THE SCHRÖDINGER EQUATION WITH A MOVING POINT INTERACTION IN THREE DIMENSIONS

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ABSTRACT. In the case of a single point interaction we improve, by using different methods, the existence theorem for the unitary evolution generated by a Schrödinger operator with moving point interactions obtained by Dell’Antonio, Figari and Teta.

1. INTRODUCTION

Let us denote by $L^2(\mathbb{R}^3)$, with the usual scalar product $\langle \cdot, \cdot \rangle_2$ and corresponding norm $\| \cdot \|_2$, the Hilbert space of square integrable measurable functions on $\mathbb{R}^3$. By $H^1(\mathbb{R}^3)$ and by $H^2(\mathbb{R}^3)$ we denote the usual Sobolev-Hilbert spaces

$$\begin{aligned} H^1(\mathbb{R}^3) &:= \{ \psi \in L^2(\mathbb{R}^3) : \nabla \psi \in L^2(\mathbb{R}^3) \}, \\ H^2(\mathbb{R}^3) &:= \{ \psi \in H^1(\mathbb{R}^3) : \Delta \psi \in L^2(\mathbb{R}^3) \}. \end{aligned}$$

Let

$$H \equiv -\Delta : H^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)$$

be the self-adjoint operator giving the Hamiltonian of a free quantum particle in $\mathbb{R}^3$. For any $y \in \mathbb{R}^3$ let us consider the symmetric operator H_y° obtained by restricting H to the subspace

$$\{ \psi \in H^2(\mathbb{R}^3) : \psi(y) = 0 \}.$$

Such a symmetric operator has defect indices $(1, 1)$. Any of its self-adjoint extensions different from H itself describes a point interaction centered at y . One has the following (see [1], section I.1.1 as regards $H_{\alpha,y}$, and see [9] as regards $F_{\alpha,y}$).

Theorem 1.1. *Any self-adjoint extension of H_y° different from H itself is given by*

$$\begin{aligned} &H_{\alpha,y} : D(H_{\alpha,y}) \rightarrow L^2(\mathbb{R}^3), \\ &D(H_{\alpha,y}) := \\ &\{ \psi \in L^2(\mathbb{R}^3) : \psi(x) = \psi_\lambda(x) + \Gamma_\alpha(\lambda)^{-1} \psi_\lambda(y) \mathcal{G}_\lambda(x - y), \psi_\lambda \in H^2(\mathbb{R}^3) \}, \\ &(H_{\alpha,y} + \lambda)\psi := (H + \lambda)\psi_\lambda, \end{aligned}$$

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the definition being λ -independent. Here $\alpha \in \mathbb{R}$,

$$\Gamma_\alpha(\lambda) = \alpha + \frac{\sqrt{\lambda}}{4\pi}, \quad \mathcal{G}_\lambda(x) = \frac{e^{-\sqrt{\lambda}|x|}}{4\pi|x|},$$

and $\lambda > 0$ is chosen in such a way that $\Gamma_\alpha(\lambda) \neq 0$. The kernel of the resolvent of $H_{\alpha,y}$ is given by

$$(H_{\alpha,y} + \lambda)^{-1}(x_1, x_2) = \mathcal{G}_\lambda(x_1 - x_2) + \Gamma_\alpha(\lambda)^{-1} \mathcal{G}_\lambda(x_1 - y) \mathcal{G}_\lambda(x_2 - y).$$

The quadratic form associated with $H_{\alpha,y}$ is

$$F_{\alpha,y} : D(F_y) \times D(F_y) \rightarrow \mathbb{R},$$

$$D(F_y) :=$$

$$\{\psi \in H^1(\mathbb{R}^3) : \psi(x) = \psi_\lambda(x) + q_\psi \mathcal{G}_\lambda(x - y), \psi_\lambda \in H^1(\mathbb{R}^3), q_\psi \in \mathbb{C}\},$$

$$(F_{\alpha,y} + \lambda)(\psi, \phi) = \langle \nabla \psi_\lambda, \nabla \phi_\lambda \rangle_2 + \lambda \langle \psi_\lambda, \phi_\lambda \rangle_2 + \Gamma_\alpha(\lambda) \bar{q}_\psi q_\phi.$$

