EXISTENCE OF STRONG TRAVELLING WAVE PROFILES TO 2×2 SYSTEMS OF VISCOUS CONSERVATION LAWS

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ABSTRACT. In this paper, we prove the existence of strong travelling wave profiles for a class of 2×2 viscous conservation laws when the corresponding inviscid systems are hyperbolic. Besides some technical assumptions, the only main assumption is the hyperbolicity. Therefore, the existence theory can be applied to systems which are not strictly hyperbolic. Moreover, the characteristic fields can be neither genuinely nonlinear nor linearly degenerate.

1. INTRODUCTION

Consider a 2×2 system of hyperbolic conservation laws with artificial viscosity

$$(1.1) \quad \begin{cases} u_t + f(u, v)_x = \varepsilon u_{xx}, \\ v_t + g(u, v)_x = \varepsilon v_{xx}, \end{cases}$$

where $\varepsilon > 0$ is a constant. In what follows, we assume that $f \in C^{2m+1}(\mathbb{R}^2)$ and $g \in C^{2n+1}(\mathbb{R}^2)$, where m and n are any non-negative integers. A lot of work has been done on the existence of solutions to the 2×2 systems of conservation laws with two constant end states; cf. [6, 7, 8, 9, 10] and the references therein. In this paper, we assume that the flux functions $f(u, v)$ and $g(u, v)$ satisfy the following conditions:

$$(H1) \quad \frac{\partial f(u, v)}{\partial v} \leq 0, \quad \frac{\partial g(u, v)}{\partial u} \leq 0.$$

(H2) Set

$$A_i = \left\{ (u, v) : \frac{\partial^i f(u, v)}{\partial v^i} = 0, \quad (u, v) \in \mathbb{R}^2 \right\}, \quad i = 1, 2, \dots, 2m,$$

$$B_j = \left\{ (u, v) : \frac{\partial^j g(u, v)}{\partial u^j} = 0, \quad (u, v) \in \mathbb{R}^2 \right\}, \quad j = 1, 2, \dots, 2n.$$

Assume

$$A_1 \subset A_i, \quad i = 2, 3, \dots, 2m, \quad B_1 \subset B_j, \quad j = 2, 3, \dots, 2n;$$

$$(H3) \quad \left. \frac{\partial^{2m+1} f(u, v)}{\partial v^{2m+1}} \right|_{A_1} < 0, \quad \left. \frac{\partial^{2n+1} g(u, v)}{\partial u^{2n+1}} \right|_{B_1} < 0.$$

(H4) For every bounded set U in $\mathbb{R}$, the sets $\{f(u, v) : u \in \mathbb{R}, v \in U\}$ and $\{g(u, v) : u \in U, v \in \mathbb{R}\}$ are bounded.

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About these assumptions, condition (H1) implies that the system when $\epsilon = 0$ is hyperbolic, that is, the matrix $(\nabla_{(u,v)} f(u, v), \nabla_{(u,v)} g(u, v))$ has two real eigenvalues $\lambda_1(u, v) \leq \lambda_2(u, v)$ with corresponding right eigenvectors $r_1(u, v)$ and $r_2(u, v)$. In the following discussion, $r_i(u, v) \cdot \nabla_{(u,v)} \lambda_i$, $i = 1, 2$, may change sign so that the characteristic fields can be neither genuinely nonlinear nor linearly degenerate; cf. [5]. The conditions (H2) and (H3) are used to prove some monotone properties of the function (u, v) along the travelling wave profile where some higher derivatives of the solution (u, v) are considered. Basically, the condition (H2) ensures that the higher order derivatives $\frac{\partial^i f(u, v)}{\partial v^i} = 0$ (or $\frac{\partial^i g(u, v)}{\partial u^i} = 0$) up to the order $2m$ (or $2n$) at the points when $\frac{\partial f(u, v)}{\partial v} = 0$ (or $\frac{\partial g(u, v)}{\partial u} = 0$), while (H3) ensures that the next derivatives have a definite sign. Finally, the last condition (H4) is used to obtain the boundedness on the travelling wave solution.

