THE UNIVERSAL CENTRAL EXTENSION OF THE THREE-POINT $\mathfrak{sl}_2$ LOOP ALGEBRA

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ABSTRACT. We consider the three-point loop algebra,

$$L = \mathfrak{sl}_2 \otimes \mathbb{K}[t, t^{-1}, (t-1)^{-1}],$$

where $\mathbb{K}$ denotes a field of characteristic 0 and t is an indeterminate. The universal central extension $\widehat{L}$ of L was determined by Bremner. In this note, we give a presentation for $\widehat{L}$ via generators and relations, which highlights a certain symmetry over the alternating group A_4 . To obtain our presentation of $\widehat{L}$, we use the realization of L as the tetrahedron Lie algebra.

1. INTRODUCTION

Throughout this paper, $\mathbb{K}$ will denote a field of characteristic 0. Consider the Lie algebra $\mathfrak{sl}_2$ over $\mathbb{K}$, and for an indeterminate t , the polynomial algebra $\mathbb{K}[t]$ localized at t and $t-1$:

$$\mathcal{A} = \mathbb{K}[t, t^{-1}, (t-1)^{-1}].$$

The *loop algebra* corresponding to $\mathfrak{sl}_2$ and $\mathcal{A}$ is the Lie algebra

$$L = \mathfrak{sl}_2 \otimes \mathcal{A},$$

with product $[x \otimes a, y \otimes b] = [x, y] \otimes ab$. (Unless specifically noted otherwise, all tensor products are over $\mathbb{K}$.)

The algebra $\mathcal{A}$ is a natural choice for a coordinate algebra. Any three distinct points $\alpha_1, \alpha_2, \alpha_3$ in $\mathbb{K} \cup \{\infty\}$ can be taken by the group $PGL_2(\mathbb{K})$ to the points $\infty, 0, 1$ respectively, so the algebra of rational functions over $\mathbb{K}$ with poles only at $\alpha_1, \alpha_2, \alpha_3$ is isomorphic to the algebra $\mathcal{A}$.

Our primary focus here is on central extensions of L , so we begin by recalling a few relevant definitions. A *central extension* of a Lie algebra $\mathcal{L}$ is a pair $(\mathcal{K}, \pi)$ consisting of a Lie algebra $\mathcal{K}$ and a surjective Lie algebra homomorphism $\pi : \mathcal{K} \rightarrow \mathcal{L}$ whose kernel lies in the center of $\mathcal{K}$. Given central extensions $(\mathcal{K}, \pi)$ and $(\mathcal{K}', \pi')$ of $\mathcal{L}$, by a *homomorphism* (resp. *isomorphism*) from $(\mathcal{K}, \pi)$ to $(\mathcal{K}', \pi')$ we mean a homomorphism (resp. isomorphism) of Lie algebras $\varphi : \mathcal{K} \rightarrow \mathcal{K}'$ such that $\pi = \pi' \circ \varphi$.

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A central extension $(\mathcal{K}, \pi)$ of $\mathcal{L}$ is *universal* if for any central extension $(\mathcal{K}', \pi')$ of $\mathcal{L}$, there exists a unique homomorphism from $(\mathcal{K}, \pi)$ to $(\mathcal{K}', \pi')$. A Lie algebra $\mathcal{L}$ possesses a universal central extension if and only if $\mathcal{L}$ is perfect (i.e. $\mathcal{L} = [\mathcal{L}, \mathcal{L}]$), and in this case, the universal central extension of $\mathcal{L}$ is unique up to isomorphism of central extensions. It is well established that the universal central extension plays a crucial role in representation theory; one need only look at the examples of the affine, Virasoro, and toroidal Lie algebras for affirmation of this statement (see [Ka], [MP]).

The loop algebra $L = \mathfrak{sl}_2 \otimes \mathcal{A}$ is easily seen to be perfect; therefore, L has a universal central extension which we denote by $(\widehat{L}, \pi)$. Bremner [Br] has given a detailed description of $(\widehat{L}, \pi)$. He has shown that the center of $\widehat{L}$ has dimension 2, and he has given an explicit basis and Lie bracket for $\widehat{L}$.

