VOLUME OF TRUNCATED FUNDAMENTAL DOMAINS

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1. INTRODUCTION

For $T \gg 0$, by applying Arthur's analytic truncation $\bigwedge^T$ to the constant function $\mathbf{1}$ on $G(F) \backslash G(\mathbb{A})^1$, we obtain a characteristic function of a compact subset $\mathfrak{F}(T)$ of $G(F) \backslash G(\mathbb{A})^1$. The aim of this paper is to give a general formula for the volume of $\mathfrak{F}(T)$, which is an exponential polynomial function of T . Consequently, by letting $T \rightarrow \infty$, we obtain a new way to evaluate the volume of fundamental domain for $G(F) \backslash G(\mathbb{A})^1$.

While $\mathfrak{F}(T)$ seems to be quite arbitrary, its volume admits beautiful structures:

- (i) Surprisingly enough, the coefficients of T are special values of zeta functions.
 - (ii) The volume is an alternating sum associated to standard parabolic subgroups.
- ((ii) is suggested by geometry — after all, $\mathfrak{F}(T)$ is obtained from $G(F) \backslash G(\mathbb{A})^1$ by truncating off cuspidal regions which are known to be parametrized by standard parabolic subgroups.)

Besides all basic properties of Arthur's analytic truncation, our method is based on a fundamental formula of Jacquet-Lapid-Rogawski on an integration of truncated Eisenstein series associated to cusp forms. Their formula, an advanced version of the Rankin-Selberg method, is obtained using the so-called Bernstein principle via regularized integration over cones.

2. ARTHUR'S TRUNCATION AND INTEGRAL OF TRUNCATED EISENSTEIN SERIES

Let G be a reductive group and $P = MN$ be a parabolic subgroup. Fix a good maximal compact subgroup $K \subset G(\mathbb{A})$ such that the Iwasawa decomposition $G(\mathbb{A}) = N(\mathbb{A})M(\mathbb{A})K$ holds. Then under suitable normalization of Haar measures, we have

$$\int_{G(\mathbb{A})^1} f(g) dg = \int_{N(\mathbb{A})} \int_{A_P} \int_{M(\mathbb{A})^1} \int_K f(namk) e^{-2\langle \rho_P, H_P(a) \rangle} dndadm dk.$$

An element $T \in \mathfrak{a}_0$ is said to be *sufficiently positive* and denoted by $T \gg 0$ if for all $\alpha \in \Delta_0$, $\langle \alpha, T \rangle \gg 0$ are large enough. Fix such a T . Let $\phi : G(F) \backslash G(\mathbb{A}) \rightarrow \mathbb{C}$ be a smooth function. We define *Arthur's analytic truncation* $\bigwedge^T \phi$ (for ϕ with respect

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to the parameter T) to be the function on $G(F)\backslash G(\mathbb{A})$ given by

$$\left(\bigwedge^T \phi\right)(g) := \sum_{P: \text{standard}} (-1)^{\dim(Z_P/Z_G)} \sum_{\delta \in P(F)\backslash G(F)} \phi_P(\delta g) \cdot \hat{\tau}_P(H_P(\delta g) - T),$$

where ϕ_P denotes the constant term of ϕ along with the standard parabolic subgroup P , $\hat{\tau}_P$ is the characteristic function of the so-called positive cone in $\mathfrak{a}_P$, and $H_P(g) := \log_M m_P(g)$ is an element in $\mathfrak{a}_P$. (Since $\hat{\tau}_P(\lambda)$ depends only on the projection λ_P of λ in $\mathfrak{a}_P$, it does not matter whether we view H_P as a map onto $\mathfrak{a}_P$ or a map to $\mathfrak{a}_0$.)

Fundamental properties of Arthur's truncation may be summarized as

Theorem 1 ([A1, 2]). *For a sufficiently positive T in $\mathfrak{a}_0$, we have*

- (1) $\bigwedge^T \phi$ is rapidly decreasing, if ϕ is an automorphic form on $G(F)\backslash G(\mathbb{A})$,
- (2) $\bigwedge^T \circ \bigwedge^T = \bigwedge^T$,
- (3) $\bigwedge^T$ is self-adjoint,
- (4) $\bigwedge^T \mathbf{1}$ is a characteristic function of a compact subset of $G(F)\backslash G(\mathbb{A})^1$.

