\nabla u|^2 |u|^{s-2} dx + \frac{2(s-2)}{s^2} \int_{\mathbb{R}^3} |\nabla |u|^{\frac{s}{2}}|^2 dx \\ &= \int_{\mathbb{R}^3} p \nabla \cdot (|u|^{s-2} u) dx = \int_{\mathbb{R}^3} p u \cdot \nabla |u|^{s-2} dx \\ &\leq \frac{2(s-2)}{s} \int_{\mathbb{R}^3} |p| |u|^{\frac{s}{2}-1} |\nabla |u|^{\frac{s}{2}}| dx \equiv I(t). \end{aligned}$$

Here we used the fact that $\int_{\mathbb{R}^3} (u \cdot \nabla u) \cdot |u|^{s-2} u dx = 0$. Using the Littlewood-Paley decomposition (2.3), we decompose p as follows:

$$(3.3) \quad p = \sum_{j=-\infty}^{+\infty} \Delta_j p = \sum_{j < -N} \Delta_j p + \sum_{j=-N}^N \Delta_j p + \sum_{j > N} \Delta_j p.$$

Here N is a positive integer to be chosen later. Substituting (3.3) into $I(t)$, $I(t)$ can be written as

$$\begin{aligned}
 I(t) &\leq 2 \int_{\mathbb{R}^3} \left| \sum_{j < -N} \Delta_j p \right|^{\frac{1}{2}} |p|^{\frac{1}{2}} |u|^{\frac{s}{2}-1} |\nabla |u|^{\frac{s}{2}}| dx \\
 &\quad + 2 \int_{\mathbb{R}^3} \left| \sum_{j=-N}^N \Delta_j p \right|^{\frac{1}{2}} |p|^{\frac{1}{2}} |u|^{\frac{s}{2}-1} |\nabla |u|^{\frac{s}{2}}| dx \\
 &\quad + 2 \int_{\mathbb{R}^3} \left| \sum_{j > N} \Delta_j p \right|^{\frac{1}{2}} |p|^{\frac{1}{2}} |u|^{\frac{s}{2}-1} |\nabla |u|^{\frac{s}{2}}| dx \\
 (3.4) \quad &\equiv I_1(t) + I_2(t) + I_3(t).
 \end{aligned}$$

In the following, we estimate each term on the right-hand of (3.4) separately. We first consider $I_1(t)$. By the Hölder inequality, (2.4), and (2.7), we have

$$\begin{aligned}
 |I_1(t)| &\leq C \sum_{j < -N} \|\Delta_j p\|_{\infty}^{\frac{1}{2}} \|p\|^{\frac{1}{2}} \|u\|^{\frac{s-2}{s}} \|\nabla |u|^{\frac{s}{2}}\|_2 \\
 &\leq C \sum_{j < -N} 2^{\frac{3j}{s}} \|p\|^{\frac{1}{s}} \|u\|^{\frac{s-2}{s}} \|\nabla |u|^{\frac{s}{2}}\|_2 \\
 (3.5) \quad &\leq C 2^{-\frac{3N}{s}} \|u\|^{\frac{s+2}{s}} \|\nabla |u|^{\frac{s}{2}}\|_2.
 \end{aligned}$$

Now we turn to estimate $I_2(t)$. From the Hölder inequality and (2.7), it follows that

$$\begin{aligned}
 |I_2(t)| &\leq C \left(\sum_{j=-N}^N \|\Delta_j p\|_{\infty} \right)^{\frac{1}{2}} \|p\|^{\frac{1}{2}} \|u\|^{\frac{s-2}{s}} \|\nabla |u|^{\frac{s}{2}}\|_2 \\
 &\leq C N^{\frac{1}{2}} \|p\|_{\dot{B}_{\infty,\infty}^0}^{\frac{1}{2}} \|p\|^{\frac{1}{2}} \|u\|^{\frac{s-2}{s}} \|\nabla |u|^{\frac{s}{2}}\|_2 \\
 (3.6) \quad &\leq C N^{\frac{1}{2}} \|p\|_{\dot{B}_{\infty,\infty}^0}^{\frac{1}{2}} \|u\|^{\frac{s}{s}} \|\nabla |u|^{\frac{s}{2}}\|_2.
 \end{aligned}$$