Our aim here is to give a presentation for $\widehat{L}$ via generators and relations that highlights a certain symmetry over the alternating group A_4 . Our point of departure is the realization of L as the tetrahedron algebra given by Hartwig and Terwilliger.

Definition 1.1 ([HT]). Let $\boxtimes$ denote the Lie algebra over $\mathbb{K}$ defined by generators $\{x_{i,j} \mid i, j \in \mathbb{I}, i \neq j\}$, $\mathbb{I} = \{0, 1, 2, 3\}$, and the following relations:

(i) For distinct $i, j \in \mathbb{I}$,

$$(1.2) \quad x_{i,j} + x_{j,i} = 0.$$

(ii) For mutually distinct $i, j, k \in \mathbb{I}$,

$$(1.3) \quad [x_{i,j}, x_{j,k}] = 2x_{i,j} + 2x_{j,k}.$$

(iii) For mutually distinct $i, j, k, \ell \in \mathbb{I}$,

$$(1.4) \quad [x_{i,j}[x_{i,j}, [x_{i,j}, x_{k,\ell}]]] = 4[x_{i,j}, x_{k,\ell}].$$

We call $\boxtimes$ the *tetrahedron algebra* or “tet” for short.

Theorem 1.5 ([HT]). *The Lie algebras $\boxtimes$ and L are isomorphic.*

We will obtain our presentation of $\widehat{L}$ as follows. First we will display a central extension $(\widehat{\boxtimes}, \pi)$ of $\boxtimes$, with $\widehat{\boxtimes}$ defined by generators and relations. Then we will modify the Lie algebra isomorphism $\sigma : \boxtimes \rightarrow L$ given in [HT] to obtain a homomorphism of Lie algebras, $\widehat{\sigma} : \widehat{\boxtimes} \rightarrow \widehat{L}$, such that the following diagram commutes:

$$(1.6) \quad \begin{array}{ccc} \widehat{\boxtimes} & \xrightarrow{\pi} & \boxtimes \\ \widehat{\sigma} \downarrow & & \downarrow \sigma \\ \widehat{L} & \xrightarrow{\pi} & L \end{array}$$

Using this and the universality of $(\widehat{L}, \pi)$, we will argue that $\widehat{\sigma} : \widehat{\boxtimes} \rightarrow \widehat{L}$ is an isomorphism and thereby determine a presentation of $\widehat{L}$ by generators and relations.

2. THE ISOMORPHISM $\sigma : \boxtimes \rightarrow L$

Let $\{e, f, h\}$ denote the canonical basis elements for $\mathfrak{sl}_2$ having products $[h, e] = 2e$, $[h, f] = -2f$, and $[e, f] = h$. The basis

$$X = 2e - h, \quad Y = -2f - h, \quad Z = h$$

is more suitable for our purposes. Following [ITW], we call X, Y, Z the *equitable basis* for $\mathfrak{sl}_2$, since

$$[X, Y] = 2X + 2Y, \quad [Y, Z] = 2Y + 2Z, \quad [Z, X] = 2Z + 2X.$$

There exists a unique $\mathbb{K}$ -algebra automorphism ι of $\mathcal{A}$ such that $t' = 1 - t^{-1}$. The map ι cyclically permutes t, t', t'' , where $t'' = (1 - t)^{-1}$, and so it has order 3. We have

$$(2.1) \quad tt' = t - 1, \quad t't'' = t' - 1, \quad t''t = t'' - 1.$$

It follows from (2.1), that a basis for the $\mathbb{K}$ -vector space $\mathcal{A}$ is given by

$$\{1\} \cup \{t^i, (t')^i, (t'')^i \mid i \in \mathbb{N}\},$$

where $\mathbb{N} = \{1, 2, 3, \dots\}$.

