A NOTE ON A PROBLEM BY RICCERI ON THE AMBROSETTI-RABINOWITZ CONDITION

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ABSTRACT. In this note we give an answer to Ricceri's open problem involving the Ambrosetti-Rabinowitz superlinear condition.

1. INTRODUCTION

Throughout this note, $\Omega \subset \mathbb{R}^N$ is a nonempty open bounded set with sufficiently regular boundary $\partial\Omega$. Denote by $\mathcal{A}$ the set of all continuous functions $g : \mathbb{R} \rightarrow \mathbb{R}$ such that, if $N \neq 1$, there exists $q > 0$ with $q < \frac{N+2}{N-2}$ if $N \geq 3$, satisfying

$$\sup_{t \in \mathbb{R}} \frac{|g(t)|}{1 + |t|^q} < +\infty.$$

For each $c > 2$ and $r \geq 0$ we put

$$\mathcal{B}(r, c) \stackrel{\text{def}}{=} \left\{ f \in \mathcal{A} : 0 < c \int_0^\xi f(t) dt \leq \xi f(\xi) \text{ for } \xi \in \mathbb{R} \setminus \{0\} \text{ with } |\xi| \geq r \right\}.$$

Note that what defines the set $\mathcal{B}(r, c)$ is the well-known Ambrosetti-Rabinowitz superlinear condition.

For every $f \in \mathcal{A}$ we put $\Phi_f(u) = \int_\Omega \left(\int_0^{u(x)} f(t) dt \right) dx$ for all $u \in W_0^{1,2}(\Omega)$. We equip $W_0^{1,2}(\Omega)$ with the usual norm $\|u\| = \left(\int_\Omega |\nabla u|^2 dx \right)^{\frac{1}{2}}$. Let us consider the following subsets of $\mathcal{A}$:

$$\begin{aligned} \mathcal{C} &\stackrel{\text{def}}{=} \left\{ f \in \mathcal{A} : \inf_{\rho > 0} \inf_{\|u\| < \rho} \frac{\sup_{\|v\| < \rho} \Phi_f(v) - \Phi_f(u)}{\rho^2 - \|u\|^2} < \frac{1}{2} \right\}, \\ \mathcal{D} &\stackrel{\text{def}}{=} \left\{ f \in \mathcal{A} : \lim_{t \rightarrow 0} \frac{f(t)}{t} = 0 \right\}. \end{aligned}$$

When $f \in \mathcal{D} \cap \mathcal{B}(r, c)$ for some $r \geq 0$ and $c > 2$, the classical Ambrosetti-Rabinowitz result (see [1]) assures that the problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

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has at least a non-zero weak solution in $W_0^{1,2}(\Omega)$. In recent years B. Ricceri has studied the above problem without assuming the condition $f \in \mathcal{D}$. In particular, he has established the following theorem.

Theorem A (Theorem 4 of [4]). *Let $f \in \mathcal{B}(r, c)$ for some $r \geq 0$ and $c > 2$. Then, for every $\rho > 0$ and every*

$$\mu > \inf_{\|u\| < \rho} \frac{\sup_{\|v\| < \rho} \Phi_f(v) - \Phi_f(u)}{\rho^2 - \|u\|^2}$$

the problem

$$\begin{cases} -\Delta u = \frac{1}{2\mu} f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

admits at least two weak solutions in $W_0^{1,2}(\Omega)$, one of which has norm less than ρ .

A comparison between Theorem A and the Ambrosetti-Rabinowitz result leads to the following problem formulated by B. Ricceri in [3].

Problem. *Does it hold the inclusion $\mathcal{B}(r, c) \subseteq \mathcal{C}$ for all $c > 2$ and $r \geq 0$?*

In this note we will prove that the inclusion in the above problem does not hold in general and that it holds only when $r = 0$. Hence, in this latter case, we can remove the condition $f \in \mathcal{D}$ in the Ambrosetti-Rabinowitz result. Clearly the previous assertion makes sense only if we prove that there exists a function $f \in \mathcal{B}(0, c)$ which does not belong to $\mathcal{D}$. In our result we will prove the existence of such a function as well.

2. THE RESULT

The following theorem gives a complete answer to Ricceri's problem. We first point out that, by the proof, it will be clear that, if $N \geq 3$ and $c \geq \frac{2N}{N-2}$, then $\mathcal{B}(r, c) = \emptyset$ for all $r \geq 0$.