In fact, for a given constant s as the wave speed, a travelling wave profile $(u(\xi), v(\xi))$ with $\xi = x - st$ to the system (1.1) satisfies the following differential equations:

$$(1.2) \quad \begin{cases} \epsilon u'' = f'(u, v) - su', \\ \epsilon v'' = g'(u, v) - sv', \end{cases}$$

where $'$ denotes differentiation with respect to ξ .

The main purpose of this paper is to show that for any given constant states (u^-, v^-) and (u^+, v^+) satisfying the Rankine-Hugoniot condition

$$(1.3) \quad s = \frac{f(u^+, v^+) - f(u^-, v^-)}{u^+ - u^-} = \frac{g(u^+, v^+) - g(u^-, v^-)}{v^+ - v^-} \neq 0,$$

there exists a solution to (1.2) with boundary condition

$$(1.4) \quad (u(-\infty), v(-\infty)) = (u^-, v^-), \quad (u(\infty), v(\infty)) = (u^+, v^+).$$

This statement will be proved in the next section by obtaining the a priori estimate on the boundedness of the travelling wave solution profiles. Note also that the result in this paper does not include the case when $s = 0$.

2. A PRIORI ESTIMATE

As in [1, 2], for the global existence of solutions to (1.2) and (1.4), it is sufficient to establish the *a priori* estimate. For this, we first consider the problem in a bounded domain with two parameters μ and L as in the following theorem; cf. [1].

Theorem 2.1. *Consider the boundary value problem*

$$(2.1) \quad \begin{cases} \epsilon u'' = \mu f'(u, v) - su', \\ \epsilon v'' = \mu g'(u, v) - sv', \end{cases}$$

with

$$(2.2) \quad (u(-L), v(-L)) = (\mu u^-, \mu v^-), \quad (u(L), v(L)) = (\mu u^+, \mu v^+).$$

Here the parameters μ and L satisfy $\mu \in [0, 1]$ and $L \geq 1$. Under the conditions (H1)-(H3), the solution $(u(\xi), v(\xi))$ of (2.1) and (2.2) has exactly one of the following properties:

1. $u(\xi)$ and $v(\xi)$ are constants on $[-L, L]$;

2. $u(\xi)$ is increasing (or decreasing) with no critical points in $[-L, L]$, while $v(\xi)$ has at most one critical point in $(-L, L)$ corresponding to a local maximum (or minimum);

3. $v(\xi)$ is increasing (or decreasing) with no critical points in $[-L, L]$, while $u(\xi)$ has at most one critical point in $(-L, L)$ corresponding to a local maximum (or minimum).

Remark 2.1. Note that when $\mu = 0$, the system (2.1) is a simple system of linear equations. Moreover, for finite $L > 0$, the problem (2.1) and (2.2) can be solved by the standard contraction mapping theorem. Since the original problem (1.2) and (1.4) correspond to the case when $\mu = 1$ and $L = \infty$, as in [1], it suffices to prove a uniform bound on the solutions to (2.1) and (2.2) which is independent of μ and L .

Proof of Theorem 2.1. Let $(u(\xi), v(\xi))$ be a non-constant solution of (2.1) on $[-L, L]$. If ξ_0 is a critical point of $u(\xi)$. In the following, we only consider the case when

$$(2.3) \quad \frac{\partial}{\partial v} f(u(\xi_0), v(\xi_0)) = 0.$$

It is because the proof for the case when (2.3) does not hold is simpler, which corresponds to $m = 0$ in the following discussion.

By using the condition (H2), (2.3) implies that

$$(2.4) \quad \frac{\partial^i}{\partial v^i} f(u(\xi_0), v(\xi_0)) = 0, \quad i = 2, 3, \dots, 2m.$$

(H3) gives

$$(2.5) \quad \frac{\partial^{2m+1}}{\partial v^{2m+1}} f(u(\xi_0), v(\xi_0)) < 0.$$

Differentiating the first equation in (2.1) with respect to ξ ($k - 2$) times, for $3 \leq k \leq 2m + 2$, we have the following expression for u^k :

$$(2.6) \quad \begin{aligned} \varepsilon u^{(k)} = & \mu \sum_{i=1}^{k-2} F_i^k(u, v, u', v', \dots, u^{(k-2)}, v^{(k-2)}) u^{(i)} \\ & + \mu \sum_{i=1}^{k-2} \frac{\partial^i f(u, v)}{\partial v^i} G_i^k(v', v'', \dots, v^{(k-1)}) \\ & + \mu \frac{\partial f(u, v)}{\partial u} u^{(k-1)} + \mu \frac{\partial^{k-1} f(u, v)}{\partial v^{k-1}} (v')^{k-1} - s u^{(k-1)}, \end{aligned}$$

where F_i^k, G_i^k are smooth functions of their variables.