Proposition 2.2 ([HT, Thm. 11.5]). *There exists an isomorphism of Lie algebras $\sigma : \boxtimes \rightarrow L$ that sends*

$$\begin{aligned} x_{1,2} &\rightarrow X \otimes 1, & x_{0,3} &\rightarrow Y \otimes t + Z \otimes (t - 1), \\ x_{2,3} &\rightarrow Y \otimes 1, & x_{0,1} &\rightarrow Z \otimes t' + X \otimes (t' - 1), \\ x_{3,1} &\rightarrow Z \otimes 1, & x_{0,2} &\rightarrow X \otimes t'' + Y \otimes (t'' - 1), \end{aligned}$$

where X, Y, Z is the equitable basis for $\mathfrak{sl}_2$.

3. A CENTRAL EXTENSION OF $\boxtimes$

In this section, we will construct a central extension $(\widehat{\boxtimes}, \pi)$ of $\boxtimes$. Later we will show that this extension is universal. From now on, we identify the symmetric group S_4 with the group of permutations of $\mathbb{I}$.

Definition 3.1. Given a sequence (i, j, k) of mutually distinct elements of $\mathbb{I}$, there exists a unique $\tau \in S_4$ such that $\tau(0) = i$, $\tau(1) = j$, and $\tau(2) = k$. The sequence (i, j, k) is said to be *even* (resp. *odd*) whenever $\tau \in A_4$ (resp. $\tau \notin A_4$), where A_4 is the alternating subgroup of S_4 .

Definition 3.2. A partition p of $\mathbb{I}$ into two (disjoint) subsets, each with two elements, is said to have *shape* $(2, 2)$.

The set $P(\mathbb{I})$ of all partitions of $\mathbb{I}$ of shape $(2, 2)$ has cardinality 3.

Definition 3.3. Let $\widehat{\boxtimes}$ denote the Lie algebra over $\mathbb{K}$ defined by generators

$$\{X_{i,j} \mid i, j \in \mathbb{I}, i \neq j\} \cup \{C_p \mid p \in P(\mathbb{I})\}$$

and the following relations:

(i) For $p \in P(\mathbb{I})$,

$$C_p \text{ is central.}$$

(ii)

$$\sum_{p \in P(\mathbb{I})} C_p = 0.$$

(iii) For distinct $i, j \in \mathbb{I}$,

$$X_{i,j} + X_{j,i} = C_p,$$

where $p \in P(\mathbb{I})$ consists of $\{i, j\}$ and its complement in $\mathbb{I}$.

(iv) For mutually distinct $i, j, k \in \mathbb{I}$ such that (i, j, k) is even,

$$[X_{i,j}, X_{j,k}] = 2X_{i,j} + 2X_{j,k}.$$

(v) For mutually distinct $i, j, k, \ell \in \mathbb{I}$,

$$[X_{i,j}[X_{i,j}, [X_{i,j}, X_{k,\ell}]] = 4[X_{i,j}, X_{k,\ell}].$$

Lemma 3.4. *There exists a surjective homomorphism of Lie algebras $\pi : \widehat{\boxtimes} \rightarrow \boxtimes$ such that*

$$\begin{aligned} \pi(X_{i,j}) &= x_{i,j}, & i, j \in \mathbb{I}, i \neq j, \\ \pi(C_p) &= 0, & p \in P(\mathbb{I}). \end{aligned}$$

Proof. Compare the defining relations for $\boxtimes$ and $\widehat{\boxtimes}$ given in Definitions 1.1 and 3.3. $\square$

Lemma 3.5. *For mutually distinct $i, j, k \in \mathbb{I}$ such that (i, j, k) is odd, in the algebra $\widehat{\boxtimes}$ we have*

$$(3.6) \quad [X_{i,j}, X_{j,k}] = 2X_{i,j} + 2X_{j,k} + 2C_p,$$

where $p \in P(\mathbb{I})$ consists of $\{i, k\}$ and its complement in $\mathbb{I}$.