Denote by $\mathfrak{F}(T)$ the compact subset of $G(F)\backslash G(\mathbb{A})^1$ whose characteristic function is given by $\bigwedge^T \mathbf{1}$ of (4).

Corollary 2. *Let $T \gg 0$ be a fixed element in $\mathfrak{a}_0$. If ϕ is an automorphic form on $G(F)\backslash G(\mathbb{A})$,*

$$\int_{G(F)\backslash G(\mathbb{A})^1} \bigwedge^T \phi(g) dg = \int_{\mathfrak{F}(T)} \phi(g) dg.$$

Proof. By (1), $\int_{G(F)\backslash G(\mathbb{A})^1} \bigwedge^T \phi(g) dg$ is well defined. Moreover,

$$\begin{aligned} & \int_{G(F)\backslash G(\mathbb{A})^1} \bigwedge^T \phi(g) dg \\ &= \int_{G(F)\backslash G(\mathbb{A})^1} \mathbf{1}(g) \cdot \left(\bigwedge^T \circ \bigwedge^T\right) \phi(g) dg \quad (\text{by (2) above}) \\ &= \int_{G(F)\backslash G(\mathbb{A})^1} \bigwedge^T \mathbf{1}(g) \cdot \bigwedge^T \phi(g) dg \\ & \quad (\text{by (3) above since } \bigwedge^T \phi(g) \text{ is rapidly decreasing}) \\ &= \int_{G(F)\backslash G(\mathbb{A})^1} \left(\bigwedge^T \circ \bigwedge^T\right) \mathbf{1}(g) \cdot \phi(g) dg \\ & \quad (\text{by (3) again since } \phi \text{ is of moderate growth and } \bigwedge^T \mathbf{1} \text{ is compactly supported}) \\ &= \int_{G(F)\backslash G(\mathbb{A})^1} \bigwedge^T \mathbf{1}(g) \cdot \phi(g) dg \quad (\text{by (2) again}) \\ &= \int_{\mathfrak{F}(T)} \phi(g) dg \quad (\text{by (4)}). \end{aligned} \quad \square$$

Recall the following formula of Jacquet-Lapid-Rogawski.

Theorem 3 ([JLR, Corollary 17]). *Let $P = MN$ be a parabolic subgroup and let (M, σ) be a cuspidal datum. Let $E(g, \varphi, \lambda)$ be an Eisenstein series where $\varphi \in$*

$\mathcal{A}_P(G)_\sigma$. Then

$$\int_{G(F) \backslash G(\mathbb{A})^1} \bigwedge^T E(g, \varphi, \lambda) dg$$

is equal to zero if P is not a minimal parabolic. If P is minimal, it is equal to

$$v \sum_{w \in \Omega} \frac{e^{\langle w\lambda - \rho, T \rangle}}{\prod_{\alpha \in \Delta_0} \langle w\lambda - \rho, \alpha^\vee \rangle} \int_{M(F) \backslash M(\mathbb{A})^1 \times K} M(w, \lambda) \varphi(mk) dm dk,$$

where $v = \text{Vol}\left(\left\{\sum_{\alpha \in \Delta_0} a_\alpha \alpha^\vee : 0 \leq a_\alpha < 1\right\}\right)$.

Here the volume v appears because the measure on $\mathfrak{a}_0$ is defined using co-weights $\{\varpi_\alpha^\vee : \alpha \in \Delta_0\}$. So v is equal to the order of the fundamental group of Φ [Hu, page 68] for simply laced groups. For example, $v = n$ for SL_n ; $v = 2$ for E_7 , $v = 4$ for D_l , $v = 3$ for E_6 .