Moreover the essential spectrum of $H_{\alpha,y}$ is purely absolutely continuous,

$$\sigma_{ess}(H_{\alpha,y}) = \sigma_{ac}(H_{\alpha,y}) = [0, \infty),$$

$$\alpha < 0 \quad \Rightarrow \quad \sigma_{pp}(H_{\alpha,y}) = -(4\pi\alpha)^2,$$

$$\alpha \geq 0 \quad \Rightarrow \quad \sigma_{pp}(H_{\alpha,y}) = \emptyset.$$

Suppose now that the point y is not fixed but describes a curve $y : \mathbb{R} \rightarrow \mathbb{R}^3$, thus producing the family of self-adjoint operators $H_{\alpha,y}(t) \equiv H_{\alpha,y(t)}$. Then one is interested in showing that the time-dependent Hamiltonian $H_{\alpha,y}(t)$ generates a strongly continuous unitary propagator $U_{t,s}$. Note that both the operator and the form domain of $H_{\alpha,y}(t)$ are strongly time dependent. This renders inapplicable the known general theorems (see [3], [6]), and such a generation problem is not trivial.

By exploiting the explicit form of $H_{\alpha,y}(t)$ and in the case of several moving point interactions, Dell'Antonio, Figari and Teta obtained in [2] the following (here we state their results in the simpler case of a single point interaction).

Theorem 1.2. *Suppose that*

$$y \in C^3(\mathbb{R}; \mathbb{R}^3), \quad \psi \in C_0^\infty(\mathbb{R}^3), \quad \psi(y(s)) = 0.$$

Then there exists a unique strongly continuous unitary propagator

$$U_{t,s} : L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)$$

such that

$$(1.1) \quad \left(i \frac{d}{dt} U_{t,s} \psi, \phi \right)_t = F_{\alpha,y}(t)(U_{t,s} \psi, \phi)$$

for all $\phi \in D(F_y(t))$.

Here $(\cdot, \cdot)_t$ denotes the duality between $D(F_y(t))$ and its strong dual. Moreover the solution $\psi(t) := U_{t,s} \psi$ has a natural representation (see [2], (14)-(21) for the details).

In the introduction of [2] the authors conjectured that $U_{t,s}$ defines a flow on $D(F_y(t))$ which is continuous with respect to the Banach topology induced by the quadratic form $F_{\alpha,y}(t)$.

Here we show, by using different methods, that if $y \in C^2(\mathbb{R}; \mathbb{R}^3)$, then this is indeed the case, and Theorem 1.2 above holds for any $\psi \in D(F_y(s))$ (see Theorem 3.1 for the precise statements).

Our proof proceeds in the following conceptually simple way. Noting that the unitary map

$$T_t\psi(x) := \psi(x + y(t))$$

transforms the equation

$$i \frac{d\psi}{dt}(t) = H\psi(t)$$

into the nonautonomous equation

$$i \frac{d\psi}{dt}(t) = H_v(t)\psi(t) \equiv (H + iv(t) \cdot \nabla) \psi(t), \quad v(t) \equiv \frac{dy}{dt}(t),$$

we consider the point perturbations (at $y = 0$) of H_v , where v is a given, time-independent vector in $\mathbb{R}^3$. We realize (see Theorem 2.3) that the form domains $D(F_v)$ of such singular perturbations $H_{v,\alpha}$ of H_v are v -independent. Indeed one has $D(F_v) \equiv D(F_0)$, where $D(F_0)$ is the form domain of $H_{\alpha,y}$, $y = 0$. This allows, in the case the vector v is time-dependent, the application of Kiszyński's theorem (see the Appendix), thus obtaining a strongly continuous unitary propagator $\tilde{U}_{t,s}$ which is also a strongly continuous propagator on $D(F_0)$ with respect to the Banach topology induced by the quadratic form associated with $H_{\alpha,y}$, $y = 0$. Moreover $F_{v,\alpha}$, the quadratic form associated with $H_{v,\alpha}$, and $F_{\alpha,0}$, the quadratic form associated with $H_{\alpha,0}$, are related by the identity (see Theorem 2.3 again)

$$F_{v,\alpha} = F_{\alpha,0} + Q_v,$$

where Q_v is the quadratic form associated with the natural extension of $iv \cdot \nabla$ to $D(F_0)$ (see Remark 2.4). This allows us to show (see Theorem 3.1) that

$$U_{t,s} := T_t^{-1} \tilde{U}_{t,s} T_s$$

satisfies (1.1) for any $\psi \in D(F_{\alpha,y}(s))$ and is a continuous flow from $D(F_{\alpha,y}(s))$ onto $D(F_{\alpha,y}(t))$. In the case $y \in C^3(\mathbb{R}; \mathbb{R}^3)$ we also show that $U_{t,s}$ maps $\tilde{D}(H_{\alpha,y}(s))$ onto $\tilde{D}(H_{\alpha,y}(t))$, where

$$\tilde{D}(H_{\alpha,y}(t)) := V_t D(H_{\alpha,y}(t)), \quad V_t \psi(x) := e^{iv(t) \cdot x/2} \psi(x).$$