Proof. The sequence (k, j, i) is even since (i, j, k) is odd. Therefore, by Definition 3.3(iv),

$$[X_{k,j}, X_{j,i}] = 2X_{k,j} + 2X_{j,i}.$$

Evaluating this using (i)-(iii) of Definition 3.3, we obtain (3.6). $\square$

Lemma 3.7. *The following subspaces of $\widehat{\boxtimes}$ coincide:*

- (i) the kernel of π ,
- (ii) $\text{Span}\{C_p \mid p \in P(\mathbb{I})\}$,
- (iii) the center of $\widehat{\boxtimes}$.

Proof. Set $\mathcal{C} = \text{Span}\{C_p \mid p \in P(\mathbb{I})\}$. We first show $\mathcal{C} = \text{Ker}(\pi)$. We have $\mathcal{C} \subseteq \text{Ker}(\pi)$ by Lemma 3.4. To establish equality, observe that $\mathcal{C}$ is an ideal in $\widehat{\boxtimes}$, and let $\pi' : \widehat{\boxtimes} \rightarrow \widehat{\boxtimes}/\mathcal{C}$ denote canonical surjection with kernel $\mathcal{C}$. Since $\pi'(C_p) = 0$ for $p \in P(\mathbb{I})$, it follows from Definition 3.3 and Lemma 3.5 that the elements $\{\pi'(X_{i,j}) \mid i, j \in \mathbb{I}, i \neq j\}$ satisfy the defining relations (1.2)–(1.4) for $\boxtimes$. Therefore, there exists a Lie algebra homomorphism $\gamma : \boxtimes \rightarrow \widehat{\boxtimes}/\mathcal{C}$ such that $\gamma(x_{i,j}) = \pi'(X_{i,j})$ for all distinct $i, j \in \mathbb{I}$. From the construction, the following diagram commutes:

$$\begin{array}{ccc} \widehat{\boxtimes} & \xrightarrow{\pi} & \boxtimes \\ \text{id} \downarrow & & \downarrow \gamma \\ \widehat{\boxtimes} & \xrightarrow{\pi'} & \widehat{\boxtimes}/\mathcal{C} \end{array}$$

We may now argue

$$\mathcal{C} = \text{Ker}(\pi') = \text{Ker}(\gamma \circ \pi) \supseteq \text{Ker}(\pi),$$

which implies that $\mathcal{C} = \text{Ker}(\pi)$. Next we prove that the center $Z(\widehat{\boxtimes}) = \mathcal{C}$. We have $\mathcal{C} \subseteq Z(\widehat{\boxtimes})$ by Definition 3.3(i). To obtain the reverse inclusion, it suffices to show that the image of $Z(\widehat{\boxtimes})$ under π is zero. This image is contained in $Z(\boxtimes)$ since π is surjective. But $\boxtimes$ is isomorphic to L and $Z(L) = 0$, so $Z(\boxtimes) = 0$. Therefore the

image of $Z(\widehat{\boxtimes})$ under π is zero and consequently $Z(\widehat{\boxtimes}) \subseteq \mathcal{C}$. From these comments, we find that $Z(\widehat{\boxtimes}) = \mathcal{C}$. $\square$

Corollary 3.8. *The pair $(\widehat{\boxtimes}, \pi)$ is a central extension of $\boxtimes$.*

Proof. The map $\pi : \widehat{\boxtimes} \rightarrow \boxtimes$ is a surjective homomorphism of Lie algebras whose kernel is contained in the center of $\widehat{\boxtimes}$. $\square$

4. THE UNIVERSAL CENTRAL EXTENSION OF $L = \mathfrak{sl}_2 \otimes \mathcal{A}$

In this section, we give a detailed description of the universal central extension $(\widehat{L}, \pi)$ of L . In the next section we will use this description to define the map $\widehat{\sigma} : \widehat{\boxtimes} \rightarrow \widehat{L}$ mentioned in Section 1.