By (2.1), (2.3)-(2.6) and (H1)-(H3), we have

$$(2.7) \quad u'(\xi_0) = u''(\xi_0) = \dots = u^{(2m+1)}(\xi_0) = 0.$$

Now by using (2.5) again, (2.6) and (2.7) imply that there are three possibilities of solution behavior at $\xi = \xi_0$:

- (a) $u^{(2m+2)}(\xi_0) < 0, v'(\xi_0) > 0$;
- (b) $u^{(2m+2)}(\xi_0) > 0, v'(\xi_0) < 0$;
- (c) $u^{(2m+2)}(\xi_0) = 0, v'(\xi_0) = 0$.

By the uniqueness of solution to (2.1) and (2.2) (cf. Lemma 4.1 in [1]) case (c) corresponds to $(u(\xi), v(\xi))$ being constant in $[-L, L]$. In other words, the critical

point of $u(\xi)$ should be either a local minimum or a local maximum for the non-constant function $u(\xi)$.

By symmetry, the same argument applies to the function $v(\xi)$. That is, if ξ_0 is a critical point of the non-constant function $v(\xi)$ and

$$(2.8) \quad \frac{\partial}{\partial u} g(u(\xi_0), v(\xi_0)) = 0,$$

then either $v^{(2n+2)}(\xi_0) < 0$, $u'(\xi_0) > 0$ or $v^{(2n+2)}(\xi_0) > 0$, $u'(\xi_0) < 0$ for $n \geq 1$.

On the other hand, if ξ_0 is a critical point of the non-constant $v(\xi)$ and

$$\frac{\partial}{\partial u} g(u(\xi_0), v(\xi_0)) \neq 0,$$

then either $v''(\xi_0) < 0$, $u'(\xi_0) > 0$ or $v''(\xi_0) > 0$, $u'(\xi_0) < 0$.

Therefore, if ξ_0 is a critical point of the non-constant function $u(\xi)$ (or $v(\xi)$), then ξ_0 must be either a point of local maximum or minimum of $u(\xi)$ (or $v(\xi)$) with $v'(\xi_0) \neq 0$ (or $u'(\xi_0) \neq 0$).

We now turn to proving that non-constant function $u(\xi)$ has at most one critical point in $(-L, L)$ by contradiction. Assume that there are two different critical points, denoted by τ , η of $u(\xi)$ with $-L < \tau < \eta < L$. The above argument implies that τ is a local maximum (or minimum) and η is a local minimum (or maximum) with $u'(\xi) < 0$ (or $u'(\xi) > 0$) for $\xi \in (\tau, \eta)$. Since we have $v'(\tau) > 0$ (or $v'(\tau) < 0$) and $v'(\eta) < 0$ (or $v'(\eta) > 0$); this implies that $v(\xi)$ attains a local maximum (or minimum) at some point $\xi_1 \in (\tau, \eta)$. The above discussion again implies that $u'(\xi_1) > 0$ (or $u'(\xi_1) < 0$), which contradicts the property of $u(\xi)$ in the interval (τ, η) . By symmetry, the same argument shows that the non-constant function $v(\xi)$ has at most one critical point in $(-L, L)$.