3. MAIN RESULT

Now we consider the special case of the G split, semi-simple group, $\varphi = 1$: The constant term of $E(g, 1, \lambda)$ along the Borel subgroup is given by

$$E_B(g, 1, \lambda) = \sum_{w \in \Omega} m(w, \lambda) e^{\langle w\lambda + \rho, H(g) \rangle},$$

where $\rho = \rho_B$, and

$$m(w, \lambda) = \prod_{\alpha > 0, w\alpha < 0} \frac{\xi_F(\langle \lambda, \alpha^\vee \rangle)}{\xi_F(\langle \lambda, \alpha^\vee \rangle + 1)},$$

by the Gindikin-Karpelevich formula. Here ξ_F is the Dedekind zeta function with the usual γ -factors, namely, $\xi_F(s) = (2^{-2r_2} \pi^{-N} d_F)^{\frac{s}{2}} \Gamma(\frac{s}{2})^{r_1} \Gamma(s)^{r_2} \zeta_F(s)$, where $\zeta_F(s)$ is the Dedekind zeta function, $[F : \mathbb{Q}] = N = r_1 + 2r_2$, and d_F is the discriminant. Also r_1 (r_2) is the number of real (complex) places of F , resp.

Then $\zeta_{-1} = \text{res}_{s=1} \zeta_F(s) = \text{Vol}(F^* \backslash \mathbf{I}_F^1) = \frac{2^{r_1} (2\pi)^{r_2} h R}{w d_F^{\frac{1}{2}}}$, where h is the class number, R the regulator, and w the number of roots of unity in F . We see that

$$\zeta_{-1} = \text{res}_{s=1} \xi_F(s) = \frac{2^{r_1} h R}{w} = (2\pi)^{-r_2} d_F^{\frac{1}{2}} \cdot \text{Vol}(F^* \backslash \mathbf{I}_F^1).$$

Hence we have

$$\begin{aligned} & \int_{G(F) \backslash G(\mathbb{A})^1} \bigwedge^T E(g, 1, \lambda) dg \\ (*) \quad &= v \sum_{w \in \Omega} \frac{e^{\langle w\lambda - \rho, T \rangle}}{\prod_{\alpha \in \Delta_0} \langle w\lambda - \rho, \alpha^\vee \rangle} m(w, \lambda) \int_{M(F) \backslash M(\mathbb{A})^1} dm, \end{aligned}$$

since $e^{\langle w\lambda + \rho, H(m) \rangle} = 1$ for $m \in M(\mathbb{A})^1$. We look at the residue at $\lambda = \rho$. Langlands [La] showed that the residue of $E(g, 1, \lambda)$ at $\lambda = \rho$ is constant. Hence the residue is given by the residue of the constant term along the Borel subgroup. Note that $\langle \rho, \alpha^\vee \rangle = 1$ for $\alpha \in \Delta_0$ and $\langle \rho, \alpha^\vee \rangle > 1$ if $\alpha \notin \Delta_0$, $\alpha > 0$. Hence the residue of $E(g, 1, \lambda)$ at $\lambda = \rho$ is given by

$$\text{res}_{\lambda=\rho} E(g, 1, \lambda) = \lim_{\lambda \rightarrow \rho} \prod_{\alpha \in \Delta_0} (\langle \lambda, \alpha^\vee \rangle - 1) \cdot m(w_0, \lambda),$$

where w_0 is the longest Weyl group element, i.e.,

$$m(w_0, \lambda) = \prod_{\alpha > 0} \frac{\xi_F(\langle \lambda, \alpha^\vee \rangle)}{\xi_F(\langle \lambda, \alpha^\vee \rangle + 1)}.$$

For $i > 1$, let $n_i = \#\{\alpha > 0 : \langle \rho, \alpha^\vee \rangle = i\} - \#\{\alpha > 0 : \langle \rho, \alpha^\vee \rangle = i-1\}$. Then

$$\text{res}_{\lambda=\rho} E(g, 1, \lambda) = \xi_{-1}^r \cdot \prod_{i>1} \xi_F(i)^{n_i},$$

where r is the rank of G .