By results of Kassel [K] (see also [KL] and [BK]), the universal central extension of $\mathcal{L} := \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{B}$ for any finite-dimensional, complex simple Lie algebra $\mathfrak{g}$ and any commutative, associative algebra $\mathcal{B}$ with 1 over $\mathbb{C}$ is given by $\widehat{\mathcal{L}} = (\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{B}) \oplus (\Omega_{\mathcal{B}}^1/d\mathcal{B})$, where $\Omega_{\mathcal{B}}^1/d\mathcal{B}$ is the space of Kähler differentials modulo the exact forms of $\mathcal{B}$. This description enabled Bremner [Br] to show that the universal central extension of any n -point loop algebra over $\mathfrak{g}$ has an $(n-1)$ -dimensional kernel. The same argument works over any field $\mathbb{K}$ of characteristic 0. Applying this result to our loop algebra $L = \mathfrak{sl}_2 \otimes \mathcal{A}$, we see that $\dim_{\mathbb{K}} \widehat{L}/L = 2$.

An alternative description of $\widehat{L}$ can be found in [ABG]. Let $\mathcal{S}$ denote the subspace of $\mathcal{A} \otimes \mathcal{A}$ spanned by the elements $a \otimes b + b \otimes a$ and $ab \otimes c + bc \otimes a + ca \otimes b$ for all $a, b, c \in \mathcal{A}$. Let $\langle \mathcal{A}, \mathcal{A} \rangle = (\mathcal{A} \otimes \mathcal{A})/\mathcal{S}$, and write $\langle a, b \rangle = (a \otimes b) + \mathcal{S}$. From the construction we know that

$$(4.1) \quad \langle a, b \rangle + \langle b, a \rangle = 0,$$

$$(4.2) \quad \langle ab, c \rangle + \langle bc, a \rangle + \langle ca, b \rangle = 0$$

for all $a, b, c \in \mathcal{A}$. Then

$$\widehat{L} = (\mathfrak{sl}_2 \otimes \mathcal{A}) \oplus \langle \mathcal{A}, \mathcal{A} \rangle,$$

where $\langle \mathcal{A}, \mathcal{A} \rangle$ is central and

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab + (x | y) \langle a, b \rangle$$

for all $x, y \in \mathfrak{sl}_2$ and $a, b \in \mathcal{A}$. Here $(x | y)$ denotes the Killing form of $\mathfrak{sl}_2$. Thus, finding $\widehat{L}$ amounts to computing $\langle \mathcal{A}, \mathcal{A} \rangle$ explicitly.

Using the relations (4.1), (4.2), it is not difficult to show, just as in the affine case (see [Ka], for example), that

$$\langle f^m, f^n \rangle = m \delta_{m+n, 0} \langle f, f^{-1} \rangle$$

for $f = t, t'$, or t'' and all integers m, n , where δ is the Kronecker delta. Letting $g = f'$ and using $f' = 1 - f^{-1}$, we have

$$\begin{aligned} \langle f^m, g^n \rangle &= \sum_{k=0}^n (-1)^k \binom{n}{k} \langle f^m, f^{-k} \rangle \\ &= m (-1)^m \binom{n}{m} \langle f, f^{-1} \rangle \end{aligned}$$

for nonnegative integers m, n . From (2.1) we find that $tt't'' = -1$. Using this and (4.2), we see that $\langle \mathcal{A}, \mathcal{A} \rangle$ is spanned by $\langle t, t^{-1} \rangle$ and $\langle t', (t')^{-1} \rangle$ and that

$$\langle t, t^{-1} \rangle + \langle t', (t')^{-1} \rangle + \langle t'', (t'')^{-1} \rangle = 0.$$

Since $\widehat{L}/L$ has dimension 2, the space $\langle \mathcal{A}, \mathcal{A} \rangle$ has dimension 2. Consequently, $\langle t, t^{-1} \rangle$ and $\langle t', (t')^{-1} \rangle$ form a basis for $\langle \mathcal{A}, \mathcal{A} \rangle$.