2. THE OPERATOR $-\Delta + iL_v$ WITH A POINT INTERACTION

Let us consider the linear operator $-\Delta + iL_v$, where

$$L_v \psi := \sum_{k=1}^3 v_k \partial_k \psi, \quad v \equiv (v_1, v_2, v_3) \in \mathbb{R}^3.$$

Since, for any $\epsilon > 0$,

$$\begin{aligned} \|L_v \psi\|_2^2 &\leq \sum_{1 \leq k, j \leq 3} |v_k v_j \langle \partial_{kj}^2 \psi, \psi \rangle_2| \leq 3|v|^2 |\langle \Delta \psi, \psi \rangle_2| \\ &\leq \frac{3}{2} |v|^2 \left(\epsilon \|\Delta \psi\|_2^2 + \frac{1}{\epsilon} \|\psi\|_2^2 \right), \end{aligned}$$

by Kato-Rellich's theorem one has that

$$H_v := -\Delta + iL_v : H^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)$$

is self-adjoint. Moreover, since, for any $\epsilon > 0$,

$$|\langle L_v \psi, \psi \rangle_2| \leq |v| \|\nabla \psi\|_2 \|\psi\|_2 \leq \frac{|v|}{2} \left(\epsilon \|\nabla \psi\|_2^2 + \frac{1}{\epsilon} \|\psi\|_2^2 \right),$$

H_v has lower bound $-|v|^2/4$.

Now we look for the self-adjoint extensions of the symmetric operator H_v° obtained by restricting H_v to the kernel of the continuous and surjective linear map

$$\tau : H^2(\mathbb{R}^3) \rightarrow \mathbb{C}, \quad \tau\psi := \psi(0).$$

Theorem 2.1. *Any self-adjoint extension of H_v° different from H_v itself is given by*

$$\begin{aligned} H_{v,\alpha} &: D(H_{v,\alpha}) \rightarrow L^2(\mathbb{R}^3), \\ D(H_{v,\alpha}) &:= \{ \psi \in L^2(\mathbb{R}^3) : \psi = \psi_\lambda + \Gamma_{v,\alpha}(\lambda)^{-1} \psi_\lambda(0) \mathcal{G}_\lambda^v, \psi_\lambda \in H^2(\mathbb{R}^3) \}, \\ (H_{v,\alpha} + \lambda)\psi &:= (H_v + \lambda)\psi_\lambda, \end{aligned}$$

the definition being λ -independent. Here $\alpha \in \mathbb{R}$,

$$\Gamma_{v,\alpha}(\lambda) = \alpha + \frac{\sqrt{\lambda - |v|^2/4}}{4\pi}, \quad \mathcal{G}_\lambda^v(x) = \frac{e^{-\sqrt{\lambda - |v|^2/4}|x|}}{4\pi|x|} e^{iv \cdot x/2},$$

and $\lambda > |v|^2/4$ is chosen in such a way that $\Gamma_{v,\alpha}(\lambda) \neq 0$. The kernel of the resolvent of $H_{v,\alpha}$ is given by

$$(H_{v,\alpha} + \lambda)^{-1}(x_1, x_2) = \mathcal{G}_\lambda^v(x_1 - x_2) + \Gamma_{v,\alpha}(\lambda)^{-1} \mathcal{G}_\lambda^v(x_1) \mathcal{G}_\lambda^{-v}(x_2).$$

Moreover the spectrum of $H_{v,\alpha}$ is purely absolutely continuous,

$$\begin{aligned} \sigma_{ess}(H_{v,\alpha}) &= \sigma_{ac}(H_{v,\alpha}) = [-|v|^2/4, \infty), \\ \alpha < 0 &\Rightarrow \sigma_{pp}(H_{v,\alpha}) = -(4\pi\alpha)^2 - |v|^2/4, \\ \alpha \geq 0 &\Rightarrow \sigma_{pp}(H_{v,\alpha}) = \emptyset. \end{aligned}$$

Proof. Let us define the bounded linear operators

$$\check{G}(\lambda) : L^2(\mathbb{R}^3) \rightarrow \mathbb{C}, \quad \check{G}(\lambda) := \tau(H_v + \lambda)^{-1},$$

and

$$G(\lambda) : \mathbb{C} \rightarrow L^2(\mathbb{R}^3), \quad G(\lambda) := \check{G}(\lambda)^*.$$

