MINIMAL RANK AND REFLEXIVITY OF OPERATOR SPACES

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ABSTRACT. Let $\mathcal{S}$ be an n -dimensional space of linear operators between the linear spaces U and V over an algebraically closed field $\mathbb{F}$. Improving results of Larson, Ding, and Li and Pan we show the following.

Theorem. *Let $S_1, \dots, S_n$ be a basis of $\mathcal{S}$. Assume that every nonzero operator in $\mathcal{S}$ has rank larger than n . Then a linear operator $T : U \rightarrow V$ belongs to $\mathcal{S}$ if and only if for every $u \in U$, Tu is a linear combination of $S_1u, \dots, S_nu$.*

1. INTRODUCTION

Let U and V be linear spaces over a field $\mathbb{F}$, and let $\mathcal{L}(U, V)$ denote the space of all linear operators from U to V . Let $\mathcal{S} \subset \mathcal{L}(U, V)$ be a linear subspace of operators. An operator $T \in \mathcal{L}(U, V)$ is *locally linearly dependent* on $\mathcal{S}$, if $Tu \in \mathcal{S}u = \{Su : S \in \mathcal{S}\}$ for all $u \in U$. The *reflexive closure* of $\mathcal{S}$ is the space $\text{ref}(\mathcal{S})$ of all $T \in \mathcal{L}(U, V)$ that are locally linearly dependent on $\mathcal{S}$.

The space $\mathcal{S}$ is (algebraically) *reflexive* if $\text{ref}(\mathcal{S}) = \mathcal{S}$. Various aspects of reflexive operator spaces were studied in [5, 6, 7, 8, 10]. In this paper we are concerned with the relation between minimal rank and reflexivity. Let

$$\rho(\mathcal{S}) = \min\{\text{rank}(A) : 0 \neq A \in \mathcal{S}\}$$

and let

$$f_{\mathbb{F}}(n) = \sup\{\rho(\mathcal{S}) : \dim \mathcal{S} = n, \text{ref}(\mathcal{S}) \neq \mathcal{S}\}.$$

Larson [7] proved that if $\mathcal{S}$ is nonreflexive, then $\rho(\mathcal{S}) < \infty$. Ding [6] gave a quantitative version of Larson's result by showing that $f_{\mathbb{F}}(n) \leq n^2$. This was further improved in the complex case by Li and Pan [8] who proved $f_{\mathbb{C}}(n) \leq 2n - 2$. In [10] it was shown that if $|\mathbb{F}| \geq n + 3$, then any n -dimensional nonreflexive space either contains a nonzero operator of rank $\leq 2n - 3$, or all of its nonzero members have rank $2n - 2$. In particular $f_{\mathbb{F}}(n) \leq 2n - 2$.

The study of reflexivity of finite-dimensional operator spaces is closely related to questions concerning locally linearly dependent operator spaces. An n -dimensional subspace $\mathcal{R} \subset \mathcal{L}(U, V)$ is *locally linearly dependent* if $\dim \mathcal{R}u < n$ for all $u \in U$. Locally linearly dependent spaces and reflexive spaces have many applications in operator theory and in ring theory (see e.g. [1, 4, 9]). In [9] it was shown that such

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$\mathcal{R}$ must contain a nonzero operator of rank at most $n - 1$. Let $\mathcal{S}$ be a nonreflexive n -dimensional space of operators and assume that $T \in \text{ref}(\mathcal{S}) - \mathcal{S}$. Then clearly $\mathcal{R} = \text{Span}\{T\} + \mathcal{S}$ is locally linearly dependent. In [10] we used this fact together with a structural result for minimal locally linearly dependent operator spaces to obtain the above-mentioned $f_{\mathbb{F}}(n) \leq 2n - 2$ bound. However, one cannot expect this approach to yield sharp estimates, since the condition $T \in \text{ref}(\mathcal{S}) - \mathcal{S}$ is in general stronger than local linear dependence of $\text{Span}\{T\} + \mathcal{S}$. In fact, we conjecture that every n -dimensional nonreflexive space contains a nonzero operator of rank at most n . Our main result establishes this conjecture for algebraically closed fields.

Theorem 1.1. *If $\mathbb{F}$ is algebraically closed, then*

$$(1) \quad \lfloor \frac{n}{2} - \sqrt{n} \rfloor \leq f_{\mathbb{F}}(n) \leq n .$$

The upper bound in Theorem 1.1 is proved in section 2. In section 3 we prove the lower bound, and in addition give a simple construction which in particular shows that if $\mathbb{F}$ is finite or $\mathbb{F} = \mathbb{Q}$, then $f_{\mathbb{F}}(n) \geq n$.