Finally we estimate $I_3(t)$. By the Hölder inequality and (2.7), we have

$$\begin{aligned}
 |I_3(t)| &\leq C \sum_{j > N} \|\Delta_j p\|_3^{\frac{1}{2}} \|p\|_{\frac{3s}{4}}^{\frac{1}{2}} \|u\|^{\frac{s-2}{\frac{3s}{2}}} \|\nabla |u|^{\frac{s}{2}}\|_2 \\
 (3.7) \quad &\leq C \sum_{j > N} \|\Delta_j p\|_3^{\frac{1}{2}} \|u\|^{\frac{s}{\frac{3s}{2}}} \|\nabla |u|^{\frac{s}{2}}\|_2.
 \end{aligned}$$

Note that

$$\|u\|_{3s} \leq C \|\nabla |u|^{\frac{s}{2}}\|_2^{\frac{2}{s}},$$

which together with the interpolation inequality implies that

$$(3.8) \quad \|u\|_{\frac{3s}{2}} \leq \|u\|^{\frac{1}{s}} \|u\|_{3s}^{\frac{1}{2}} \leq C \|u\|^{\frac{1}{s}} \|\nabla |u|^{\frac{s}{2}}\|_2^{\frac{1}{s}}.$$

On the other hand, by (2.4), (2.7), and the Hölder inequality, we have

$$\begin{aligned}
 \sum_{j > N} \|\Delta_j p\|_3 &\leq C \sum_{j > N} 2^{-j} \|\Delta_j \nabla p\|_3 \leq C 2^{(-2+\frac{6}{s})N} \|\nabla p\|_{\frac{s}{2}} \\
 &\leq C 2^{(-2+\frac{6}{s})N} \|\nabla u\| \|u\|_{\frac{s}{2}} \\
 (3.9) \quad &\leq C 2^{(-2+\frac{6}{s})N} \|u\|_{\frac{4-s}{2}}^{\frac{4-s}{2}} \|\nabla u\| \|u\|^{\frac{s-2}{2}} \|u\|_{\frac{s}{2}}.
 \end{aligned}$$

From (3.7)-(3.9), we obtain

$$(3.10) \quad I_3(t) \leq C 2^{(-1+\frac{3}{s})N} \|u\|_s \| |\nabla u| |u|^{\frac{s-2}{2}} \|_2^{\frac{1}{2}} \|\nabla |u|^{\frac{s}{2}}\|_2^{\frac{3}{2}}.$$

Thus, by summing up (3.2), (3.5), (3.6), and (3.10), we obtain

$$(3.11) \quad \begin{aligned} & \frac{1}{s} \frac{d}{dt} \|u(t)\|_s^s + \int_{\mathbb{R}^3} |\nabla u|^2 |u|^{s-2} dx + \frac{2(s-2)}{s^2} \int_{\mathbb{R}^3} |\nabla |u|^{\frac{s}{2}}|^2 dx \\ & \leq C \left(2^{-\frac{3N}{s}} \|u\|_s^{\frac{s+2}{2}} \|\nabla |u|^{\frac{s}{2}}\|_2 + N^{\frac{1}{2}} \|p\|_{\dot{B}_{\infty,\infty}^0}^{\frac{1}{2}} \|u\|_s^{\frac{s}{2}} \|\nabla |u|^{\frac{s}{2}}\|_2 \right. \\ & \quad \left. + 2^{(-1+\frac{3}{s})N} \|u\|_s \| |\nabla u| |u|^{\frac{s-2}{2}} \|_2^{\frac{1}{2}} \|\nabla |u|^{\frac{s}{2}}\|_2^{\frac{3}{2}} \right). \end{aligned}$$

Then by the Young inequality, (3.11) implies that

$$(3.12) \quad \begin{aligned} & \frac{d}{dt} \|u(t)\|_s^s + \int_{\mathbb{R}^3} |\nabla u|^2 |u|^{s-2} dx + \int_{\mathbb{R}^3} |\nabla |u|^{\frac{s}{2}}|^2 dx \\ & \leq \left(C_1 2^{-\frac{6N}{s(s+2)}} \|u\|_s \right)^{s+2} + C_2 N \|p\|_{\dot{B}_{\infty,\infty}^0} \|u\|_s^s \\ & \quad + \left(C_3 2^{-(1-\frac{3}{s})N} \|u\|_s \right)^{\frac{4}{3}} \|\nabla |u|^{\frac{s}{2}}\|_2^2. \end{aligned}$$