Finally, we will show that among non-constant functions $u(\xi)$ and $v(\xi)$, at most one of them has critical points in $(-L, L)$. Again, this can be proved by contradiction. Assume that τ is a critical point of $u(\xi)$ and η is a critical point of $v(\xi)$. The above argument shows that $u(\xi)$ and $v(\xi)$ cannot attain local extremums at the same point. Without loss of generality, we can assume $\tau < \eta$. By the above discussion, if τ is a local maximum (or minimum) point of $u(\xi)$, then $v'(\tau) > 0$ (or $v'(\tau) < 0$). This in turn implies that η must be a local maximum (or minimum) point of $v(\xi)$. However, in this case, we also have $u'(\eta) > 0$ (or $u'(\eta) < 0$) which implies that τ cannot be a local maximum (or minimum) point of $u(\xi)$. This contradiction implies that at most one of the non-constant functions $u(\xi)$ and $v(\xi)$ can have a critical point in $(-L, L)$. This completes the proof of the theorem.

The next theorem gives the *a priori* estimate on the boundedness of the solutions to (2.1) and (2.2).

Theorem 2.2 (*A priori estimate*). *Let $(u(\xi), v(\xi))$ be a solution of (2.1) and (2.2). Then the solution has the following properties:*

- (i) *There exists a constant M depending only on u^- , v^- , u^+ , v^+ , f and g which is then independent of $L \geq 1$ and $\mu \in [0, 1]$, such that*

$$(2.9) \quad \sup_{(-L, L)} |u(\xi)| \leq M, \quad \sup_{(-L, L)} |v(\xi)| \leq M.$$

- (ii) When $\mu = 1$ and $L = \infty$, there exists a constant N depending only on u^-, v^-, u^+, v^+, f and g , such that

$$(2.10) \quad \sup_{(-\infty, \infty)} |u(\xi)| \leq N, \quad \sup_{(-\infty, \infty)} |v(\xi)| \leq N.$$

Proof. By Theorem 2.1, we know that at least one of $u(\xi)$ and $v(\xi)$ is monotone. If both $u(\xi)$ and $v(\xi)$ are monotone, the theorem obviously holds. Thus, in what follows, we assume only one of the functions $u(\xi)$ and $v(\xi)$ is monotone. Without loss of generality, we assume $v(\xi)$ is increasing. Then by Theorem 2.1, there exists a constant $\xi_0 \in (-L, L)$ so that $u(\xi)$ is increasing on $(-L, \xi_0)$, while it is decreasing on (ξ_0, L) . Thus, the uniform bound on $v(\xi)$ is obvious, and we only need to consider $u(\xi)$ as follows. Since the wave speed s is assumed to be non-zero, the discussion can be divided into the following two cases.

Case 1. When $s > 0$, by integrating the first equation in (2.1) over (ξ_0, L) , we have

$$(2.11) \quad \varepsilon u'(L) - \varepsilon u'(\xi_0) = -su^+ + su(\xi_0) + \mu f(\mu u^+, \mu v^+) - \mu f(u(\xi_0), v(\xi_0)).$$

Thus, $u'(\xi_0) = 0$ and $u'(L) \leq 0$ yield

$$(2.12) \quad \begin{aligned} u(\xi_0) &= u^+ + \frac{\mu}{s} f(u(\xi_0), v(\xi_0)) - \frac{\mu}{s} f(\mu u^+, \mu v^+) + \frac{\varepsilon}{s} u'(L) \\ &\leq u^+ + \frac{\mu}{s} f(u(\xi_0), v(\xi_0)) - \frac{\mu}{s} f(\mu u^+, \mu v^+). \end{aligned}$$

By noting that $|v(\xi_0)|$ and μv^+ are bounded by $\max\{|v^-|, |v^+|\}$, the assumption (H4) on f yields that the right-hand side of (2.12) is bounded by a constant I which is independent of $L \geq 1$, $\mu \in [0, 1]$ and $\varepsilon > 0$. Hence $|u(\xi)|$ is bounded by $\max\{|u_-|, |u_+|, |I|\}$.

Case 2. When $s < 0$, by integrating the first equation in (2.1) over $(-L, \xi_0)$, we have

$$(2.13) \quad \varepsilon u'(\xi_0) - \varepsilon u'(-L) = -su(\xi_0) + su^- + \mu f(u(\xi_0), v(\xi_0)) - \mu f(\mu u^-, \mu v^-).$$

Hence, $u'(\xi_0) = 0$ and $u'(-L) \geq 0$ give

$$(2.14) \quad \begin{aligned} u(\xi_0) &= u^- + \frac{\mu}{s} f(u(\xi_0), v(\xi_0)) - \frac{\mu}{s} f(\mu u^-, \mu v^-) + \frac{\varepsilon}{s} u'(-L) \\ &\leq u^- + \frac{\mu}{s} f(u(\xi_0), v(\xi_0)) - \frac{\mu}{s} f(\mu u^-, \mu v^-). \end{aligned}$$

Similar to Case 1, (2.14) and (H4) imply that $u(\xi)$ is bounded independent of μ and L .