2. THE UPPER BOUND

Let $\mathbb{F}$ be algebraically closed and let $\mathcal{S}$ be an n -dimensional subspace of $\mathcal{L}(U, V)$ such that $\text{ref}(\mathcal{S}) \neq \mathcal{S}$. We have to show that there exists a $0 \neq S \in \mathcal{S}$ such that $\text{rank} S \leq n$. By standard considerations (see e.g. Proposition 2.1 in [9]) we may assume that U and V are finite dimensional. We thus write $U = \mathbb{F}^k$, $V = \mathbb{F}^\ell$ and identify $\mathcal{L}(U, V)$ with $M_{\ell \times k}(\mathbb{F})$ —the space of $\ell \times k$ matrices over $\mathbb{F}$. Let $S_1, \dots, S_n$ be a basis of $\mathcal{S}$. Let $u = (u_1, \dots, u_k)$ be a vector of k independent variables and let $\mathbb{K}$ denote the function field $\mathbb{F}(u_1, \dots, u_k)$. For a column vector $v \in \mathbb{K}^\ell$ and a row vector $w \in \mathbb{K}^k$ we write $v \otimes w$ for the matrix product $v \cdot w \in M_{\ell \times k}(\mathbb{K})$.

Choose an operator $T \in \text{ref}(\mathcal{S}) - \mathcal{S}$. If $\dim \mathcal{S}a < \dim \mathcal{S} = n$ for all $a \in \mathbb{F}^k$, then by [3, 9], $\mathcal{S}$ contains a nonzero matrix of rank at most $n - 1$. We thus assume that $\dim \mathcal{S}a = n$ for some $a \in \mathbb{F}^k$. This implies that $S_1u, \dots, S_nu$ are linearly independent vectors in $\mathbb{K}^\ell$. Since $Ta \in \text{Span}\{S_1a, \dots, S_na\}$ for all $a \in \mathbb{F}^k$, it follows (by the infinitude of $\mathbb{F}$) that $Tu, S_1u, \dots, S_nu$ are linearly dependent in $\mathbb{K}^\ell$. Thus, by Cramer's rule, there exist homogeneous polynomials $p(u), p_1(u), \dots, p_n(u)$ of the same degree d , such that $p(u) \neq 0$ and such that

$$(2) \quad p(u)Tu = \sum_{i=1}^n p_i(u)S_iu .$$

We may assume that $\text{g.c.d.}(p(u), p_1(u), \dots, p_n(u)) = 1$. Furthermore, $T \notin \mathcal{S}$ implies that $d \geq 1$. Differentiating (2) we obtain

$$(3) \quad p(u)T + Tu \otimes dp(u) = \sum_{i=1}^n p_i(u)S_i + \sum_{i=1}^n S_iu \otimes dp_i(u),$$

where $df(u) = (\frac{\partial f}{\partial u_1}(u), \dots, \frac{\partial f}{\partial u_k}(u))$. Therefore

$$(4) \quad p(u)T - \sum_{i=1}^n p_i(u)S_i = \sum_{i=1}^n S_iu \otimes \left(dp_i(u) - \frac{p_i(u)}{p(u)} dp(u) \right) .$$

The right-hand side is a sum of at most n rank one matrices in $M_{\ell \times k}(\mathbb{K})$, hence

$$(5) \quad \text{rank}(p(u)T - \sum_{i=1}^n p_i(u)S_i) \leq n .$$

If $(p_1(a), \dots, p_n(a)) = 0$ for all $a \in \mathbb{F}^k$ such that $p(a) = 0$, then by the Nullstellensatz, $p(u)$ must divide some power of each of the $p_i(u)$'s. Since $d \geq 1$, this implies that $p(u), p_1(u), \dots, p_n(u)$ have a nontrivial common factor, a contradiction. Therefore, there exists an $a \in \mathbb{F}^k$ such that $p(a) = 0$ but $(p_1(a), \dots, p_n(a)) \neq 0$. By (5)

$$0 < \text{rank}\left(\sum_{i=1}^n p_i(a)S_i\right) \leq n .$$

3. THE LOWER BOUND

We shall need a characterization of reflexivity due essentially to Azoff [2]. Consider the standard bilinear form on $M_{\ell \times k}(\mathbb{F})$ given by $(A, B) = \text{tr}(AB^T)$. For a subspace $S \subset M_{\ell \times k}(\mathbb{F})$ let

$$\mathcal{S}^\perp = \{B \in M_{\ell \times k}(\mathbb{F}) : (A, B) = 0 \text{ for all } A \in \mathcal{S}\} .$$

Let $R_d = \{A \in M_{\ell \times k}(\mathbb{F}) : \text{rank} A \leq d\}$. The following result is a variant of Proposition 2.2 in [2].