Since $s > 3$, we can choose N in (3.12) so that $\left(C_3 2^{-(1-\frac{3}{s})N} \|u\|_s \right)^{\frac{4}{3}} \leq \frac{1}{2}$, i.e.,

$$N \geq \frac{s}{s-3} \left(\frac{\log^+(C_3 \|u\|_s)}{\log 2} + 1 \right),$$

where $\log^+ t = \log t$ for $1 \leq t$ and $\log^+ t = 0$ for $0 < t < 1$. On the other hand, note that $\frac{6}{s(s+2)} > \frac{s-3}{s}$, where we have

$$C_1 2^{-\frac{6N}{s(s+2)}} \|u\|_s \leq C.$$

Then the inequality (3.12) implies that

$$(3.13) \quad \frac{d}{dt} \|u(t)\|_s^s \leq C \|u(t)\|_s^s \|p(t)\|_{\dot{B}_{\infty,\infty}^0} \log(\|u(t)\|_s + e) + C,$$

for all $0 < t < T$. By the Gronwall inequality, we have

$$(3.14) \quad \|u(t)\|_s^s \leq (\|u(0)\|_s^s + CT) \exp \left(C \int_0^t \|p(s)\|_{\dot{B}_{\infty,\infty}^0} \log(\|u(s)\|_s + e) ds \right).$$

Defining $Z(t) = \log(\|u(t)\|_s + e)$, inequality (3.14) implies that

$$(3.15) \quad Z(t) \leq \log(\|u(0)\|_s + (CT)^{\frac{1}{s}} + e) + C \int_0^t \|p(s)\|_{\dot{B}_{\infty,\infty}^0} Z(s) ds.$$

Applying the Gronwall inequality to $Z(t)$ again, we have

$$(3.16) \quad Z(t) \leq \log(\|u(0)\|_s + (CT)^{\frac{1}{s}} + e) \exp \left(C \int_0^t \|p(s)\|_{\dot{B}_{\infty,\infty}^0} ds \right),$$

which implies the desired estimate (3.1).

Now we are in a position to complete the proof of Theorem 1.2. Since $u_0 \in L^2(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ for some $q > 3$, $u_0 \in L^s(\mathbb{R}^3)$ for any $s \in (3, q)$. We now fix s

so that $s \in (3, \min(q, 4))$. Then from Proposition 2.1, it follows that there exists $T_* > 0$ and the smooth solution v of (1.1) satisfying

$$v(t) \in BC([0, T_*]; L^s(\mathbb{R}^3)), \quad v(0) = u_0.$$

Since the weak solution u satisfies the energy inequality, we may apply Serrin's uniqueness criterion [11] to conclude that

$$u \equiv v \quad \text{on} \quad [0, T_*).$$

Thus it is sufficient to show that $T_* = T$. Suppose that $T_* < T$. Without loss of generality, we may assume that T_* is the maximal existence time for $v(t)$. By Proposition 2.1 again, we have

$$(3.17) \quad \|u(t)\|_s \geq \frac{C}{(T_* - t)^{\frac{s-3}{2s}}}, \quad \forall t \in (0, T_*).$$

On the other hand, by the priori estimate (3.1), we have

$$(3.18) \quad \sup_{0 \leq t < T_*} \|u(t)\|_s \leq C(T, \|u_0\|_s),$$

which contradicts with (3.17). Thus $T = T_*$. This completes the proof of Theorem 1.2. $\square$

Proof of Corollary 1.3. By (2.4), we have

$$(3.19) \quad \|\triangle_j p\|_\infty \leq C2^j \|\triangle_j p\|_3 \leq C \|\triangle_j \nabla p\|_3,$$

which by the definition of Besov spaces implies that

$$(3.20) \quad \int_0^T \|p(t)\|_{\dot{B}_{\infty,\infty}^0} dt \leq C \int_0^T \|\nabla p(t)\|_{\dot{B}_{3,\infty}^0} dt.$$

Then from Theorem 1.2, we conclude the proof. $\square$

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