Therefore, the proof of the theorem is complete.

Similar to the argument in [1, 2], the uniform boundedness of solutions to the problem (2.1) and (2.2) immediately gives the existence of travelling wave profiles stated in the following theorem.

Theorem 2.3. *Under the assumptions (H1)-(H4), (1.2) and (1.4) admits a solution $(u(\xi), v(\xi))$ when the speed of wave $s \neq 0$.*

Remark 2.2. When the assumptions (H1) and (H3) are replaced by the following ones denoted by (H5) and (H6):

$$(H5) \quad \frac{\partial f(u, v)}{\partial v} \geq 0, \quad \frac{\partial g(u, v)}{\partial u} \geq 0,$$

(H6) $\frac{\partial^{2m+1} f(u,v)}{\partial v^{2m+1}} \Big|_{A_1} > 0$, $\frac{\partial^{2n+1} g(u,v)}{\partial u^{2n+1}} \Big|_{B_1} > 0$, the existence of travelling wave solutions to (1.2) and (1.4) with non-zero wave speed still holds.

Example. Here we use the following system to illustrate Theorem 2.3 on the existence of travelling wave profiles. Consider

$$(2.15) \quad \begin{cases} u_t - \frac{1}{2m+1} (v^{2m+1})_x = \varepsilon u_{xx}, \\ v_t - (u + \frac{1}{3} Au^3)_x = \varepsilon v_{xx}, \end{cases}$$

where $m \geq 1$ is an integer and $A \geq 0$ is a constant. Note that the above system when $\varepsilon = 0$ and $A = 0$ was studied in [3, 4].

A travelling wave with speed $s \neq 0$ to the system (2.15) satisfies the differential equations

$$(2.16) \quad \begin{cases} \varepsilon u'' = -su' - \frac{1}{2m+1} (v^{2m+1})', \\ \varepsilon v'' = -sv' - (u + \frac{1}{3} Au^3)', \end{cases}$$

with boundary condition

$$(2.17) \quad (u(-\infty), v(-\infty)) = (u^-, v^-), \quad (u(\infty), v(\infty)) = (u^+, v^+).$$

Note that the two eigenvalues of (2.15) are

$$(2.18) \quad \lambda_{\mp} = \mp(1 + Au^2)^{\frac{1}{2}} v^m,$$

with corresponding right eigenvectors given by

$$(2.19) \quad r_{\mp} = \left(v^m, \pm(1 + Au^2)^{\frac{1}{2}} \right)^T.$$

Thus,

$$(2.20) \quad \nabla \lambda_{\mp} \cdot r_{\mp} = \mp v^{m-1} (1 + Au^2)^{-\frac{1}{2}} \left(Auv^{m+1} \pm m(1 + Au^2)^{\frac{3}{2}} \right).$$

Since $\lambda_- = \lambda_+$ when $v = 0$, the system is not strictly hyperbolic. Moreover, the first (or second) characteristic field is linearly degenerate (cf. [5]) when

$$\begin{aligned} v^{m-1} \left(Auv^{m+1} + m(1 + Au^2)^{\frac{3}{2}} \right) &= 0 \\ \left(\text{or } v^{m-1} \left(Auv^{m+1} - m(1 + Au^2)^{\frac{3}{2}} \right) &= 0 \right). \end{aligned}$$

Thus, when $A > 0$ and $m \geq 2$, the characteristic fields of the system (2.15) are neither genuinely nonlinear nor linearly degenerate. However, it is straightforward to check that the assumptions (H1)-(H4) hold so that the system (2.15) admits a travelling wave solution $(u(\xi), v(\xi))$ for any given end states at $\pm\infty$ in (2.17).

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