The next result is now apparent.

Theorem 4.3 ([Br], [ABG]). *Let C denote a two-dimensional vector space over $\mathbb{K}$. Let $\mathbf{c}, \mathbf{c}'$ denote a basis for C and define $\mathbf{c}''$ so that*

$$\mathbf{c} + \mathbf{c}' + \mathbf{c}'' = 0.$$

Then the following (i)–(iii) hold:

- (i) *There exists a Lie algebra*

$$\widehat{L} = L \oplus C$$

with product

$$[\widehat{L}, C] = 0,$$

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab + (x | y) \langle a, b \rangle$$

for $x, y \in \mathfrak{sl}_2$ and $a, b \in \mathcal{A}$, where $(x | y)$ is the Killing form for $\mathfrak{sl}_2$, and where $\langle \cdot, \cdot \rangle : \mathcal{A} \times \mathcal{A} \rightarrow C$ is $\mathbb{K}$ -bilinear and satisfies

$\langle \cdot, \cdot \rangle$	1	t^j	$(t')^j$	$(t'')^j$
1	0	0	0	0
t^i	0	0	$(-1)^i i \binom{j}{i} \mathbf{c}$	$(-1)^{j+1} j \binom{i}{j} \mathbf{c}''$
$(t')^i$	0	$(-1)^{j+1} j \binom{i}{j} \mathbf{c}$	0	$(-1)^i i \binom{j}{i} \mathbf{c}'$
$(t'')^i$	0	$(-1)^i i \binom{j}{i} \mathbf{c}''$	$(-1)^{j+1} j \binom{i}{j} \mathbf{c}'$	0

for $i, j \in \mathbb{N}$.

- (ii) *There exists a homomorphism of Lie algebras $\pi : \widehat{L} \rightarrow L$ that has kernel C and acts as the identity on L .*
 (iii) *The pair $(\widehat{L}, \pi)$ is the universal central extension of L .*

5. A HOMOMORPHISM $\widehat{\sigma} : \widehat{\boxtimes} \rightarrow \widehat{L}$

Lemma 5.1. *There exists a unique Lie algebra homomorphism $\widehat{\sigma} : \widehat{\boxtimes} \rightarrow \widehat{L}$ specified by the following tables:*

p	Image of C_p under $\widehat{\sigma}$
$\{0, 1\}\{2, 3\}$	$-4\mathbf{c}''$
$\{0, 2\}\{1, 3\}$	$-4\mathbf{c}$
$\{0, 3\}\{1, 2\}$	$-4\mathbf{c}'$

i	j	Image of $X_{i,j}$ under $\widehat{\sigma}$
1	2	$X \otimes 1 - 4\mathfrak{c}'$
2	1	$-X \otimes 1$
2	3	$Y \otimes 1 - 4\mathfrak{c}''$
3	2	$-Y \otimes 1$
3	1	$Z \otimes 1 - 4\mathfrak{c}$
1	3	$-Z \otimes 1$
0	3	$Y \otimes t + Z \otimes (t-1) + 4\mathfrak{c}$
3	0	$-Y \otimes t - Z \otimes (t-1) + 4\mathfrak{c}''$
0	1	$Z \otimes t' + X \otimes (t'-1) + 4\mathfrak{c}'$
1	0	$-Z \otimes t' - X \otimes (t'-1) + 4\mathfrak{c}$
0	2	$X \otimes t'' + Y \otimes (t''-1) + 4\mathfrak{c}''$
2	0	$-X \otimes t'' - Y \otimes (t''-1) + 4\mathfrak{c}'$

where X, Y, Z is the equitable basis for $\mathfrak{sl}_2$.