Now by Corollary 2,

$$\int_{G(F) \backslash G(\mathbb{A})^1} \bigwedge^T E(g, 1, \lambda) dg = \int_{\mathfrak{F}(T)} E(g, 1, \lambda) dg.$$

Hence

$$\begin{aligned} \text{res}_{\lambda=\rho} \left(\int_{G(F) \backslash G(\mathbb{A})^1} \bigwedge^T E(g, 1, \lambda) dg \right) &= \int_{\mathfrak{F}(T)} \text{res}_{\lambda=\rho} E(g, 1, \lambda) dg \\ &= \text{Vol}(\mathfrak{F}(T)) \cdot \text{res}_{\lambda=\rho} E(g, 1, \lambda). \end{aligned}$$

So the residue at $\lambda = \rho$ of the left-hand side of (*) is

$$\xi_{-1}^r \prod_{i>1} \xi_F(i)^{n_i} \cdot \text{Vol}(\mathfrak{F}(T)).$$

On the other hand, the residue at $\lambda = \rho$ of the right-hand side of (*) is

$$\text{Vol}(T(F) \backslash T(\mathbb{A})^1) \cdot \lim_{\lambda \rightarrow \rho} \left(v \sum_{w \in \Omega} e^{\langle w\lambda - \rho, T \rangle} m(w, \lambda) \prod_{\alpha \in \Delta_0} \frac{\langle \lambda, \alpha^\vee \rangle - 1}{\langle w\lambda, \alpha^\vee \rangle - 1} \right).$$

Here $\text{Vol}(T(F) \backslash T(\mathbb{A})^1) = \left(\text{Vol}(F^* \backslash \mathbf{I}_F^1) \right)^r = \zeta_{-1}^r$. Clearly $\langle w\lambda, \alpha^\vee \rangle = \langle \lambda, (w^{-1}\alpha)^\vee \rangle$.

Also given $w \in \Omega$, the sets $\{\alpha \in \Delta_0 : w\alpha < 0\}$ and $\{\alpha \in \Delta_0 : w\alpha \in \Delta_0\}$ are disjoint. Let W_0 be the set of all $w \in \Omega$ such that

$$\{\alpha \in \Delta_0 : w\alpha < 0\} \cup \{\alpha \in \Delta_0 : w\alpha \in \Delta_0\} = \Delta_0.$$

Let $r_w = \#\{\alpha \in \Delta_0 : w\alpha < 0\}$. For $i > 1$, let

$$n_{i,w} = \#\{\alpha > 0, w\alpha < 0 : \langle \rho, \alpha^\vee \rangle = i\} - \#\{\alpha > 0, w\alpha < 0 : \langle \rho, \alpha^\vee \rangle = i-1\}.$$

Then

$$\begin{aligned} \lim_{\lambda \rightarrow \rho} \left(\sum_{w \in \Omega} e^{\langle w\lambda - \rho, T \rangle} m(w, \lambda) \prod_{\alpha \in \Delta_0} \frac{\langle \lambda, \alpha^\vee \rangle - 1}{\langle w\lambda, \alpha^\vee \rangle - 1} \right) \\ = \sum_{w \in W_0} e^{\langle w\rho - \rho, T \rangle} \frac{\xi_{-1}^r \prod_{i>1} \xi_F(i)^{n_{i,w}}}{\prod_{\alpha \in \Delta_0, w^{-1}\alpha \notin \Delta_0} (\langle w\rho, \alpha^\vee \rangle - 1)}. \end{aligned}$$

Note that $w = 1 \in W_0$ and it gives rise to 1. Also $w = w_0 \in W_0$ and it gives rise to $(-2)^{-r} \cdot \xi_{-1}^r \prod_{i>1} \xi_F(i)^{n_i} \cdot e^{\langle -2\rho, T \rangle}$.

Therefore,

$$\begin{aligned} \text{Vol}(\mathfrak{F}(T)) &= v \cdot (2\pi)^{r_2 r} d_F^{-\frac{r}{2}} \prod_{i>1} \xi_F(i)^{-n_i} \\ &\cdot \sum_{w \in W_0} \xi_{-1}^r \frac{\prod_{i>1} \xi_F(i)^{n_{i,w}}}{\prod_{\alpha \in \Delta_0, w^{-1}\alpha \notin \Delta_0} (\langle w\rho, \alpha^\vee \rangle - 1)} \cdot e^{\langle w\rho - \rho, T \rangle}. \end{aligned}$$

We claim that *there is one to one correspondence between W_0 and the standard parabolic subgroups of G* . The correspondence is as follows: Given $w \in W_0$, let $J_w = \{\alpha \in \Delta_0 : w\alpha \in \Delta_0\}$. Let $P = P_w$ be the standard parabolic subgroup corresponding to J_w .

