A SHORT PROOF OF BING’S CHARACTERIZATION OF S^3

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ABSTRACT. We give a short proof of Bing’s characterization of S^3 : a compact, connected 3-manifold M is S^3 if and only if every knot in M is isotopic into a ball.

Let M be a closed orientable 3-manifold. We assume familiarity with the basic notions of irreducible and prime 3-manifolds (see, e.g., [3] or [4]) and the basic results about Heegaard splittings of compact 3-manifolds (see, e.g., [7]). By *genus* we always mean Heegaard genus. A knot $k \subset M$ (that is, a smooth embedding of the circle into M) is called *irreducible* if its exterior $E(k) = M \setminus N(k)$ is an irreducible 3-manifold. In his own words, Bing’s Theorem [1, Theorem 1] is:

Theorem 1 (Bing). *A compact, connected 3-manifold M is topologically S^3 if each simple closed curve in M lies in a topological cube in M .*

By “topological cube” Bing meant what we usually call a ball. Clearly, any knot in S^3 is contained in a ball. If a knot k in a manifold $M \not\cong S^3$ is contained in a ball (say B), then by considering the boundary of B we see that k is not irreducible. Thus, Theorem 1 follows from:

Theorem 2. *Any compact, connected 3-manifold admits an irreducible knot.*

In [5, Theorem 8.1] Jaco and Rubinstein gave a very short proof of Theorem 1 for irreducible manifolds, but their proof relies on the existence of 0-efficient triangulations. The purpose of this note is giving a short, elementary proof of Theorem 1.

1. THE PROOF

We prove Theorem 2; as remarked above Theorem 1 follows.

Case One: M is prime. First, when M has genus at most one, let k be a knot on a Heegaard torus (in M) with $E(k)$ a Seifert fibered space over the disk with 2 exceptional fibers, which is irreducible.

Second, when M has genus two or more, then $M \not\cong S^2 \times S^1$ and hence is irreducible. Let $M = V_1 \cup_{\Sigma} V_2$ be a minimal genus Heegaard splitting of M . By Waldhausen [8] (see also [7, Theorem 3.8]) Σ is irreducible. Let k be a core of a 1-handle in V_1 . Then Σ is an irreducible Heegaard surface for $E(k)$; Haken [2] (see also [7, Theorem 3.4]) showed that every Heegaard splitting of a reducible manifold is reducible; hence, $E(k)$ is irreducible.

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Remark 3. In Case One, $\partial E(k)$ is incompressible. For manifolds of genus one or less, this is so by construction of k . For manifold of genus two or more, if $\partial E(k)$ compressed, then (since $E(k)$ is irreducible) $E(k)$ would be a solid torus; but that implies M has genus at most one, contradiction.

Case Two: M is composite. By Kneser [6] M has a prime decomposition as $M \cong M_1 \# \cdots \# M_n$ with M_i prime ($i = 1, \dots, n$). Let $k_i \subset M_i$ be the knot obtained in Case One, let $k = \#_{i=1}^n k_i \subset M$ be their connected sum, and let $\mathcal{A} \subset E(k)$ be a collection of annuli that decomposes k into its summands, that is, the components of $E(k)$ cut open along $\mathcal{A}$ are homeomorphic to $E(k_i)$ ($i = 1, \dots, n$).

Let S be a sphere in $E(k)$; we will prove that S bounds a ball. By isotopy of S , minimize $S \cap \mathcal{A}$. Assume that $S \cap \mathcal{A} \neq \emptyset$. Since $\chi(S) = 2$, S cut open along $\mathcal{A}$ has disk components, and let D be such a disk. Then D is contained in some component of $E(k)$ cut open along $\mathcal{A}$ (which is homeomorphic to $E(k_i)$, for some i). By Remark 3, $\partial E(k_i)$ is incompressible and hence D is boundary parallel, contradicting the minimality assumption. Hence $S \cap \mathcal{A} = \emptyset$, and S is contained in a component of $E(k)$ cut open along $\mathcal{A}$. By the construction in Case One S bounds a ball. Thus every sphere in $E(k)$ bounds a ball and k is an irreducible knot, completing the proof of Theorems 2 and 1.

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