Proof. It is routine to verify that the elements in the above tables satisfy the defining relations for $\widehat{\boxtimes}$ given in Definition 3.3. $\square$

Lemma 5.2. *The diagram in (1.6) commutes.*

Proof. By Definition 3.3, the following is a generating set for $\widehat{\boxtimes}$:

$$\{X_{i,j} \mid i, j \in \mathbb{I}, i \neq j\} \cup \{C_p \mid p \in P(\mathbb{I})\}.$$

We chase these generators around the diagram using the maps in Proposition 2.2, Lemma 3.4, Theorem 4.3(ii), and Lemma 5.1. For each generator the image under the composition $\pi \circ \widehat{\sigma}$ coincides with the image under the composition $\sigma \circ \pi$. $\square$

Theorem 5.3. *The Lie algebra homomorphism $\widehat{\sigma} : \widehat{\boxtimes} \rightarrow \widehat{L}$ from Lemma 5.1 is an isomorphism.*

Proof. We first show that $\widehat{\sigma}$ is surjective. The map $\pi : \widehat{\boxtimes} \rightarrow \boxtimes$ is surjective, and the map $\sigma : \boxtimes \rightarrow L$ is an isomorphism, so the composite map $\sigma \circ \pi : \widehat{\boxtimes} \rightarrow L$ is surjective. Therefore by Lemma 5.2, the composition $\pi \circ \widehat{\sigma} : \widehat{\boxtimes} \rightarrow L$ is surjective. The kernel of $\pi : \widehat{L} \rightarrow L$ is the space C from Theorem 4.3, and C is contained in the image of $\widehat{\sigma}$, so $\widehat{\sigma}$ is surjective. We now argue that $\widehat{\sigma}$ is injective. As before, set $\mathcal{C} = \text{Span}\{C_p \mid p \in P(\mathbb{I})\}$. The map $\pi : \widehat{\boxtimes} \rightarrow \boxtimes$ has kernel $\mathcal{C}$ by Lemma 3.7, and the map $\sigma : \boxtimes \rightarrow L$ is an isomorphism, so the composition $\sigma \circ \pi : \widehat{\boxtimes} \rightarrow L$ has kernel $\mathcal{C}$. By this and Lemma 5.2, the composition $\pi \circ \widehat{\sigma} : \widehat{\boxtimes} \rightarrow L$ has kernel $\mathcal{C}$. Consequently, the kernel of $\widehat{\sigma}$ is contained in $\mathcal{C}$. From the first table of Lemma 5.1, and the fact that $\mathfrak{c}, \mathfrak{c}'$ form a basis for C , we find that the restriction of $\widehat{\sigma}$ to $\mathcal{C}$ is injective. Therefore $\widehat{\sigma}$ is injective and hence an isomorphism. $\square$

Corollary 5.4. *The central extension $(\widehat{\boxtimes}, \pi)$ of $\boxtimes$ given in Corollary 3.8 is universal.*

Proof. By Lemma 5.2, the diagram in (1.6) commutes. The map σ is an isomorphism by Proposition 2.2, and $\widehat{\sigma}$ is an isomorphism by Theorem 5.3. The result follows from this, since $(\widehat{L}, \pi)$ is the universal central extension of L . $\square$

6. THE DEFINING RELATIONS FOR $\widehat{\boxtimes}$ REVISITED

Definition 3.3 gives the defining relations for the Lie algebra $\widehat{\boxtimes}$. In this section, we re-express these relations, this time using a notation that makes explicit the role of the alternating group A_4 .

We will view S_4 as acting on $\mathbb{I}$ from the right; this means that when applying a product $\alpha\beta$, we first apply α and then β . We consider the following normal subgroup of S_4 :

$$N = \{(01)(23), (02)(31), (03)(12), e\}.$$

This subgroup is contained in A_4 and is therefore a normal subgroup of A_4 . Let N' denote the set of nonidentity elements of N . For each $\eta \in N'$ let $[\eta]$ denote the partition of $\mathbb{I}$ consisting of the orbits of η . Note that the map $\eta \rightarrow [\eta]$ is a bijection from N' to $P(\mathbb{I})$. Two elements of A_4 ,

$$\zeta = (01)(23), \quad \vartheta = (012),$$

play a distinguished role in our computations. The first belongs to N' , while the second one does not. Together they generate A_4 .