Conversely, let $J \subset \Delta_0$. Then there is the set D_J of distinguished coset representatives of W/W_J . The elements in D_J are characterized by the property $wJ > 0$. Let w_J be the element of the maximal length in D_J . Let w_l be the element of the maximal length in W , and let $w_{l,J}$ be the element of the maximal length in W_J . Then $w_J = w_l w_{l,J}$, and $W_0 = \{w_J : J \subset \Delta\}$ [Ca, Prop. 1.14].

We claim that *if $w \in W_0$, $\langle w\rho, \alpha^\vee \rangle < 0$ for all $\alpha \in \Delta_0 - wJ_w$* : Suppose that for $\alpha \in \Delta_0 - wJ_w$, $w_l(\alpha) = -\beta$ for some $\beta \in \Delta_0$. Since $w^{-1} = w_{l,J_w} w_l$, $w^{-1}(\alpha) = -w_{l,J_w}(\beta)$. If $\beta \in J_w$, then $-w_{l,J_w}(\beta) \in J_w$ and $w^{-1}(\alpha) \in J_w$. But since $\alpha \notin wJ_w$, $w^{-1}(\alpha) \notin J_w$. Contradiction. Hence $\beta \notin J_w$ and $w_{l,J_w} > 0$. Therefore $w^{-1}(\alpha) < 0$. Hence

$$\prod_{\alpha \in \Delta_0, w^{-1}\alpha \notin \Delta_0} (\langle w\rho, \alpha^\vee \rangle - 1) = (-1)^{\text{rank } P_w} \prod_{\alpha \in \Delta_0 - wJ_w} (1 - \langle w\rho, \alpha^\vee \rangle).$$

Theorem 4. *Let $\mathfrak{F}(T)$ be the truncated domain for a sufficiently positive T . Then*

$$\begin{aligned} \text{Vol}(\mathfrak{F}(T)) &= v \cdot (2\pi)^{r_2 r} d_F^{-\frac{r}{2}} \prod_{i>1} \xi_F(i)^{-n_i} \\ &\cdot \sum_{w \in W_0} (-1)^{\text{rank } P_w} \xi_{-1}^{r_w} \frac{\prod_{i>1} \xi_F(i)^{n_{i,w}}}{\prod_{\alpha \in \Delta_0 - wJ_w} (1 - \langle w\rho, \alpha^\vee \rangle)} \cdot e^{\langle w\rho - \rho, T \rangle}. \end{aligned}$$

Note that as $T \rightarrow \infty$, only the term corresponding to $w = 1$ survives and the limit is exactly the volume of the fundamental domain given by Langlands [La], namely,

Corollary 5 (Langlands). *If $\mathfrak{F}$ is the fundamental domain, then*

$$\text{Vol}(\mathfrak{F}) = v \cdot (2\pi)^{r_2 r} d_F^{-\frac{r}{2}} \cdot \prod_{i>1} \xi_F(i)^{-n_i}.$$

We note that if $w \in W_0$, $r_w + \sum n_{i,w} = 0$, and $-\sum n_i = r$, the rank of G , $-\sum n_i$ is the number of positive roots such that $\langle \rho, \alpha^\vee \rangle = 1$. It is exactly r . Similarly, $-\sum n_{i,w}$ is the number of positive roots such that $w\alpha < 0$, and $\langle \rho, \alpha^\vee \rangle = 1$. It is exactly $r_w = \text{rank } P_w$. So $r_w + \sum (-n_i + n_{i,w}) = r$ for each $w \in W_0$. Then we can write the formula in Theorem 4 as

Corollary 6. *Let $\mathfrak{F}(T)$ be the truncated domain for a sufficiently positive T . Then*

$$\begin{aligned} \text{Vol}(\mathfrak{F}(T)) &= v \cdot (2\pi)^{r_2 r} d_F^{-\frac{r}{2}} \cdot \sum_{J \subset \Delta_0} (-1)^{\text{rank } P_J} \frac{\xi_{-1}^{\text{rank } P_J} \prod_{i>1} \xi_F(i)^{-n_i + n_{i,w_J}}}{\prod_{\alpha \in \Delta_0 - w_J J} (1 - \langle w_J \rho, \alpha^\vee \rangle)} \cdot e^{\langle w_J \rho - \rho, T \rangle}. \end{aligned}$$

Remark. The formula, as it stands, suggests that volumes of cuspidal regions corresponding to parabolic subgroups can be written in terms of special zeta values as well.