Since

$$\mathcal{G}_\lambda^v(x) = \frac{e^{-\sqrt{\lambda - |v|^2/4}|x|}}{4\pi|x|} e^{iv \cdot x/2}$$

is the Green function of $H_v + \lambda$ (see e.g. [5]), one obtains

$$\check{G}(\lambda)\psi = \langle \mathcal{G}_\lambda^{-v}, \psi \rangle_2, \quad G(\lambda)q = q \mathcal{G}_\lambda^v.$$

Since (see [7], Lemma 2.1)

$$(\mu - \lambda)(H_v + \lambda)^{-1}G(\mu) = G(\lambda) - G(\mu),$$

one obtains (see [7], Lemma 2.2)

$$\begin{aligned} &(\mu - \lambda)\check{G}(\lambda)G(\mu) \\ &= \tau(G(\lambda) - G(\mu)) = \tau(G(\nu) - G(\mu)) - \tau(G(\nu) - G(\lambda)) \\ &= \frac{\sqrt{\mu - |v|^2/4}}{4\pi} - \frac{\sqrt{\nu - |v|^2/4}}{4\pi} - \left(\frac{\sqrt{\lambda - |v|^2/4}}{4\pi} - \frac{\sqrt{\nu - |v|^2/4}}{4\pi} \right) \\ &= \frac{\sqrt{\mu - |v|^2/4}}{4\pi} - \frac{\sqrt{\lambda - |v|^2/4}}{4\pi}. \end{aligned}$$

The thesis about $H_{v,\alpha}$ and its resolvent then follows from Theorem 2.1 in [7]. As regards the spectral properties of $H_{v,\alpha}$ one proceeds as in [1], Theorem 1.1.4 . $\square$

Remark 2.2. Note that, as expected, $H_{v,\alpha}$ converges in norm resolvent sense to $H_{\alpha,0}$ as $|v| \downarrow 0$.

Theorem 2.3. *The quadratic form associated with $H_{v,\alpha}$ is*

$$F_{v,\alpha} : D(F_0) \times D(F_0) \rightarrow \mathbb{R}, \quad F_{v,\alpha} = F_{\alpha,0} + Q_v,$$

where $D(F_0)$ is the domain of the quadratic form $F_{\alpha,0}$ associated with $H_{\alpha,y}$, $y = 0$, (see Theorem 1.1) and

$$Q_v : D(F_0) \times D(F_0) \rightarrow \mathbb{R},$$

$$Q_v(\psi, \phi) := \langle iL_v \psi_\lambda, \phi_\lambda \rangle_2 + \bar{q}_\psi \langle \mathcal{G}_\lambda, iL_v \phi_\lambda \rangle_2 + q_\phi \langle iL_v \psi_\lambda, \mathcal{G}_\lambda \rangle_2.$$

Proof. Given ψ and ϕ in $D(H_{v,\alpha})$ put

$$q_\psi := \Gamma_{v,\alpha}(\lambda)^{-1} \psi_\lambda(0), \quad q_\phi := \Gamma_{v,\alpha}(\lambda)^{-1} \phi_\lambda(0).$$

Then

$$\begin{aligned} \langle (H_{v,\alpha} + \lambda) \psi, \phi \rangle_2 &= \langle (H_v + \lambda) \psi_\lambda, \phi_\lambda \rangle_2 + q_\phi \langle (H_v + \lambda) \psi_\lambda, \mathcal{G}_\lambda^\vee \rangle_2 \\ &= \langle (H_v + \lambda) \psi_\lambda, \phi_\lambda \rangle_2 + \Gamma_{v,\alpha}(\lambda) \bar{q}_\psi q_\phi. \end{aligned}$$

Thus one is lead to define the quadratic form

$$F_{v,\alpha} : D(F_v) \times D(F_v) \rightarrow \mathbb{R}$$

by

$$D(F_v) := \{ \psi \in H^1(\mathbb{R}^3) : \psi = \psi_\lambda + q_\psi \mathcal{G}_\lambda^\vee, \quad \psi_\lambda \in H^1(\mathbb{R}^3), \quad q_\psi \in \mathbb{C} \},$$

$$\begin{aligned} (F_{v,\alpha} + \lambda)(\psi, \phi) \\ := \langle (-\Delta + iL_v + \lambda)^{1/2} \psi_\lambda, (-\Delta + iL_v + \lambda)^{1/2} \phi_\lambda \rangle_2 + \Gamma_{v,\alpha}(\lambda) \bar{q}_\psi q_\phi. \end{aligned}$$