Claim 3.1. *For any subspace $\mathcal{S} \subset M_{\ell \times k}(\mathbb{F})$*

$$\text{ref}(\mathcal{S})/\mathcal{S} \cong \mathcal{S}^\perp / \text{Span}(\mathcal{S}^\perp \cap R_1) .$$

In particular, $\mathcal{S}$ is reflexive iff $\mathcal{S}^\perp$ is generated by rank one elements.

Proof. For $u \in \mathbb{F}^k$ let $z(u) = \{A \in M_{\ell \times k}(\mathbb{F}) : Au = 0\}$. Clearly $z(u) = \mathbb{F}^\ell \otimes u^\perp$. It follows that

$$\text{ref}(\mathcal{S}) = \bigcap_{u \in \mathbb{F}^k} (\mathcal{S} + z(u)) = \bigcap_{u \in \mathbb{F}^k} (\mathcal{S} + \mathbb{F}^\ell \otimes u^\perp) .$$

Therefore

$$\begin{aligned} \text{ref}(\mathcal{S})^\perp &= \sum_{u \in \mathbb{F}^k} (\mathcal{S} + \mathbb{F}^\ell \otimes u^\perp)^\perp \\ &= \sum_{u \in \mathbb{F}^k} (\mathcal{S}^\perp \cap \mathbb{F}^\ell \otimes \text{Span}\{u\}) = \text{Span}(\mathcal{S}^\perp \cap R_1) . \end{aligned}$$

It follows that

$$\mathcal{S}^\perp / \text{Span}(\mathcal{S}^\perp \cap R_1) = \mathcal{S}^\perp / \text{ref}(\mathcal{S})^\perp \cong \text{ref}(\mathcal{S})/\mathcal{S} .$$

Let $\mathbb{F}$ be algebraically closed. To prove the lower bound in Theorem 1.1, we show that there exists an n -dimensional nonreflexive operator space $\mathcal{S}$ such that $\rho(\mathcal{S}) \geq \lfloor \frac{n}{2} - \sqrt{n} \rfloor = r$. Let $k = \lfloor \frac{n}{2} \rfloor$ and let $\mathcal{S}$ be a generic n -dimensional subspace of $M_k(\mathbb{F})$. Recall that $R_d = \{A \in M_k(\mathbb{F}) : \text{rank} A \leq d\}$ is an irreducible affine variety of codimension $(k-d)^2$ in $M_k(\mathbb{F})$. Since

$$\dim \mathcal{S}^\perp = k^2 - n < (k-1)^2 = \text{codim} R_1$$

it follows that $\mathcal{S}^\perp \cap R_1 = 0$, hence $\mathcal{S}$ is nonreflexive by Claim 3.1.

On the other hand

$$\dim \mathcal{S} = n < (k-r+1)^2 = \text{codim} R_{r-1} .$$

Hence $\rho(\mathcal{S}) \geq r$. □

For certain nonalgebraically closed fields, such as $\mathbb{F} = \mathbb{F}_q$ or $\mathbb{F} = \mathbb{Q}$, the lower bound can be significantly improved.

Claim 3.2. *Let $n > 1$ and suppose there exists a (not necessarily associative) n -dimensional division algebra A over the field $\mathbb{F}$. Then $f_{\mathbb{F}}(n) \geq n$.*

Proof. For $a \in A$ let $T_a \in \mathcal{L}(A)$ be given by $T_a(b) = ab$. Let $\mathcal{S} = \{T_a : a \in A\} \subset \mathcal{L}(A)$. Clearly $\rho(\mathcal{S}) = \dim \mathcal{S} = n$. On the other hand $\mathcal{S}b = A$ for all $0 \neq b \in A$ implies that $\text{ref}(\mathcal{S}) = \mathcal{L}(A)$, hence $\mathcal{S}$ is nonreflexive. $\square$

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