Theorem 6.1. $\widehat{\boxtimes}$ is isomorphic to the Lie algebra over $\mathbb{K}$ that has generators

$$X_\alpha \quad (\alpha \in A_4), \quad C_\eta \quad (\eta \in N')$$

and the following relations:

(i) For $\eta \in N'$,

$$C_\eta \text{ is central.}$$

(ii)

$$\sum_{\eta \in N'} C_\eta = 0.$$

(iii) For $\alpha \in A_4$,

$$X_\alpha + X_{\zeta\alpha} = C_{\alpha^{-1}\zeta\alpha}.$$

(iv) For $\alpha \in A_4$,

$$[X_\alpha, X_{\vartheta\alpha}] = 2X_\alpha + 2X_{\vartheta\alpha}.$$

(v) For $\alpha \in A_4$ and for $\eta \in N'$, $\eta \neq \zeta$,

$$[X_\alpha, [X_\alpha, [X_\alpha, X_{\eta\alpha}]]] = 4[X_\alpha, X_{\eta\alpha}].$$

An isomorphism with the presentation in Definition 3.3 is given by

$$\begin{aligned} X_\alpha &\rightarrow X_{\alpha(0),\alpha(1)}, & \alpha &\in A_4, \\ C_\eta &\rightarrow C_{[\eta]}, & \eta &\in N'. \end{aligned}$$

Proof. Up to notation, the above presentation of $\widehat{\boxtimes}$ is the same as the one given in Definition 3.3. $\square$

7. CONCLUDING REMARKS

We conclude with some comments relating our results to the Onsager Lie algebra. This algebra was introduced in a seminal paper [O] in which the free energy of the two-dimensional Ising model was computed exactly. Since then it has been widely investigated by both the physics and mathematics communities in connection with solvable lattice models, representation theory, Kac-Moody Lie algebras, tridiagonal pairs, and partially orthogonal polynomials. In [P], Perk showed that the Onsager Lie algebra has a presentation by generators A, B and the following relations:

$$\begin{aligned} [A, [A, [A, B]]] &= 4[A, B], \\ [B, [B, [B, A]]] &= 4[B, A]. \end{aligned}$$

Let Ω (resp. Ω' , resp. Ω'') denote the subalgebra of $\hat{\boxtimes}$ generated by $x_{0,1}$ and $x_{2,3}$ (resp. $x_{0,2}$ and $x_{1,3}$, resp. $x_{0,3}$ and $x_{1,2}$). It was shown in [HT] that each of the Lie algebras Ω , Ω' , Ω'' is isomorphic to the Onsager algebra, and that $\hat{\boxtimes}$ is their direct sum. Our results lead to a similar decomposition of $\hat{\boxtimes}$ as follows: Let $\mathcal{O}$ (resp. $\mathcal{O}'$, resp. $\mathcal{O}''$) denote the subalgebra of $\hat{\boxtimes}$ generated by $X_{0,1}$ and $X_{2,3}$ (resp. $X_{0,2}$ and $X_{1,3}$, resp. $X_{0,3}$ and $X_{1,2}$). Using the commuting diagram (1.6) and the tables in Lemma 5.1, we find that each of the algebras $\mathcal{O}$, $\mathcal{O}'$, $\mathcal{O}''$ is isomorphic to the Onsager algebra and that $\hat{\boxtimes}$ is the direct sum $\mathcal{O} \oplus \mathcal{O}' \oplus \mathcal{O}'' \oplus \mathcal{C}$, where $\mathcal{C} := \text{Span}\{C_p \mid p \in P(\mathbb{I})\}$ is the two-dimensional center of $\hat{\boxtimes}$. Using the isomorphism $\hat{\sigma} : \hat{\boxtimes} \rightarrow \hat{L}$ in Theorem 5.3, we obtain a corresponding decomposition for the universal central extension $\hat{L}$ of the loop algebra L .

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