It is then straightforward to check that $F_{v,\alpha}$ is closed and bounded from below. Thus $F_{v,\alpha}$ is the quadratic form associated with $H_{v,\alpha}$. Since

$$\mathcal{G}_\lambda^\vee - \mathcal{G}_\lambda \in H^1(\mathbb{R}^3),$$

one obtains $D(F_v) \equiv D(F_0)$. Re-writing the quadratic form above by using the decomposition entering in the definition of $D(F_0)$ and noting that

$$\forall \psi \in H^1(\mathbb{R}^3) \quad (F_v + \lambda)(\mathcal{G}_\lambda^\vee - \mathcal{G}_\lambda, \psi) = \langle \mathcal{G}_\lambda, iL_v \psi \rangle_2,$$

one obtains

$$\begin{aligned} (F_{v,\alpha} + \lambda)(\psi, \phi) &= \langle \nabla \psi_\lambda, \nabla \phi_\lambda \rangle_2 + \lambda \langle \psi_\lambda, \phi_\lambda \rangle_2 + Q_v(\psi, \phi) \\ &\quad + (\Gamma_{v,\alpha}(\lambda) + (F_{v,\alpha} + \lambda)(\mathcal{G}_\lambda^\vee - \mathcal{G}_\lambda, \mathcal{G}_\lambda^\vee - \mathcal{G}_\lambda)) \bar{q}_\psi q_\phi. \end{aligned}$$

Since L_v is skew-adjoint one has $\langle L_v \psi, \psi \rangle_2 = 0$ for any real-valued $\psi \in H^1(\mathbb{R}^3)$. Thus, by taking a real-valued $J_\epsilon \in C_0^\infty(\mathbb{R}^3)$ such that J_ϵ weakly converges to the

Dirac mass at the origin as $\epsilon \downarrow 0$, one obtains (here $*$ denotes convolution)

$$\begin{aligned}
 & (F_{v,\alpha} + \lambda)(\mathcal{G}_\lambda^\vee - \mathcal{G}_\lambda, \mathcal{G}_\lambda^\vee - \mathcal{G}_\lambda) = \langle \mathcal{G}_\lambda, iL_v(\mathcal{G}_\lambda^\vee - \mathcal{G}_\lambda) \rangle_2 \\
 & = \lim_{\epsilon \downarrow 0} \langle iL_v \mathcal{G}_\lambda * J_\epsilon, \mathcal{G}_\lambda^\vee - \mathcal{G}_\lambda * J_\epsilon \rangle_2 = \lim_{\epsilon \downarrow 0} \langle iL_v \mathcal{G}_\lambda * J_\epsilon, \mathcal{G}_\lambda^\vee \rangle_2 \\
 & = \lim_{\epsilon \downarrow 0} \langle (H_v + \lambda)(\mathcal{G}_\lambda^\vee - \mathcal{G}_\lambda) * J_\epsilon, \mathcal{G}_\lambda^\vee \rangle_2 \\
 & = \lim_{\epsilon \downarrow 0} \langle \mathcal{G}_\lambda^\vee - \mathcal{G}_\lambda, (H_v + \lambda)G_\lambda^\vee * J_\epsilon \rangle_2 = \lim_{\epsilon \downarrow 0} \langle \mathcal{G}_\lambda^\vee - \mathcal{G}_\lambda, J_\epsilon \rangle_2 \\
 & = (\mathcal{G}_\lambda^\vee - \mathcal{G}_\lambda)(0) = \frac{\sqrt{\lambda} - \sqrt{\lambda - |\mathbf{v}|^2/4}}{4\pi} \equiv \Gamma_\alpha(\lambda) - \Gamma_{v,\alpha}(\lambda),
 \end{aligned}$$

and the proof is done. $\square$

Remark 2.4. Let J_ϵ be a real-valued, compactly supported smooth function weakly converging to the Dirac mass a zero as $\epsilon \downarrow 0$. For any $\psi = \psi_\lambda + q_\psi \mathcal{G}_\lambda$ and $\phi = \phi_\lambda + q_\phi \mathcal{G}_\lambda$, let us define $\psi_\epsilon := \psi_\lambda + q_\psi \mathcal{G}_\lambda * J_\epsilon$ and $\phi_\epsilon := \phi_\lambda + q_\phi \mathcal{G}_\lambda * J_\epsilon$. Then, since L_v is skew-adjoint, one has