Theorem 1. *Let $c \in]2, \frac{2N}{N-2}[$ if $N \geq 3$ (resp. $c > 2$ if $N = 1, 2$) and $r > 0$. Then, one has:*

- i) $\mathcal{B}(r, c) \setminus \mathcal{C} \neq \emptyset$;
- ii) $\mathcal{B}(0, c) \subseteq \mathcal{C}$;
- iii) $\mathcal{B}(0, c) \setminus \mathcal{D} \neq \emptyset$.

Proof. We restrict ourselves to the case $N \geq 3$. The cases $N = 1, 2$ can be treated in the same manner with some slight modification. To verify i) fix $q \in]c - 1, \frac{N+2}{N-2}[$. Define

$$f(t) = at + b|t|^{q-1}t,$$

for all $t \in \mathbb{R}$, where a is a positive real number satisfying

$$\frac{1}{a} < \mu_1 := \sup_{u \in W_0^{1,2}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |u(x)|^2 dx}{\|u\|^2}$$

and

$$b > a \left(\frac{c}{2} - 1 \right) \left(1 - \frac{c}{q+1} \right)^{-1} r^{1-q}.$$

From the definition of f , the inequalities in condition i) become

$$0 < \frac{ca}{2}\xi^2 + b\frac{c}{q+1}|\xi|^{q+1} \leq a\xi^2 + b|\xi|^{q+1}$$

which are satisfied if $|\xi| \geq r$, taking the choice of b into account. Consequently, we have $f \in \mathcal{B}(r, c)$. Let us show that $f \notin \mathcal{C}$. In view of Theorem 2.1 of [2], it suffices to check that the functional

$$\Psi(u) = \frac{1}{2}\|u\|^2 - \Phi_f(u),$$

$u \in W_0^{1,2}(\Omega)$, has no local minima. In fact, if $v \in W_0^{1,2}(\Omega)$ was a local minimum for Ψ , then, taking into account that Ψ is of class C^2 (this latter fact comes out from the condition $q > 1$), we should have

$$\Psi''(v)(u, u) \geq 0$$

for all $u \in W_0^{1,2}(\Omega)$. Hence

$$\int_{\Omega} |\nabla u(x)|^2 dx - a \int_{\Omega} |u(x)|^2 dx - q \int_{\Omega} |v(x)|^{q-1} |u(x)|^2 dx \geq 0$$

for all $u \in W_0^{1,2}(\Omega)$. In particular, choosing $u = v_1$, where v_1 is an eigenvector related to the first eigenvalue of the laplacian $\lambda_1 = \frac{1}{\mu_1}$, we should have, being $a\mu_1 > 1$,

$$\begin{aligned} 0 &= \int_{\Omega} |\nabla v_1(x)|^2 dx - \frac{a}{a\mu_1} \int_{\Omega} |v_1(x)|^2 dx \\ &> \int_{\Omega} |\nabla v_1(x)|^2 dx - a \int_{\Omega} |v_1(x)|^2 dx - q \int_{\Omega} |v(x)|^{q-1} |v_1(x)|^2 dx \geq 0, \end{aligned}$$

which is absurd.

To prove ii) let $f \in \mathcal{B}(0, c)$. Then, there exists $q \in]1, \frac{N+2}{N-2}[$ such that

$$M := \sup_{t \in \mathbb{R}} \frac{|f(t)|}{1 + |t|^q} < +\infty.$$

Hence,

$$(1) \quad 0 < \int_0^\xi f(t) dt \leq M \left(|\xi| + \frac{1}{q+1} |\xi|^{q+1} \right)$$

for all $\xi \in \mathbb{R}$. At this point, consider the real function

$$g(\xi) = \frac{\int_0^\xi f(t) dt}{|\xi|^c}$$

defined for all $\xi \in \mathbb{R} \setminus \{0\}$. One has $g \in C^1(\mathbb{R} \setminus \{0\})$ and

$$g'(\xi) = \frac{|\xi|^c f(\xi) - c|\xi|^{c-2} \xi \int_0^\xi f(t) dt}{|\xi|^{2c}} = \frac{\xi^2 f(\xi) - c\xi \int_0^\xi f(t) dt}{|\xi|^{c+2}}$$

for all $\xi \in \mathbb{R} \setminus \{0\}$. So, from the fact that $f \in \mathcal{B}(0, c)$, we easily deduce that g is positive, decreasing in $\mathbb{R}_-$ and increasing in $\mathbb{R}_+$. Then, for a suitable positive number L , we have