We look at three examples.

Example 1. $G = Sp_4$. Let $\Delta_0 = \{e_1 - e_2, 2e_2\}$. There are 4 positive roots $\{e_1 \pm e_2, 2e_1, 2e_2\}$ and 8 Weyl group elements $\{1, (12), c_1, c_2, (12)c_1, (12)c_2, (12)c_1c_2, c_1c_2\}$, where c_i 's are sign changes. Then $\rho = 2e_1 + e_2$. In this case, $\{e_1, e_1 + e_2\}$ is a basis of the co-weight lattice. Let

$$T = x \cdot e_1 + y \cdot (e_1 + e_2), \quad x, y \gg 0.$$

Then there are 4 standard parabolic subgroups corresponding to

$$J_1 = \Delta_0, \quad J_{c_1} = \{2e_2\}, \quad J_{(12)c_1c_2} = \{e_1 - e_2\}, \quad J_{c_1c_2} = \emptyset.$$

Accordingly, $v = 1$, and

$$\begin{aligned} \text{Vol}(\mathfrak{F}(T)) &= (2\pi)^{2r_2} d_F^{-1} \left(\xi_F(2) \xi_F(4) \right. \\ &\quad \left. - \frac{1}{4} \cdot \xi_{-1} \xi_F(2) \cdot e^{-4x-4y} - \frac{1}{3} \cdot \xi_{-1} \xi_F(2) \cdot e^{-3x-6y} + \frac{1}{4} \cdot \xi_{-1}^2 \cdot e^{-4x-3y} \right). \end{aligned}$$

Example 2. $G = SL_4$. Let $\Delta_0 = \{e_1 - e_2, e_2 - e_3, e_3 - e_4\}$. There are 6 positive roots and 24 Weyl group elements. Here $\rho = \frac{1}{2}(3e_1 + e_2 - e_3 - 3e_4)$, and $\{\frac{1}{4}(3e_1 - e_2 - e_3 - e_4), \frac{1}{2}(e_1 + e_2 - e_3 - e_4), \frac{1}{4}(e_1 + e_2 + e_3 - 3e_4)\}$ is a basis of the co-weight lattice. Let

$$T = x \cdot \frac{1}{4}(3e_1 - e_2 - e_3 - e_4) + y \cdot \frac{1}{2}(e_1 + e_2 - e_3 - e_4) + z \cdot \frac{1}{4}(e_1 + e_2 + e_3 - 3e_4),$$

with $x, y, z \gg 0$. Then there are 8 standard parabolic subgroups corresponding to

$$\begin{aligned} J_1 &= \Delta_0, \quad J_{(1432)} = \{e_2 - e_3, e_3 - e_4\}, \quad J_{(13)(24)} = \{e_1 - e_2, e_3 - e_4\}, \\ J_{(1234)} &= \{e_1 - e_2, e_2 - e_3\}, \quad J_{(1423)} = \{e_3 - e_4\}, \quad J_{(14)} = \{e_2 - e_3\}, \\ J_{(1324)} &= \{e_1 - e_2\}, \quad J_{(14)(23)} = \emptyset. \end{aligned}$$