$$\begin{aligned}
 & \lim_{\epsilon \downarrow 0} \langle iL_v \psi_\epsilon, \phi_\epsilon \rangle_2 = \lim_{\epsilon \downarrow 0} (\langle iL_v \psi_\lambda, \phi_\lambda \rangle_2 + \bar{q}_\psi \langle \mathcal{G}_\lambda * J_\epsilon, iL_v \phi_\lambda \rangle_2 \\
 & \quad + q_\phi \langle iL_v \psi_\lambda, \mathcal{G}_\lambda * J_\epsilon \rangle_2 - i\bar{q}_\psi q_\phi \langle L_v \mathcal{G}_\lambda * J_\epsilon, \mathcal{G}_\lambda * J_\epsilon \rangle_2) \\
 & = \lim_{\epsilon \downarrow 0} Q_v(\psi_\epsilon, \phi_\epsilon) = Q_v(\psi, \phi).
 \end{aligned}$$

Thus Q_v is the natural extension to $D(F_0)$ of the quadratic form associated with iL_v .

3. THE SCHRÖDINGER EQUATION WITH A MOVING POINT INTERACTION

Let us now consider a differentiable curve $y : \mathbb{R} \rightarrow \mathbb{R}^3$ and put $\mathbf{v}(t) \equiv \frac{dy}{dt}(t)$. Thus one has the families of self-adjoint operators and associated quadratic forms

$$\begin{aligned}
 & H_{\alpha,y}(t) : D(H_{\alpha,y}(t)) \rightarrow L^2(\mathbb{R}^3), \\
 & F_{\alpha,y}(t) : D(F_y(t)) \times D(F_y(t)) \rightarrow \mathbb{R}, \\
 & H_{v,\alpha}(t) : D(H_{\alpha,y}(t)) \rightarrow L^2(\mathbb{R}^3), \\
 & F_{v,\alpha}(t) : D(F_0) \times D(F_0) \rightarrow \mathbb{R}.
 \end{aligned}$$

Now we can state our main result.

Theorem 3.1. *Let $y \in C^2(\mathbb{R}; \mathbb{R}^3)$. Then there is a unique strongly continuous unitary propagator*

$$U_{t,s} : L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3), \quad (t, s) \in \mathbb{R}^2,$$

such that

1)

$$U_{t,s} D(F_{\alpha,y}(s)) = D(F_{\alpha,y}(t));$$

2) each $U_{t,s}$ is strongly continuous as a map from $D(F_y(s))$ onto $D(F_y(t))$ with respect to the Banach topologies induced by the bounded from below closed quadratic forms $F_{\alpha,y}(s)$ and $F_{\alpha,y}(t)$ respectively;

3)

$$\forall \psi \in D(F_y(s)), \quad t \mapsto F_{\alpha,y}(t)(U_{t,s}\psi, U_{t,s}\psi) \quad \text{is in } C(\mathbb{R}; \mathbb{R});$$

4)

$$\forall \psi \in D(F_y(s)), \quad t \mapsto U_{t,s}\psi \quad \text{is in } C^1(\mathbb{R}; D(F_y(\cdot))^*),$$

where $D(F_y(t))^*$ denotes the dual of $D(F_y(t))$ with respect to the $L^2(\mathbb{R}^3)$ scalar product;

5)

$$\forall \psi \in D(F_y(s)), \forall \phi \in D(F_y(t)), \quad \left(i \frac{d}{dt} U_{t,s} \psi, \phi \right)_t = F_{\alpha,y}(t)(U_{t,s} \psi, \phi),$$

where $(\cdot, \cdot)_t$ denotes the duality between $D(F_y(t))$ and $D(F_y(t))^*$.

If $y \in C^3(\mathbb{R}; \mathbb{R}^3)$, then

6)

$$U_{t,s} \tilde{D}(H_{\alpha,y}(s)) = \tilde{D}(H_{\alpha,y}(t)),$$

where

$$\tilde{D}(H_{\alpha,y}(t)) := V_t D(H_{\alpha,y}(t)), \quad V_t \psi(x) := e^{i v(t) \cdot x / 2} \psi(x).$$

Proof. By Theorem 2.3 we have that $y \in C^2(\mathbb{R}; \mathbb{R}^3)$ implies that

$$\forall \psi, \phi \in D(F_0), \quad t \mapsto F_{v,\alpha}(t)(\psi, \phi) \quad \text{is in } C^1(\mathbb{R}).$$