$$(2) \quad \int_0^\xi f(t) dt \leq L|\xi|^c$$

for all $\xi \in \mathbb{R}$ with $|\xi| \leq 1$. Now, from (1) we can find a positive constant M_1 such that

$$(3) \quad \int_0^\xi f(t)dt \leq M_1 |\xi|^{q+1}$$

for all $\xi \in \mathbb{R} \setminus [-1, 1]$. Hence, putting together (2) and (3), we have

$$(4) \quad \int_0^\xi f(t)dt \leq L|\xi|^c + M_1 |\xi|^{q+1}$$

for all $\xi \in \mathbb{R}$. At this point, from (4), for suitable positive constants C_1, C_2 we have

$$\begin{aligned} \frac{\sup_{\|u\| \leq \rho} \Phi_f(u)}{\rho^2} &\leq \frac{\sup_{\|u\| \leq \rho} \int_\Omega (L|u(x)|^c + M_1|u(x)|^{q+1}) dx}{\rho^2} \\ &\leq \frac{\sup_{\|u\| \leq \rho} C_1 \|u\|^c + C_2 \|u\|^{q+1}}{\rho^2} = C_1 \rho^{c-2} + C_2 \rho^{q-1} \rightarrow 0 \end{aligned}$$

as $\rho \rightarrow 0^+$. This implies $f \in \mathcal{C}$.

We finally prove iii). Let $c_1 \in]c, \frac{2N}{N-2}[$ and $p = \log_2(\frac{c_1}{c_1-c})$. Put

$$f(t) = |t|^{c_1-2}t + g(t)t$$

for all $t \in \mathbb{R}$, where

$$g(t) = \begin{cases} \frac{(t-2^{-n})(2^{-n}+2^{-(nc_1+p)}-t)}{2^{-2(nc_1+p+1)}} & \text{if } n \in \mathbb{N}, t \in [2^{-n}, 2^{-n} + 2^{-(nc_1+p)}], \\ 0 & \text{otherwise.} \end{cases}$$

By an easy calculation we see that

$$\lim_{n \rightarrow +\infty} \frac{f(2^{-n} + 2^{-(nc_1+p+1)})}{2^{-n} + 2^{-(nc_1+p+1)}} = 1;$$

then $f \notin \mathcal{D}$. On the other hand, one has $f \in \mathcal{A}$. Now, let $\xi \neq 0$. It is easily seen that

$$\int_0^\xi f(t)dt > 0.$$

Moreover, if $\xi < 0$, then we have

$$\xi f(\xi) - c \int_0^\xi f(t)dt = (1 - \frac{c}{c_1})|\xi|^{c_1} > 0.$$

While, if $\xi > 0$, denoting by n the small positive integer such that $2^{-n} < \xi$, we have

$$\begin{aligned}
 & \xi f(\xi) - c \int_0^\xi f(t) dt \\
 & \geq \frac{c_1 - c}{c_1} |\xi|^{c_1} - c \sum_{k=n}^{\infty} \int_{2^{-k}}^{2^{-(kc_1+p)}} t \frac{(t - 2^{-k})(2^k + 2^{-(kc_1+p)} - t)}{2^{-2(kc_1+p+1)}} \\
 & \geq \frac{c_1 - c}{c_1} |\xi|^{c_1} - \frac{c}{6} \sum_{k=n}^{\infty} 2^{-(kc_1+p)} (2^{-(k-1)} + 2^{-(kc_1+p)}) \\
 & \geq \frac{c_1 - c}{c_1} 2^{-nc_1} - 2^{-p} \frac{c}{3} \sum_{k=n}^{\infty} 2^{kc_1} = 2^{-nc_1} \left(\frac{c_1 - c}{c_1} - 2^{-p} \frac{c}{3(1 - 2^{-c_1})} \right) \\
 & \geq 2^{-nc_1} \left(\frac{c_1 - c}{c_1} - 2^{-p} \right) = 0.
 \end{aligned}$$

Consequently, $f \in \mathcal{B}(0, c)$. The proof is now complete. $\square$

Remark. From the above proof it is easy to check that, in point of fact, the following inclusion holds:

$$\left\{ f \in \mathcal{A} : 0 < c \int_0^\xi f(t) dt \leq \xi f(\xi) \text{ for } \xi \in \mathbb{R} \setminus \{0\} \text{ with } |\xi| \leq r \right\} \subseteq \mathcal{C}$$

for all $r > 0$ and $c > 2$.

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