Accordingly, $v = 4$, and

$$\begin{aligned} \text{Vol}(\mathfrak{F}(T)) &= 4 \cdot (2\pi)^{3r_2} d_F^{-\frac{3}{2}} \cdot \left(\xi_F(2) \xi_F(3) \xi_F(4) - \frac{1}{4} \cdot \xi_{-1} \xi_F(2) \xi_F(3) \cdot e^{-x-2y-3z} \right. \\ &\quad \left. - \frac{1}{4} \cdot \xi_{-1} \xi_F(2)^2 \cdot e^{-2x-4y-2z} - \frac{1}{4} \cdot \xi_{-1} \xi_F(2) \xi_F(3) \cdot e^{-3x-2y-z} \right. \\ &\quad \left. + \frac{1}{6} \cdot \xi_{-1}^2 \xi_F(2) \cdot e^{-2x-4y-3z} + \frac{1}{9} \cdot \xi_{-1}^2 \xi_F(2) \cdot e^{-3x-3y-3z} \right. \\ &\quad \left. + \frac{1}{6} \cdot \xi_{-1}^2 \xi_F(2) \cdot e^{-3x-4y-2z} - \frac{1}{8} \cdot \xi_{-1}^3 \cdot e^{-3x-4y-3z} \right). \end{aligned}$$

Example 3. More generally, let $G = \text{SL}_n$. Let $\Delta_0 = \{e_1 - e_2, \dots, e_{n-1} - e_n\}$. In this case, $n_i = -1$ for $i = 2, \dots, n$; $\rho = \frac{1}{2}((n-1)e_1 + (n-3)e_2 + \dots + (n-1)e_n)$. Any parabolic subgroup is associated to a parabolic subgroup with the Levi subgroup whose derived group is $\text{SL}_{m_1} \times \dots \times \text{SL}_{m_k}$, where $m_1 + \dots + m_k + m = n$. We assume that $m_1 \geq \dots \geq m_k \geq 2$ and $k \geq 2$. It corresponds to the subset

$$\begin{aligned} J = \{e_1 - e_2, \dots, e_{m_1-1} - e_{m_1}, e_{m_1+1} - e_{m_1+2}, \dots, e_{m_1+m_2-1} - e_{m_1+m_2}, \dots, \\ e_{m_1+\dots+m_{k-1}+1} - e_{m_1+\dots+m_k}\}. \end{aligned}$$

Then $\text{rank } P_J = m + k - 1$, and w_J is given by the combination of

$$\begin{pmatrix} 1 & \cdots & m_1 \\ n - m_1 + 1 & \cdots & n \end{pmatrix} \begin{pmatrix} m_1 + 1 & \cdots & m_1 + m_2 \\ n - m_1 - m_2 + 1 & \cdots & n - m_1 \end{pmatrix} \\ \begin{pmatrix} m_1 + \cdots + m_{k-1} + 1 & \cdots & m_1 + \cdots + m_{k-1} + m_k \\ m + 1 & \cdots & m + m_k \end{pmatrix} \begin{pmatrix} n - m + 1 & \cdots & n \\ m & \cdots & 1 \end{pmatrix}.$$

We can show that

$$\Delta_0 - w_J J = \left\{ e_1 - e_2, \dots, e_m - e_{m+1}, e_{n-m_1} - e_{n-m_1+1}, \dots, \right. \\ \left. e_{n-m_1-\cdots-m_{k-1}} - e_{n-m_1-\cdots-m_{k-1}+1} \right\},$$

and n_{i, w_J}

$$= \begin{cases} k-1, & \text{for } 1 \leq i \leq m_k, \\ k-j, & \text{for } m_{k-j+2} + \cdots + m_k + 1 \leq i \leq m_{k-j+1} + \cdots + m_k, \\ & \text{and } 2 \leq j \leq k, \\ -1, & \text{for } m_1 + \cdots + m_k < i \leq n. \end{cases}$$

Hence if $m \geq 1$, the coefficient corresponding to J in Corollary 6 is

$$\frac{(-1)^{m+k-1}}{2^{m-1}(m_1 + m_2)(m_2 + m_3) \cdots (m_{k-1} + m_k)(m_k + 1)} \cdot \left(\xi_{-1}^{m+k-1} \prod_{j=1}^k \prod_{i=2}^{m_j} \xi_F(i) \right).$$

If $m = 0$, then it is

$$\frac{(-1)^{k-1}}{(m_1 + m_2)(m_2 + m_3) \cdots (m_{k-1} + m_k)} \cdot \left(\xi_{-1}^{k-1} \prod_{j=1}^k \prod_{i=2}^{m_j} \xi_F(i) \right).$$

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