Let $T > 0$. By Kisyński's theorem (see the Appendix) applied to the family of strictly positive self-adjoint operators

$$H_{v,\alpha}(t) + \lambda, \quad t \in [-T, T], \quad \lambda > (4\pi \min(0, \alpha))^2 + \frac{1}{4} \sup_{t \in [-T, T]} |v(t)|,$$

one knows that $H_{v,\alpha}(t)$, $t \in [-T, T]$, generates a strongly continuous unitary propagator $\tilde{U}_{t,s}^T$, $(s, t) \in [-T, T]^2$. By unicity if $T' > T$, then $\tilde{U}_{s,t}^T = U_{s,t}^{T'}$ for any $(s, t) \in [-T, T]^2 \subset [-T', T']^2$. Thus we obtain a unique strongly continuous unitary propagator $\tilde{U}_{t,s}$, $(s, t) \in \mathbb{R}^2$, generated by the family $H_{v,\alpha}(t)$, $t \in \mathbb{R}$. Such a propagator is also a strongly continuous propagator on $D(F_0)$ with respect to the Banach topology induced by the bounded from below closed quadratic form $F_{\alpha,0}$.

Considering the unitary map

$$T_t : L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3), \quad T_t \psi(x) := \psi(x + y(t)),$$

we then define the strongly continuous unitary propagator

$$U_{t,s} := T_t^{-1} \tilde{U}_{t,s} T_s.$$

Since T_t is a bounded operator from $D(F_y(t))$ onto $D(F_0)$, one has that $U_{t,s}$ is a bounded operator from $D(F_y(s))$ onto $D(F_y(t))$ with respect to the Banach topologies induced by the bounded from below closed quadratic forms $F_{\alpha,y}(s)$ and $F_{\alpha,y}(t)$, respectively. Moreover, for all $\psi \in D(F_y(s))$, the map

$$t \mapsto F_{\alpha,y}(t)(U_{t,s} \psi, U_{t,s} \psi) \equiv F_{\alpha,0}(\tilde{U}_{t,s} T_s \psi, \tilde{U}_{t,s} T_s \psi)$$

is continuous. Let us now show that, for all $\psi \in D(F_y(s))$ and for all $\phi \in D(F_y(t))$, one has

$$\left(i \frac{d}{dt} U_{t,s} \psi, \phi \right)_t = F_{\alpha,y}(t)(U_{t,s} \psi, \phi).$$

For any $\psi \in D(F_y(s))$ and $\phi \in D(F_y(t))$ there exist $\tilde{\psi}$ and $\tilde{\phi}$ in $D(F_0)$ such that $T_s^{-1} \tilde{\psi} = \psi$ and $T_t^{-1} \tilde{\phi} = \phi$. Thus equivalently we need to show that

$$\left(i \frac{d}{dt} T_t^{-1} \tilde{U}_{t,s} \tilde{\psi}, T_t^{-1} \tilde{\phi} \right)_t = F_{\alpha,y}(t)(T_t^{-1} \tilde{U}_{t,s} \tilde{\psi}, T_t^{-1} \tilde{\phi}) \equiv F_{\alpha,0}(\tilde{U}_{t,s} \tilde{\psi}, \tilde{\phi}).$$

Since

$$\begin{aligned} \left(i \frac{d}{dt} T_t^{-1} \tilde{U}_{t,s} \tilde{\psi}, T_t^{-1} \tilde{\phi} \right)_t &= \left(i T_t \frac{d}{dt} T_t^{-1} \tilde{U}_{t,s} \tilde{\psi}, \tilde{\phi} \right) \\ &= \left(i T_t \left(\frac{d}{dt} T_t^{-1} \right) \tilde{U}_{t,s} \tilde{\psi}, \tilde{\phi} \right) + \left(i \frac{d}{dt} \tilde{U}_{t,s} \tilde{\psi}, \tilde{\phi} \right) \\ &= \left(i T_t \left(\frac{d}{dt} T_t^{-1} \right) \tilde{U}_{t,s} \tilde{\psi}, \tilde{\phi} \right) + F_{v,\alpha}(t)(\tilde{U}_{t,s} \tilde{\psi}, \tilde{\phi}), \end{aligned}$$

by Theorem 2.3 we need to show that

$$\left(i T_t \frac{d}{dt} T_t^{-1} \tilde{\psi}, \tilde{\phi} \right) = -Q_v(\tilde{\psi}, \tilde{\phi}).$$

This is obviously true in the case where either $\tilde{\psi}$ or $\tilde{\phi}$ is in $H^1(\mathbb{R}^3)$ and, by taking J_ϵ as in Remark 2.4,

$$\begin{aligned} \left(T_t \frac{d}{dt} T_t^{-1} \mathcal{G}_\lambda, \mathcal{G}_\lambda \right) &= \lim_{\epsilon \downarrow 0} \left\langle T_t \frac{d}{dt} T_t^{-1} \mathcal{G}_\lambda * J_\epsilon, \mathcal{G}_\lambda * J_\epsilon \right\rangle_2 \\ &= - \lim_{\epsilon \downarrow 0} \langle L_v \mathcal{G}_\lambda * J_\epsilon, \mathcal{G}_\lambda * J_\epsilon \rangle_2 = 0. \end{aligned}$$

Thus point 5) is proven. Point 6) follows from Kisyński's theorem again by noting that if $y \in C^3(\mathbb{R}; \mathbb{R}^3)$, then $\tilde{U}_{t,s}$ maps $D(H_{v,\alpha}(s))$ onto $D(H_{v,\alpha}(t))$ and that

$$D(H_{v,\alpha}(t)) \equiv T_t V_t D(H_{\alpha,y}(t)).$$

□

4. APPENDIX: KISYŃSKI'S THEOREM

For the reader's convenience in this Appendix we recall Kisyński's theorem. For the proof we refer to Kisyński's original paper [4] (see in particular [4], section 8; see also [8], section II.7).

Let us recall that the double family $U_{t,s}$, $(t, s) \in [T_1, T_2]^2$, is said to be a strongly continuous propagator on the Hilbert space $\mathcal{H}$ if each $U_{t,s}$ is a bounded operator on $\mathcal{H}$, the map $(t, s) \mapsto U_{t,s}$ is strongly continuous, $U_{s,s} = 1$ and the Chapman-Kolmogorov equation

$$U_{t,r} U_{r,s} = U_{t,s}$$

holds. Such a propagator is then said to be unitary if each $U_{t,s}$ is unitary.

Theorem 4.1. *Let $A(t)$, $t \in [T_1, T_2]$, be a family of strictly positive self-adjoint operators on the Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ with time-independent form domain $\mathcal{H}_+$. Suppose that*

$$\forall \psi, \phi \in \mathcal{H}_+ \quad t \mapsto F(t)(\psi, \phi) \quad \text{is in } C^k([T_1, T_2]; \mathbb{R}),$$

where $F(t)$ denotes the quadratic form associated with $A(t)$.

If $k = 1$, then there is a unique strongly continuous unitary propagator

$$U_{t,s} : \mathcal{H} \rightarrow \mathcal{H}, \quad (s, t) \in [T_1, T_2]^2,$$

such that

1)

$$U_{t,s} \mathcal{H}_+ = \mathcal{H}_+;$$

2) $U_{t,s}$ is a strongly continuous propagator on $(\mathcal{H}_+, \langle \cdot, \cdot \rangle_+)$, where $\langle \cdot, \cdot \rangle_+$ denotes any of the equivalent scalar products

$$\langle \psi, \phi \rangle_{t,+} := F(t)(\psi, \phi);$$

3)

$$\forall \psi \in \mathcal{H}_+ \quad t \mapsto U_{t,s}\psi \quad \text{is in } C^1([T_1, T_2]; \mathcal{H}_-),$$

where $(\mathcal{H}_-, \langle \cdot, \cdot \rangle_-)$ is the completion of $\mathcal{H}$ endowed with any of the equivalent scalar products

$$\langle \psi, \phi \rangle_{t,-} := \langle A(t)^{-1/2}\psi, A(t)^{-1/2}\phi \rangle;$$

4)

$$\forall \psi, \phi \in \mathcal{H}_+ \quad \left(i \frac{d}{dt} U_{t,s}\psi, \phi \right) = F(t)(U_{t,s}\psi, \phi),$$

where $(\cdot, \cdot)$ denotes the duality between $\mathcal{H}_+$ and $\mathcal{H}_-$. If $k = 2$, then

5)

$$U_{t,s}D(A(s)) = D(A(t)),$$

6)

$$\forall \psi \in D(A(s)) \quad t \mapsto U_{t,s}\psi \quad \text{is in } C^1([T_1, T_2]; \mathcal{H}) \cap C([T_1, T_2]; D(A(\cdot))),$$

7)

$$i \frac{d}{dt} U_{t,s}\psi = A(t)U_{t,s}\psi.$$

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